Fleas and Fuzzy Logic - A Survey

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Dedicated to Professor Gert H. Müller on the occasion of his 80$^{\mathrm{th}}$ birthday

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Abstract:

Several logics generalizing the logic BL (basic fuzzy logic) are surveyed and reasons for their investigation are discussed; results on the new flea logic are listed.

Keywords:
mathematical fuzzy logic, basic fuzzy logic, monoidal t-norm logic, hoop logic, non-commutative fuzzy logic.

This paper is a companion to my [12]. The latter is a rather technical paper developing a generalized (mathematical) fuzzy logic. Main definitions and results of [12] are reproduced here without any proofs; but this is preceded by a survey on the basic propositional fuzzy logic BL and its generalizations and by an attempt to explain why these generalizations are worth of study.

1. Basic fuzzy logic. (See [8] for details.) The term "fuzzy logic" is understood in its narrow sense – symbolic logical systems, with a comparative notion of truth. We restrict ourselves to *truth-functional* logic, i.e. semantics of connectives is given by their truth functions. The standard domain of truth degrees is the real unit interval $[0, 1]$ with its natural order. Each continuous t-norm $*$ can be taken as the truth-function of conjunction; thus its residuum is taken to be the truth function of implication. Some other connectives are defined, notably $\neg\varphi$ is $\varphi \to 0$. The three most important continuous t-norms are Łukasiewicz ($x * y = \max(0, x + y - 1)$), Gödel $x * y = \min(x \cdot y)$, and product t-norm ($x * y = x \cdot y$).

A continuous t-norm $*$ and an evaluation e of propositional variables unique defines the value $e_*(\varphi)$ for any propositional formula φ built from the variables using conections $\&, \to$ and the truth constant 0. The formula φ is a t-tautology (tautology of the continuous t-norm logic) of $e_*(\varphi) = 1$ for each e and each $*$.

In general, truth derees need not to be linearly ordered; the domain of truth degrees may just be a bounded lattice (partially ordered set in which each pair of elements has sup and inf and having a largest and a least element).

A *BL-algebra* is an algebra $\mathbf{L} = (L, \wedge, \vee, *, \to, 0_L, 1_L)$ where $(L, \wedge, \vee, 0_L, 1_L)$ is a bounded lattice (with $0_L, 1_L$ as least and largest element, $*$ is a binary operation which is associative, commutative and $1 * x = x$ for all $x, \to$ is its residuum (i.e. $z \leq x \to y$ iff $x * z \leq y$) and the following axioms of divisibility and prelinearity are satisfied: $x \wedge y = x * (x \to y)$, $(x \to y) \vee (y \to x) = 1$.

These are properties satisfied by each continuous t-norm and its residuum. Each BL-algebra $\mathbf{L}$ serves as possible algebra of truth functions; one defines $e_{\mathbf{L}}(\varphi)$ for each $\mathbf{L}$ and each evaluation of propositional variables by elements of $\mathbf{L}$ in the obvious way; φ is a BL-tautology if $e_{\mathbf{L}}(\varphi) = 1_L$ for each $\mathbf{L}$ and e.

Seven (schemes of) BL-tautologies (A1)–(A7) (similar to those listed below) are taken for axioms of the basic fuzzy propositional logic BL; modus ponens is the deduction rule. This gives the notion of *provability* of a formula φ in BL (notation: $BL \vdash \varphi$.)

BL-algebras have *subdirect representation* property: each BL-algebra $\mathbf{L}$ is a subalgebra of a direct product of linearly ordered BL-algebras (BL-chains). This is crucial for (a part of) the following.

Completeness theorem. The following are equivalent: (i) $BL \vdash \varphi$, (ii) φ is a BL-tautology, (iii) φ is a BL-chain tautology ($e_{\mathbf{L}}(\varphi) = 1$ for each BL-chain $\mathbf{L}$ and each $\mathbf{L}$-evaluation e),, (iv) φ is a t-tautology. (See [8] and [2].)

Moral: You are free to work with not necessarily linearly ordered domains of truth degrees or just to insist on continuous t-norms on $[0, 1]$; but this gives the same logical truths.

Note that the corresponding predicate calculus is well elaborated (see [8]); but it will not be discussed here.

2. Generalizations. BL has been generalized in three directions: First, a t-norm has residuum (and hence leads to a logic with reasonable implication) iff it is left continuous; the corresponding logic is MTL (monoidal t-norm logic, [4]) and the definition of the corresponding MTL-algebras results from the definition of BL-algebras by omitting the axiom of divisibility. A natural example of a t-norm which is left continuous but not continuous is Fodor's nilpotent minimum $x * y = 0$ for $x, y \leq \frac{1}{2}$, otherwise $x * y = \min(x, y)$. (See [14] for extensive theory of t-norms.)

