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2003

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Abstract:

Dempster rule of combination of belief functions is shown not to commute with restriction to a sublanguage – badly for one version of the rule and less badly, but still for an alternative version.

Keywords:
Dempster-Shafer theory, belief functions, Dempster rule

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For basic information on Dempster-Shafer theory (belief function, basic belief [probability] assignment] etc.) see [1]. Note that Shafer deals only with regular belief functions, i.e. assigning 0 to empty set. This is generalized in [2] (see also [3]). Here we deal with general (both regular and singular) belief functions. If m is basic belief assignment then the corresponding belief function is $bel(A) = \sum_{B \subset A} m(B)$. Note that $bel(A) + bel(W - A) \leq 1 + m(\emptyset)$. Dempster rule is a famous operation assigning to two belief functions bel_1, bel_2 on the same (finite) set W their combination $bel_1 \oplus bel_2$. It can be defined using the corresponding basic belief assignments m_1, m_2, defining, for $A \subseteq W$, $(bel_1 \oplus bel_2)(A) = \sum_{B \cap C \subseteq A} m_1(B) \cdot m_2(C)$. (Note that $bel_1 \oplus bel_2$ need not be regular even if both bel_1 an bel_2 is.) Alternatively, one can use Dempster spaces (as defined in the pioneering paper [4]) $\mathbf{D}_i = (E_i, W, \Gamma_i, \mu_i)(i = 1, 2)$ and define their product to be $\mathbf{D} = (E, W, \Gamma, \mu)$ where $E = E_1 \times E_2$, $\Gamma(e_1, e_2) = \Gamma_1(e_1) \cap \Gamma_2(e_2)$ and μ is the product measure ($\mu(e_1, e_2) = \mu_1(e_1) \cdot \mu_2(e_2)$). If bel_i is given by $\mathbf{D}_i$ then $bel_1 \oplus bel_2$ is given by $\mathbf{D}$.

When we are interested in belief functions on (classes of equivalent boolean) formulas and consider formulas built from finitely many variables $p_1, \ldots, p_n$ then we may identify formulas with subsets of 2^n (sets of n-tuples of zeros and ones) in the obvious way; then we may compute, given bel_i, the corresponding belief assignments and define $bel_1 \oplus bel_2$.

Let B_n be the algebra of formulas built from propositional variables $p_1, \ldots, p_n$. If $k < n$ then B_k is a subalgebra of B_n and the restriction of a belief function on B_n to B_k is a belief function in B_k. This is immediate using the condition of superadditivity: A function bel mapping the algebra of (classes of equivalent) formulas into $[0, 1]$ is a belief function (see [1]) iff $bel(true) = 1$ and

$$bel(\varphi_1 \vee \cdots \vee \varphi_n) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} bel(\bigwedge_{i \in I} \varphi_i).$$

A binary operation F on belief functions (on the same algebra) *commutes with restriction* if for any pair bel_1, bel_2 of belief functions on B_n and for $k < n$,

$$F(bel_1, bel_2) \upharpoonright B_k = F(bel_1 \upharpoonright B_k, bel_2 \upharpoonright B_k)$$

A trivial example is a convex combination: for $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, $F(bel_1, bel_2)(A) = \alpha bel_1(A) + \beta bel_2(A)$.

Our question is, if Dempster rule commutes with restriction, i.e. if we consider formulas from B_7, say, i.e. built using $p_1, \ldots, p_7$ and among them we take a φ containing only p_1, p_2, p_3, can we compute $(bel_1 \oplus bel_2)(\varphi)$ working only with restriction of bel_i to B_3? The answer is NO – for our definition of bel. The following is an example, with $n = 2$ and $k = 1, bel_1 = bel_2, \varphi$ is p, $bel_1(p) = 0.9, m'_1 = m'_2$ is the *b.b.a* for $bel_1 \upharpoonright B_1$. Furthermore, $bel = bel_1 \oplus bel_2, bel' = bel'_1 \oplus bel'_2$.

	$p \& q$	$p \& \neg q$	true
m_1	.2	.7	.1

	p	true
m'_1	0.9	0.1

	$p \& q$	$p \& \neg q$	true
$p \& q$	$p \& q$ .04	false .14	$p \& q$ .02
$p \& \neg q$	false .14	$p \& \neg q$ .49	$p \& \neg q$ .07
true	$p \& q$ .02	$p \& \neg q$ .07	true .01

	p	true
p	p .81	p .09
true	p .09	true .01

	false	$p\&q$	$p\&\neg q$	true
$m_1 \oplus m_2$	.28	.08	.63	.01

	p	true
$m_1' \oplus m_2'$	.99	.01

Observe the following: $bel(p) = bel'(p) =$0.99, $bel(\neg p) = bel(false) = 0.28$, $bel'(\neg p) = bel'(false) = 0$.

