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Thermal Flow and Heat Transport in the Vicinity of Sources of the Geothermal Energy

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Abstract:

This paper deals with numerical modelling of the geothermal flow of groundwater in the vicinity of sources of the geothermal energy. The problem is described by a set of two partial differential equations - the heat transport equation with convection and the equation for pressure. These equations are coupled together in terms of a relation between density of the fluid and its temperature. As the density is depends on temperature only, the pressure equation is of elliptic type. The resulting system of equations is thus of parabolic-elliptic type. A suitable numerical scheme for approximation of solution to this system is proposed and it is tested on several numerical experiments which are presented in the conclusion.

Keywords:

Density-driven flow, convection-diffusion, method of characteristics

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1 Introduction

Let Ω be a bounded domain in \mathbb{R}^2 representing a vertical cut through the soil which is fully saturated by water. In this domain, the following mass balance condition is to be fulfilled :

$$n \frac{\partial \rho}{\partial t} + \frac{\partial \rho q_x}{\partial x} + \frac{\partial \rho q_z}{\partial z} = Q. \quad (1.1)$$

Here, n denotes porosity, ρ is the density of the fluid, q_x and q_z are components of the Darcy velocity vector \vec{q} , and finally, Q is a source/sinks term. Equation (1.1) will be referred to as the mass balance equation in the sequel. This equation was derived in [5] under assumption of medium incompressibility.

In the equation (1.1), the dependency of \vec{q} on pressure p is explicitly given by the Darcy law:

$$\vec{q} = -\frac{k}{\mu}(\nabla p - \rho \vec{g}), \quad (1.2)$$

where μ denotes the coefficient of dynamical viscosity of the fluid and \vec{g} is a vector of gravity acceleration. The quantity k is the permeability of the porous medium which is assumed to be isotropic. In the whole text, we will suppose that the coordinate system (Oxz) is oriented so that the vector of gravitational acceleration points in the direction of negative part of z -axis. Thus, $\vec{g} = (0, g)$, where $g = -9.81 \text{ m.s}^{-2}$. Under these assumptions the Darcy law can be written in components as

$$q_x = -\frac{k}{\mu} \frac{\partial p}{\partial x}, \quad q_z = -\frac{k}{\mu} \left(\frac{\partial p}{\partial z} - \rho g \right). \quad (1.3)$$

Substituting the Darcy law to the mass balance equation, the following equation can be derived

$$-\frac{\partial}{\partial x} \left(\rho \frac{k}{\mu} \frac{\partial p}{\partial x} \right) - \frac{\partial}{\partial z} \left(\rho \frac{k}{\mu} \frac{\partial p}{\partial z} - \frac{k}{\mu} \rho^2 g \right) = Q - n \frac{\partial \rho}{\partial t}, \quad (1.4)$$

which will be denoted as the pressure equation in the whole text. Moreover, we add the equation of heat transport which is taken from [2]

$$\overline{\rho c} \frac{\partial T}{\partial t} + \rho c \vec{q} \cdot \nabla T = \nabla \cdot (\lambda \nabla T). \quad (1.5)$$

In this equation, T is the unknown temperature, c is the heat capacity per unit mass of the fluid and λ denotes the homogenized coefficient of heat conductivity. We suppose that the heat conductivity is scalar (i.e. the medium is isotropic), however, the extension for anisotropic case is possible. The notation $\overline{\rho c} = n\rho c + (1-n)\rho_s c_s$, is used, where ρ_s and c_s denote the density and the heat capacity per unit mass of the soil, respectively. The equations are coupled by the following relation between the fluid density and temperature

$$\rho(T) = \frac{\rho_0}{1 + \beta_1(T - T_0) + \beta_2(T - T_0)^2}, \quad (1.6)$$

in which the coefficients $\rho_0 \equiv \rho(T_0)$, β_1, β_2 , are given constants. Additionally, the density could depend on pressure, but this case is not considered here.