Second, BL-algebras an particular hoops (cf. [1], [**?**]). The corresponding logic is "falsity-free" – the truth degrees may not have a least element. (Think of real product on the half-closed interval $(0, 1]$.) Studying the hoop logic one gets reasonable *conservativeness results:* a formula φ not containing negation is provable in BL iff it is provable in the corresponding hoop logic.

Third, one you give up commutativity of conjunction. This was started by [7] and the corresponding logic is in [10]. There are various examples of non-commutative conjunction in natural language (e.g. "not only – but also –"; "– and then –"). Besides, there are good mathematical reasons for the study of non-commutative analogon of BL (called $psBL$ – pseudo BL). Remarkably, here we loose subdirect representability, i.e. $psBL$-tautologies are a proper subset of $psBL$-chain-tautologies [15]. An example of a non-commutative left continuous (pseudo)-t-norm of $[0,1]$ was given by Mesiar: let $0 < a < b < 1$ and let $x * y = 0$ for $x \leq a$, $y \leq b$, $x * y = 0$ otherwise.

Now it is natural to ask if we make all those three generalizations: we get the *flea logic,* a rather weak but still well-working fuzzy logic. Even if we believe that fuzzy logic is *not* confined to the standard truth set $[0,1]$ (as explained above), it is of some interest to know whether there are examples of such general algebras on $[0,1]$ or $(0,1]$. Before we go into details, consider the following modification of Mesiar's example: let $0 < c < a < b1$, let $x * y = c$ for $c \leq x \leq a$ and $c \leq y \leq b$, $x * y = \min(x,y)$ otherwise. This is a non-commutative left continuous pseudo-t-norm on $[0,1]$ and the half-open interval $(0,1]$ is closed under $*$ (making it to a flea as we shall see).

What we have to pay in the non-commutative case is the fact that the operation $*$ now has two residua $\rightarrow$ and $\rightsquigarrow$ with the following property:

If $e_{\mathbf{L}})(\varphi) = a$ and $e_{\mathbf{L}}(\varphi \rightarrow \psi) = b$ then $e_{\mathbf{L}}(\psi) \geq b * a$,
if $e_{\mathbf{L}}(\varphi) = a$ and $e_{\mathbf{L}}(\varphi \rightsquigarrow \psi) = b$ then $e_{\mathbf{L}}(\varphi) \geq a * b$.

To show how this subtleness may be practically used remain a task for the future.

Before we close this section, two remarks: First, the reader should know that fuzzy logic uses truth degrees as an auxiliary mathematical apparatus for defining its semantics (inside a crisp metatheory). You will never say "I love you with the truth degree 0.98"; you will say "I love you" and will make (fuzzy) conclusions from it. Fuzzy logic can teach you how the truth degrees behind your assertion propagate to the conclusions you make.

Second, more informal information on fuzzy logic may be found in my [9].

In the next section we survey basics of the flea logic.

3. Flea algebras and flea logic.

Definition 1 A *flea algebra* is a structure $\mathbf{F} = (F, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 1)$ where
$(F, \wedge, \vee, 1)$ is a lattice with a greatest element,
$*$ is as binary associative operation with 1 as (both-side) unit,
$x * y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ (residuation)
$(x \rightarrow y) * x \leq x \wedge y, \quad x * (x \rightsquigarrow y) \leq x \wedge y,$
$(x \rightarrow y) \vee (y \rightarrow x) = 1, \quad (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (prelinearity)

Definition 2 The flea logic FlL is a propositional calculus with binary connectives $\wedge, \vee, \&, \rightarrow, \rightsquigarrow$.
The axioms:
$(\tilde{A}1)$ $(\psi \rightarrow \chi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$, the same for $\rightsquigarrow$,
$(\tilde{A}2)$ $(\varphi \& \psi) \rightarrow \varphi$,
$(\tilde{A}3)$ $(\varphi \& \psi) \rightarrow \psi$,
$(\tilde{A}4a)$ $(\varphi \wedge \psi) \rightarrow \varphi$,
$(\tilde{A}4b)$ $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$,
$(\tilde{A}4c)$ $((\varphi \rightarrow \psi) \& \varphi) \rightarrow (\varphi \wedge \psi)$,
$(\varphi \& (\varphi \rightsquigarrow \psi)) \rightarrow (\varphi \wedge \psi)$,
$(\tilde{A}5)$ $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightleftarrows ((\varphi \& \psi) \rightarrow \chi)$,
$(\varphi \rightsquigarrow (\psi \rightsquigarrow \chi)) \rightleftarrows ((\psi \& \varphi) \rightsquigarrow \chi)$
$(\tilde{A}6)$ $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow$
$\rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$, the same for $\rightsquigarrow$
$(\tilde{A}8)$ $(\varphi \vee \psi) \rightleftarrows$
$((\varphi \rightarrow \psi) \rightsquigarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightsquigarrow \varphi) \rightleftarrows$
$\rightleftarrows ((\varphi \rightsquigarrow \psi) \rightarrow \psi) \wedge ((\psi \rightsquigarrow \varphi) \rightarrow \varphi)$
(no $(\tilde{A}7)$!)