Let us consider restriction from B_n to B_{n-1}, or, equivalently, the belief function bel' on subsets of 2^{n-1} induced by bel on subsets of 2^n. Let A, B, C run on subsets of 2^{n-1} and U, V, W on subsets of 2^n. Define

$$ext(A)\{\langle\varepsilon_1, \ldots, \varepsilon_n\rangle | \langle\varepsilon_1, \ldots, \varepsilon_{n-1}\rangle \in A, \varepsilon_n \in \{0,1\}\},$$

$$proj(U)\{\langle\varepsilon_1, \ldots, \varepsilon_{n-1}\rangle | \text{ for some } \varepsilon_n, \langle\varepsilon_1, \ldots, \varepsilon_{n-1}\varepsilon_n\rangle \in U\},$$

(extension and projection). Clearly, $bel'(A) = bel(ext(A))$. If m is the b.b.a. of bel, put

$$m'(A) = \sum_{proj(U)=A} m(U);$$

this is the b.b.a. of bel'. Indeed, clearly $\sum_{A\subseteq 2^n} m'(A) = 1$; moreover,

$$bel'(A) = bel(ext(A)) = \sum_{U\subseteq ext(A)} m(U) = \sum_{proj(U)\subseteq A} m(U) =$$

$$= \sum_{B\subseteq A} \sum_{proj(U)=B} m(U) = \sum_{B\subseteq A} m'(B).$$

Lemma
(1) Under the above notation, $proj(U \cap V) \subseteq proj(U) \cap proj(V)$; but there are U, V for which this inclusion is proper.
(2) For each $A \subseteq 2^{n-1}, proj(U) \subseteq A$ iff $U \subseteq ext(A)$.

Proof.
(1) Evidently $(U \cap V) \subseteq U$ and hence $proj(U \cap V) \subseteq proj(U)$, similarly $proj(U \cap V) \subseteq proj(V)$ and hence $proj(U \cap V) \subseteq proj(U) \cap proj(V)$. For a counterexample take $n = 2$, $U = \{(1,0)\}, V = \{(1,1)\}$; then $U \cap V = \emptyset = proj(U \cap V)$, but $proj(U) = proj(V) = \{(1)\} = proj(U) \cap proj(V)$.
(2) is obvious. □

Theorem 1. Let, for $i = 1,2$ bel_i be a belief function on B_n and bel_i' its restriction to B_{n-1}. Let $bel = bel_1 \oplus bel_2, bel' = bel_1' \oplus bel_2'$. Then, for each $\varphi \in B_{n-1}$,

$$bel'(\varphi) \leq bel(\varphi)$$

.

Proof. Let A be the set of $(n-1)$-tuples satisfying φ; we claim $bel'(A) \leq bel(ext(A))$.

We compute:

$$(bel_1' \oplus bel_2')(A) = \sum_{B\cap C\subseteq A} m_1'(B).m_2'(C) =$$

$$= \sum_{B\cap C\subseteq A} \Big(\sum_{proj(U)=B} m_1(U). \sum_{proj(V)=C} m_2(V)\Big) =$$

$$= \sum_{proj(U)\cap proj(V)\subseteq A} m_1(U) \cdot m_2(V) \leq \sum_{proj(U\cap V)\subset A} m_1(U) \cdot m_2(V) =$$

$$= \sum_{U \cap V \subseteq ext(A)} m_1(U) \cdot m_2(V) =$$

$$= bel_1 \oplus bel_2(ext(A))$$

. □

*

So far so good; but now consider another (and more usual) definition of the belief function given by a possibly singular b.b.a. m, namely $Bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B)$ (this works for $A \neq \emptyset$; one puts $Bel(\emptyset) = 0$). This may be not-normalized ($Bel(true)$ may be < 1). A normalized belief function is then $nBel(A) = (\sum_{\emptyset \neq B \subseteq A} m(B)) / (\sum_{B \neq \emptyset} m(B))$. Dempster rule for (normalized) belief functions Bel_1, Bel_2 is then defined for $A \neq \emptyset$ as

$$(Bel_1 \oplus Bel_2)(A) = \sum_{\emptyset \neq B \cap C \subseteq A} m_1(B) \cdot m_2(C),$$

(non-empty subsets!), $(Bel_1 \oplus Bel_2)(\emptyset) = 0$. This may give a non-normalized belief function; it can be normalized by dividing by $1 - \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C)$ (if this is non-zero). Even if reasonably motivated, it has the following (unwanted?) property:

Observation. Let $m_i, Bel_i, Bel'_i, Bel, Bel'$ be as above but belief functions in the new meaning just defined; Dempster rule either normalized or not. Then no inequality can be proved for $Bel(A), Bel'(A)$; all three possibilities

$$Bel'(A) < Bel(A), Bel'(A) = Bel(A), Bel'(A) > Bel(A)$$

may occur (and similarly for $nBel', nBel$).