As we assumed that neither density nor porosity depend on pressure, the equation for pressure (1.4) is an elliptic partial differential equation with respect to pressure. Therefore, we prescribe boundary conditions of the Dirichlet, Neumann or Newton type, i.e.

$$p(x, z, t) = f_1(x, z, t) \quad \forall t \text{ and } \forall (x, z) \in S_1. \quad (1.7)$$

$$(\rho q_x n_x + \rho q_z n_z)(x, z, t) = f_2(x, z, t) \quad \forall t \text{ and } \forall (x, z) \in S_2. \quad (1.8)$$

$$(\rho q_x n_x + \rho q_z n_z)(x, z, t) = \beta(p - p_{out})(x, z, t) \quad \forall t \text{ and } \forall (x, z) \in S_3, \quad (1.9)$$

Here p_{out} denotes pressure at the outer side of boundary, β is the coefficient of proportionality and the $\vec{n} = (n_x, n_z)$ denotes the unit vector of outer normal. The heat transport equation (1.5) is a parabolic

partial differential equation and it is therefore completed by one initial condition and by boundary conditions of the Dirichlet, Neumann or Newton type, i.e.

$$T(x, z, 0) = T^0(x, z) \quad \forall (x, z) \in \Omega, \quad (1.10)$$

$$T(x, z, t) = f_4(x, z, t) \quad \forall t \text{ and } \forall (x, z) \in S_4. \quad (1.11)$$

$$\left(-\lambda \frac{\partial T}{\partial \bar{n}}\right)(x, z, t) = f_5(x, z, t) \quad \forall t \text{ and } \forall (x, z) \in S_5. \quad (1.12)$$

$$\left(-\lambda \frac{\partial T}{\partial \bar{n}}\right)(x, z, t) = \gamma(T - T_{out})(x, z, t) \quad \forall t \text{ and } \forall (x, z) \in S_6, \quad (1.13)$$

where T_{out} is the outer temperature and $\gamma > 0$ is the heat transfer coefficient. The problem will be correctly formulated if the S_1, S_2, S_3 and S_4, S_5, S_6 are two (generally different) decompositions of boundary $\partial\Omega$.

2 Weak Formulation

In the sequel, we will suppose that Ω is a bounded domain with the Lipschitz boundary. Let us introduce the space $V_T = \{f \in C^\infty(\bar{\Omega}) : f|_{S_4} = 0\}$ and the enthalpy by the following definition

$$H(T) = \int_0^T \overline{\rho c}(\tau) d\tau. \quad (2.1)$$

Moreover, let us denote the time interval $(0, \Theta)$ as I . At this moment, we are ready for the following definition.

Suppose the following input data qualification : $\rho_s, c_s, n, \lambda \in L^\infty(\Omega)$, $c > 0$, $n \in (0, 1)$, $q_x, q_z \in L^2(I; L^2(\Omega))$, $f_4 \in L^2(I; L^{1/2}(S_4))$, $f_5 \in L^2(I; L^2(S_5))$, $\gamma \in L^\infty(S_6)$, $T_{out} \in L^2(I; L^2(S_6))$, $T^0 \in W_2^1(\Omega)$. Then we say that the mapping $T \in L^2(I; W_2^1(\Omega))$ is the weak solution of the equation of heat transport if the following conditions hold

$$T(0) = T^0 \quad \text{a.e. in } \Omega, \quad (2.2)$$

$$T(t)|_{S_4} = f_4 \quad \text{a.e. in } I, \quad (2.3)$$

and the integral identity

$$\frac{d}{dt}(H(T), v) + (\rho c \vec{q} \cdot \nabla T, v) + (\lambda \nabla T, \nabla v) + \int_{S_6} \gamma T v dS = - \int_{S_5} f_5 v dS + \int_{S_6} \gamma T_{out} v dS \quad (2.4)$$

holds for all $v \in V_T$ in $\mathcal{D}'(I)$.

Further, let us define the space of the functions which fulfill the homogeneous stable boundary conditions in sense of traces $V_p = \{f \in W_2^1(\Omega) : f|_{S_1} = 0\}$. At this moment, we can formulate the following definition.