Deduction rules are modus ponens and (Imp): from $\varphi \rightarrow \psi$ infer $\varphi \rightsquigarrow \psi$ and vice versa.

A $psMTL$-algebra is just a flea algebra having a least element; the logic $psMTL$ is just the extension of FlL by the axiom $(\tilde{A}7)$ saying $\bar{0} \rightarrow \varphi$. (See [11].)

Definition 3 Let $\mathbf{F}$ be a flea and 0 an element not belonging to F. $0 \oplus \mathbf{F}$ is the algebra whose domain is $F \cup \{0\}$, and the operations are extended as follows: $x \wedge 0 = 0, x \vee 0 = x$ for all x, $0 * x = x * 0 = 0$, $0 \to x = 0 \rightsquigarrow x = 1$ for all x, and for $x \neq 0$ $x \to 0 = x \rightsquigarrow 0 = 0$.

Lemma 1 For each flea $\mathbf{F}$, $0 \oplus \mathbf{F}$ is a $psMTL$-algebra and $\mathbf{F}$ is a subalgebra of $0 \oplus \mathbf{F}$.

Corollary 1 A formula not containing $\bar{0}$ is a $psMTL$-tautology iff it is a flea tautology.

Lemma 2 The logic FlL is sound w.r.t. fleas; i.e. each axiom of FlL is a tautology of each flea and deduction rules preserve Fl-tautologicity.

Theorem 1 The class of flea algebras is a variety.

(This means that a flea algebra can be defined as an algebra stisfying a set of identities.)

Theorem 2 A flea is subdirectly representable iff the following identities are valid in it:

$$(y \to x) \vee (z \rightsquigarrow ((x \to y) * z)) = 1,$$
$$(y \rightsquigarrow x) \vee (z \to (z * (x \rightsquigarrow y))) = 1.$$

Cf. [15]. Moreover, introduce FlL^r (representable flea logic) by adding Kühr's axioms

$$(\psi \to \varphi) \vee (\chi \rightsquigarrow ((\varphi \to \psi) \& z))$$
$$(\psi \rightsquigarrow \varphi) \vee (\chi \to (\chi \& (\varphi \rightsquigarrow \psi)))$$

to FlL; clearly, FlL^r is sound for representable fleas.

Theorem 3 FlL proves the following formulas:
(1) $\varphi \to (\psi \to \varphi), \varphi \rightsquigarrow (\psi \rightsquigarrow \varphi)$
(2) $(\varphi \rightsquigarrow (\psi \to \chi)) \rightleftarrows (\psi \to \varphi \rightsquigarrow \chi))$
(3) $\varphi \to \varphi,\ \varphi \rightsquigarrow \varphi$
(4) $(\varphi \& (\varphi \rightsquigarrow \psi)) \to \psi,\ ((\varphi \to \psi) \& \varphi) \to \psi$
(5) $\varphi \to (\psi \to (\varphi \& \psi)), \varphi \rightsquigarrow (\psi \rightsquigarrow (\psi \& \varphi))$
(6) $(\varphi \to \psi) \to ((\varphi \& \chi) \to (\psi \& \chi)), (\varphi \rightsquigarrow \psi) \to ((\chi \& \varphi) \to (\chi \& \psi))$
(no (7))
(8) $((\varphi \& \psi) \& \chi) \rightleftarrows (\varphi \& (\psi \& \chi))$
(9) $(\varphi \& \psi) \to (\varphi \wedge \psi)$
(10) $(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$, the same with $\rightsquigarrow$
(11) $(\varphi \wedge \psi) \to \varphi$
(12) $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi))$, the same with $\rightsquigarrow$
(13) $\varphi \to (\varphi \vee \psi),\ (\varphi \vee \psi) \to (\psi \vee \varphi)$,
(14) $(\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$, the same with $\rightsquigarrow$
(15) $(\varphi \to \psi) \vee (\psi \to \varphi)$, the same with $\rightsquigarrow$
(16) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)$, the same with $\rightsquigarrow$.