For $Bel'(p) > Bel(p)$ consider the example above: $(Bel_1 \oplus Bel_1)(p) = .08 + .63 = .71$ (normalized: 71/72) but $(Bel'_1 \oplus Bel'_2)(p) = .99$ (normal).

Now $Bel'(\varphi) = Bel(\varphi)$ as well as $bel'(\varphi) = bel(\varphi))$ holds for all $\varphi \in B_{n-1}$ if all focal elements have the form $ext(A)$ for some $A \subseteq 2^n$ (verify!), In general, if m is a b.b.a. and bel is the corresponding belief function in our former sense then $Bel(\varphi) = bel(\varphi) - bel(false)$ is the (possibly non-normalized) belief function in the latter sense and $nBel(\varphi) = (bel(\varphi) - bel(false)/(1 - bel(false))$ is its normalisation. Now if $bel = bel_1 \oplus bel_2$ and $bel' = bel'_1 \oplus bel'_2$ in the former sense then Dempster sum in the latter sense of Bel_1, Bel_2 is $bel - bel(false)$ and its normalized $nBel$ is $(bel - bel(false))/(1 - bel(false))$; similarly for bel'_i, bel'. Put $x = bel'(\varphi), y = bel(\varphi), c = bel'(false), d = bel(false)$. Then $x \leq y$ and $c \leq d$ (by Theorem 1) and $Bel'(\varphi) \leq Bel(\varphi)$ iff $x - c \leq y - d$ iff $(d - c) \leq (y - x)$ and $nBel'(\varphi) \leq nBel(\varphi)$ iff $\frac{x-c}{1-c} \leq \frac{y-d}{1-d}$ iff $(d - c) \leq y(1 - c) - x(1 - d)$ (by elementary computations). In particular, the following is an example for $Bel'(\varphi) < Bel(\varphi)$ and for $nBel'(\varphi) < nBel(\varphi)$:

$$\begin{array}{lll} m_1(q) = 0.5 & m_1(true) = 0.5 & m'_1(p) = m'_2(p) = 0 \\ m_2(p \equiv q) = 0.5 & m_2(true) = 0.5 & m'_1(true) = m'_2(true) = 0 \end{array}$$

then

$$bel(p) = 0.25 = Bel(p) = nBel(p) \qquad bel'(p) = 0 = Bel'(p) = nBel'(p)$$

1 Conclusion.

As commonly known, the set $BEL(B_n)$ all belief functions on algebra B_n endowed with the operation $\oplus$ makes BEL to a commutative semigroup with a unit element. The natural projection of B_n onto B_{n-1} (or to $B_k, k < m$) induces a natural projection of $BEL(B_n)$ onto $BEL(B_{n-1})$. The fact that this projection does not commute with the operation $\oplus$ (i.e. it is not a semigroup homomorphism) may be disappointing; but, to take it positively, contributes to our understanding of Dempster rule, which has proven to be useful in many situation. From possible variants of the definition of Dempster rule, our first variant shows to have smoother properties than the other ones; but on the other hand, the fact that $bel(\varphi)+bel(\neg\varphi)$ may be more than 1 may be felt to be counter-intuitive (if not carefully interpreted). One can ask if there is still other natural variant of Dempster-style combination of belief functions that would commute with taking subalgebra. This remains as an interesting open problem.

Remark. In this context let us mention that in [5] (section 2.4) the authors consider a related but different problem that can be formulated using our framework as follows: Let B_n be given and let B_{n_1}, B_{n_2} be subalgebra of formulas built from propositional variables $p_1, ...p_i$ and $p_{i+1}, ...p_n$ respectively. Let Bel be a normalized belief function on B_n and let Bel_1, Bel_2 be projections to B_{n_1}, B_{n_2} respectively; let $Bel_1^{'}, Bel_2^{'}$ be their extensions to B_n and let $Bel^* = Bel_1^{'} \oplus Bel_2^{'}$. Can ones conclude $Bel(A) \leq Bel^*(A)$ or $Bel^*(A) \leq Bel(A)$? They give examples showing that the answer is *no*. (Thanks are due to D. Dubois for calling our attention to [5].)

Acknowledgement

Partial support by the grant No. AA1030004/00 of the Grant Agency of the Academy of Sciences of the Czech Republic is acknowledged.

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