Let μ is a positive constant. Suppose the following input data qualification : $k \in L^\infty(\Omega)$, $f_2 \in L^2(I; L^2(S_2))$, $\beta \in L^\infty(S_3)$, $p_{out} \in L^2(I; L^2(S_3))$ and

$$Q, \frac{\partial \rho}{\partial t} \in L^2(I; L^2(\Omega)).$$

Suppose that there exists a function $p_\Omega \in W_2^1(\Omega)$ such that $p_\Omega|_{S_1} = f_1$. Then we say that the mapping $p \in L^2(I; W_2^1(\Omega))$ is the weak solution of the pressure equation if

$$p(t) - p_\Omega \in V_p \quad \text{a.e. in } I,$$

and the integral identity

$$\int_\Omega \left[\rho \frac{k}{\mu} \nabla p \cdot \nabla w - \frac{k}{\mu} \rho^2 g \frac{\partial w}{\partial z} \right] dx dz + \int_{S_3} \beta p w dS = (Q, w) - \int_{S_2} f_2 w dS + \int_{S_3} \beta p_{out} w dS - \left(n \frac{\partial \rho}{\partial t}, w \right) \quad (2.5)$$

holds for all $w \in V_p$ in $\mathcal{D}'(I)$.

3 Discretization and Numerical Algorithm

Following the procedure described in [3], we suggest the following combination of method of characteristics and standard Galerkin scheme for approximation of the heat transport equation (1.5)

$$(\mathbf{A} + \Delta t_k \mathbf{B}) \mathbf{T}^{(k+1)} = \mathbf{A} \mathbf{T}^{(k)} \circ \varphi^{(k)} + \Delta t_k \mathbf{F}, \quad (3.1)$$

where

$$\begin{aligned} A_{ij} &= \int_{\Omega} \overline{\rho c} N_i N_j \, dx dz, \\ B_{ij} &= \int_{\Omega} \lambda \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) \, dx dz + \int_{S_6} \gamma N_i N_j \, dS, \\ F_i &= - \int_{S_5} f_5 N_i \, dS + \int_{S_6} \gamma T_{out} N_i \, dS. \end{aligned}$$

and the function

$$\varphi^{(k)}(x) := x - \Delta t_k \omega_h * \frac{\rho c \vec{q}(t_{k-1}, x)}{n \rho c + (1-n) \rho_s c_s} \quad (3.2)$$

is the Euler explicit approximation of characteristics. According the [3], the velocity field has to be smoothed by convolution with mollifier ω_h which is defined as

$$\omega_h(x) = \frac{1}{h^2} \omega_1\left(\frac{x}{h}\right), \quad \omega_1(x) = \begin{cases} \kappa \exp\left(\frac{|x|^2}{|x|^2-1}\right), & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases} \quad (3.3)$$

and the coefficient κ is chosen so that the integral of ω_1 over the whole support is unitary. This smoothing guarantees that the approximated characteristics do not intersect each other. This scheme is nothing but the implicit standard Galerkin approximation of the heat transport equation without convection which is included in the right hand side in terms of method of characteristics. As the Galerkin scheme solves just the diffusion problem there are no problems with the artificial oscillations in the case of dominant convection. The matrix of the system of linear algebraical equations is symmetric, and if we use the mass lumping technique then it will be diagonally dominant and therefore also positively definite which is advantageous for the numerical solution of this linear algebraical system.

The equation for pressure (1.4) is discretized by the standard Galerkin approach. The same triangulation and the same linear basis functions as in the case of heat transport equation are used. Assuming that the heat transport equation has been solved before we can approximate the time derivative of density on the right hand side of (2.5) by the forward difference. The standard Galerkin discretization results to the following algebraical system for the unknown pressures at the nodes of the mesh in time t_{k+1}

$$\mathbf{A} \mathbf{p}^{(k+1)} = \mathbf{F}, \quad (3.4)$$

where

$$\begin{aligned} A_{ij} &= \int_{\Omega} \rho \frac{k}{\mu} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) \, dx dz + \int_{S_3} \beta N_i N_j \, dS, \\ F_i &= \int_{\Omega} \left[Q N_i - n \frac{\partial \rho}{\partial t} N_i + \frac{k}{\mu} \rho^2 g \frac{\partial N_i}{\partial z} \right] \, dx dz - \int_{S_2} f_2 N_i \, dS + \int_{S_3} \beta p_{out} N_i \, dS \end{aligned} \quad (3.5)$$

are the coefficients of the matrix \mathbf{A} and vector \mathbf{F} . As the pressure is approximated linearly on each element the pressure gradients are element-wise constant and thus, the Darcy velocity can be easily determined in terms of (1.3).