Theorem 4 (conservativeness). $psMTL$ and FlL prove the same zero-free formulas. The same for $psMTL^r$ and FlL^r.

Corollary 2 (completeness). (1) $FlL \vdash \varphi$ iff φ is a FlL-tautology.
(2) $FlL^r \vdash \varphi$ iff φ is a FlL^r-tautology iff φ is a FlL-chain tautology.

Fleas and pseudo-BCK-algebras

BCK-algebras were introduced and studied by Imai, Iseki and Tanaka. And $psBCK$-algebras were introduced by Georgescu and Iorgulescu as a generalization of BCK-algebras not assuming commutativity. (See [7].)

A pseudo-BCK-algebra (briefly $psBCK$-algebra) is a structure $\mathbf{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ where $\leq$ is a binary relation, $\rightarrow$ and $\rightsquigarrow$ are binary operations and 1 is an element of A, such that the following is valid:

(I) $y \rightarrow z \leq (z \rightarrow x) \rightsquigarrow (y \rightarrow x),\ y \rightsquigarrow z \leq (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)$
(II) $y \leq (y \rightsquigarrow x) \rightarrow x,\ y \leq (y \rightarrow x) \rightsquigarrow x$
(III) $x \leq x$
(IV) $x \leq 1$
(V) $x \leq y$ and $y \leq x$ implies $x = y$
(VI) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$

A BCK-algebra (pedantically: reversed left-BCK-algebra) is a $psBCK$-algebra in which the identity $x \rightarrow y = x \rightsquigarrow y$ is valid.

Lemma 3 Let $\mathbf{F} = (F, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 1)$ be a flea. Then the reduct $(F, \leq, \rightarrow, \rightsquigarrow, 1)$ is a $psBCK$-algebra.

Iorgulescu studied expansions of $psBCK$-algebras by an operation $*$ satisfying reasonable conditions. The reader will easily check that our fleas are exactly Iorgulescu's "left-X-$psBCK(pRP)$-algebras" satisfying two additional conditions.

Definition 4 A *pulex*[1] is a structure $\mathbf{P} = (P, \wedge, \vee, \rightarrow, \rightsquigarrow, 1)$ such that $(P, \wedge, \vee, 1)$ is a lattice with a largest element, $(P, \leq, \rightarrow, \rightsquigarrow, 1)$ is a $psBCK$-algebra and the axioms $(\tilde{A}6), (\tilde{A}8)$ as well as provability (10) are valid, i.e.

$(x \rightarrow y) \rightarrow z \leq ((y \rightarrow z) \rightarrow z) \rightarrow z$, the same for $\rightsquigarrow$,
$x \vee y = ((x \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow x) \rightsquigarrow x) = ((x \rightsquigarrow y) \rightarrow y) \wedge (y \rightsquigarrow x) \rightarrow x)$.
$(x \rightarrow y) \rightarrow (x \rightarrow (x \wedge y))$, the same for $\rightsquigarrow$.

Thus we have the following

Corollary 3 The implicational reduct of a flea is a pulex.

Observe that the class of $psBCK$-algebras is not a variety since the class of all BCK algebras is not.

Theorem 5 The class of all $psBCK$-algebras whose order is a lattice is a variety (in the language $\wedge, \vee, \rightarrow, \rightsquigarrow, 1$).

Definition 5 The logic of pulexes has the axioms $(\tilde{A}1), (\tilde{A}4a), (\tilde{A}4b), (\tilde{A}6)$ and $(\tilde{A}8)$ of the flea logic as well as the formulas (1), (2), (10) from the list of formulas provable in FlL; the deduction rules are modus ponens and (Imp).

Theorem 6 A formula φ is provable in PulL iff it is a $\mathbf{P}$-tautology for each pulex $\mathbf{P}$ iff it is a $\mathbf{L}_{Pul}$-tautology.

Conclusion. We have presented the main definitions and theorems concerning flea algebras and hope to have shown the reader that even rather abstract these algebras and the corresponding logics behave ("jump") well. Studying them helps one to clarify the role of (left) continuity of t-norms and the corresponding (un)definability of infimum from the star operation and its reduct, the role of falsity (zero) and the role of commutativity. Moreover, it contributes to our understanding of the role of the linearity of the order of truth degrees and the role of the standard set $[0, 1]$ of truth degrees. The simple example of a flea on $(0, 1]$ is new here; proofs of all theorems are found (let's repeat) in [12], where also some open problems are formulated. One can add more, eg. the problem of decidability (and computational complexity) of the set of tautologies of the flea logics FLl, fLl^r. Also to develop the corresponding predicate calculi in the style of [8] remains a future task.

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[1] Pulex irritans is a kind of fleas.

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