Assume, that we are situated in the k -th time level and all the quantities at time t_k are known - either from the initial condition or from the previous time step. The values of all examined quantities in time t_{k+1} can be obtained using the following steps :

- **Step 1.** The solution of the heat transport equation with the initial condition $\mathbf{T}^{(k)}$ provides a new distribution of temperature - $\mathbf{T}^{(k+1)}$.
- **Step 2.** The new layer of densities is obtained by substitution of the new temperatures to (1.6).
- **Step 3.** Then, the equation for pressure can be solved in order to obtain the new layer of pressures - $\mathbf{p}^{(k+1)}$. The time derivative of pressure on the right hand side of this equation can be approximated by the backward difference as the values of densities in time t_k and t_{k+1} are known.
- **Step 4.** Compute new Darcy velocities using (1.3).

At this moment, values of all required quantities in time t_{k+1} have been evaluated. This procedure can be repeated from the step one if needed or can be stopped if the simulation time is up.

4 Results

We simulate the saturated groundwater flow in a rectangular domain of size 100×30 m. At the beginning, the temperature inside of the considered domain is 10°C and the water does not move. The boundary conditions and the material properties will be described individually depending on the problem solved.

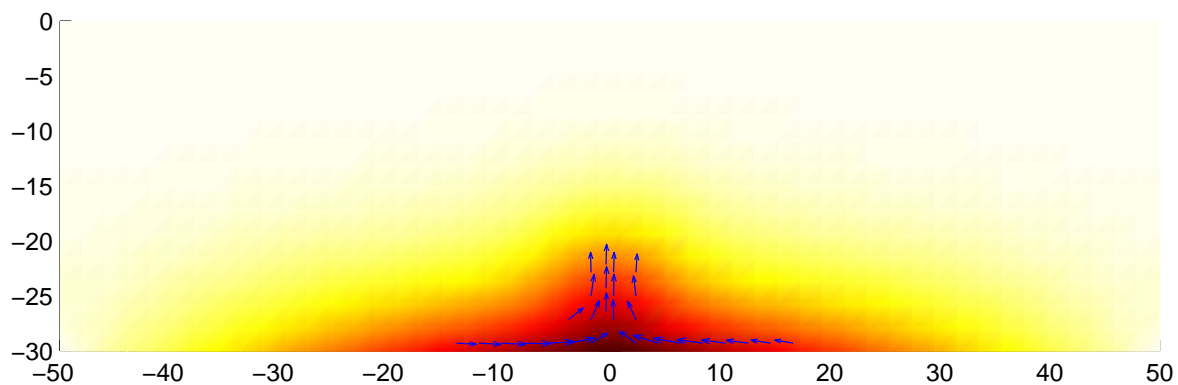


Figure 4.1: Problem 1DirHill.

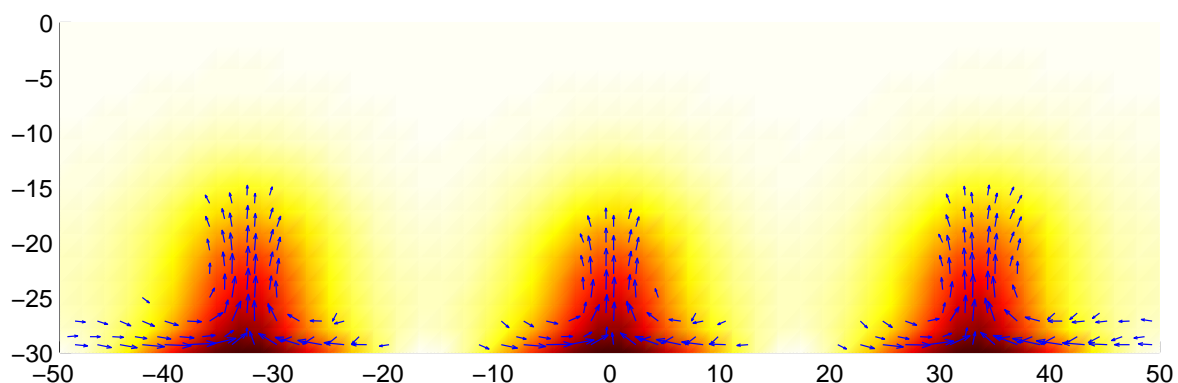


Figure 4.2: Problem 3DirHill

Problem : 1DirHill Suppose that the soil in Ω is homogeneous and isotropic. The top and the bottom boundary of Ω are impermeable. On the top part of boundary and along the sides, the

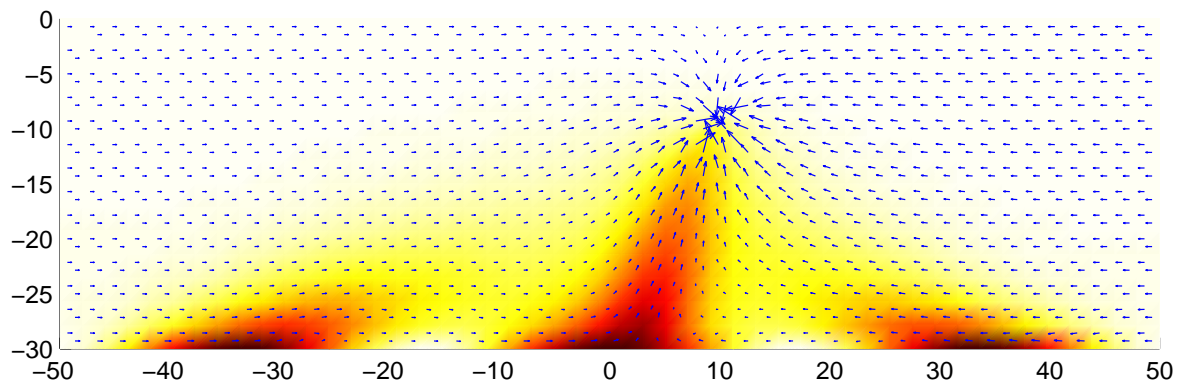


Figure 4.3: Problem 1Well.

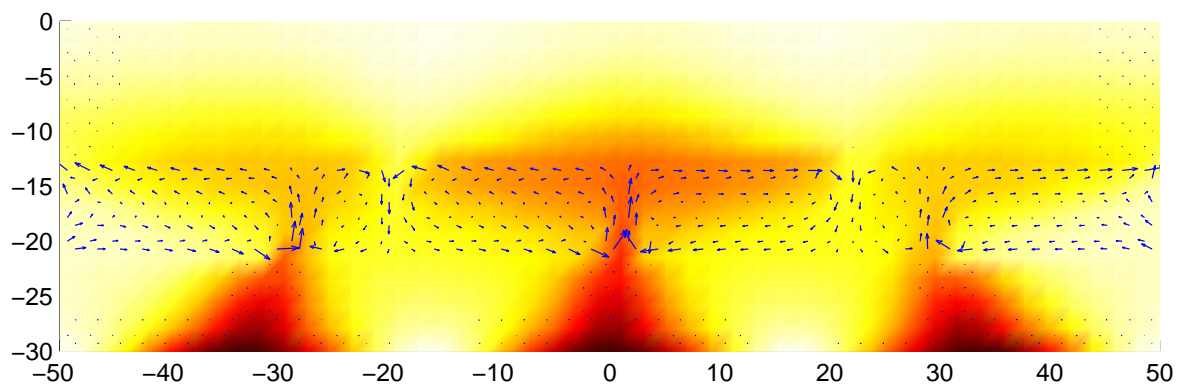


Figure 4.4: Problem HorLayer.

zero heat flux is prescribed. On the left and right side of the boundary, we prescribe the hydrostatic pressure. The water in domain Ω is heated from the bottom - in this case we prescribe the temperature on the bottom side of the $\partial\Omega$. This temperature grows up linearly from 10 °C on the sides to the 90 °C in the middle of the bottom part of the boundary $\partial\Omega$. In the figure 4.1, we can see the situation in time of 1000 days. The colours represent the temperatures – white colour belong to the 10 °C and black represents the 90 °C. The other colours are associated with the temperatures in between. The arrows show distribution of the Darcy velocities.

Problem : 3DirHill This is an example of the similar problem as described in the previous subsection. The only difference is that the temperature on the bottom part of the $\partial\Omega$ is given by a cosinus function of x-coordinate such that the temperature changes between 10 and 90 °C and has three maxima along the boundary. In the regions of these maxima, we can observe the flow to the top of the aquifer where the water is cooled and then it flows toward the bottom boundary in the regions of the minima of the temperature. The situation in Ω in time of 1000 days is shown in the figure 4.2. This example is useful for comparison with the problems **1Well** and **HorLayer** whose results are shown in figures 4.3 and 4.4. The first of the figures shows the same situation with added pumping while in the second figure there is a situation in which we added a horizontal layer of soil in which the permeability and the coefficient of heat conductivity are higher than in the rest of the aquifer.

5 Analysis of the Convergence

In this section, the convergence of the proposed numerical scheme is examined. The problem **1DirHill** was chosen as a suitable problem used for testing. As the exact solution was not available, we computed

a numerical solution on a mesh 250×75 with the timestep 1 day and this solution was used for the comparison. Then, the additional solutions of the same problem were computed on coarser meshes with larger timesteps.

Our task was to measure the distances of the individual solutions from the solution on the finest mesh in the norms of several function spaces. The main problem here is the interpolation of the solution between two different meshes. This is solved by the point-wise projection of the solution from the coarse mesh to the finer mesh. Then, the computation of the norms of the difference is done on the finer mesh.

The results of the convergence analysis are contained in the tables 5.1 and 5.2. In the table 5.1, the symbol $\|\cdot\|_{\mathcal{X}}$ denotes the \mathcal{X} -norm of the difference of the temperature on the mesh 250×75 and the temperature projected from the coarser mesh to the 250×75 -mesh. The same symbol used in table 5.2 has the analogous meaning, only the pressure difference is measured instead of the temperature.

#	Mesh	Timestep	$\ \cdot\ _{L^\infty(I;L^2(\Omega))}$	$\ \cdot\ _{L^\infty(I;W_2^1(\Omega))}$	$\ \cdot\ _{L^\infty(I;L^\infty(\Omega))}$
1	50×15	25 days	28.5641	43.1245	3.18548
2	100×30	5 days	8.33552	13.4872	0.84721
3	150×45	$2\frac{1}{2}$ days	5.41337	5.89123	0.49784
4	200×60	$1\frac{1}{3}$ days	4.09549	4.20282	0.39215

Table 5.1: The results of the convergence analysis for temperature

#	Mesh	Timestep	$\ \cdot\ _{L^\infty(I;L^2(\Omega))}$	$\ \cdot\ _{L^\infty(I;W_2^1(\Omega))}$	$\ \cdot\ _{L^\infty(I;L^\infty(\Omega))}$
1	50×15	25 days	438.598	658.184	66.8801
2	100×30	5 days	170.488	207.186	17.9188
3	150×45	$2\frac{1}{2}$ days	108.380	124.451	9.99112
4	200×60	$1\frac{1}{3}$ days	81.5763	93.0700	7.20534

Table 5.2: The results of the convergence analysis for pressure

\mathcal{EOC} of \downarrow between \rightarrow	#1 \mapsto #2	#2 \mapsto #3	#3 \mapsto #4
T in $\ \cdot\ _{L^\infty(I;L^2(\Omega))}$	1.75	1.06	0.96
p in $\ \cdot\ _{L^\infty(I;L^2(\Omega))}$	1.34	1.11	0.98
T in $\ \cdot\ _{L^\infty(I;W_2^1(\Omega))}$	1.65	2.03	1.17
p in $\ \cdot\ _{L^\infty(I;W_2^1(\Omega))}$	1.64	1.25	1.00
T in $\ \cdot\ _{L^\infty(I;L^\infty(\Omega))}$	1.88	1.30	0.82
p in $\ \cdot\ _{L^\infty(I;L^\infty(\Omega))}$	1.87	1.43	1.13

Table 5.3: Experimental orders of the convergence for temperature and pressure

We could present figures which show the log-log plots of the norms contained in the previous tables as a function of the size of the mesh. The slope of the curves would allow us to estimate the experimental orders of the convergence (\mathcal{EOC} 's) for temperature and pressure in the norms of the corresponding function spaces. The \mathcal{EOC} between two triangulations with the mesh sizes h_1 and h_2 is defined as in [1] by

$$\mathcal{EOC} = \frac{\log E(h_1) - \log E(h_2)}{\log h_1 - \log h_2}, \quad (5.1)$$

where the symbol $E(h)$ denotes some of those norms of the difference of the solution on h -mesh and the finest mesh. The experimental orders of the convergence between the individual triangulations are summarized in the table 5.3.

6 Conclusion

The presented results show that the heat transport processes in porous media are relatively slow. The heat transport can be substantially faster in the fractures, so we intend to add the model of fracture flow to the current model.

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