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Technical report No. 884

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Diffuse Interface Model of Microstructure Formation in Solidification with Anisotropy Based on Finsler Geometry

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Abstract:

In the article, we analyse the anisotropic phase-field model of microstructure growth in solidification of pure crystallic materials. The anisotropy has been previously introduced into the model using the concept of Finsler geometry which allows to define the notion of weak solution and to show the existence and uniqueness. For this purpose, we obtain two a priori estimates, and use the compactness method. Furthermore, we investigate the asymptotic behaviour of the solution when the diffuse-layer thickness tends to zero.

Keywords:

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1 Introduction

The aim of the article is to present main mathematical properties of a system of phase-field equations endowed by anisotropy. The equations represent a mathematical model of the solidification of pure crystallic substances at microscale. The mentioned physical phenomenon is accompanied by presence of an interface between phases which can move in the space and is determined intrinsicly by the state of the physical system and its boundary and initial data. Among various approaches to mathematical treatment of the problem (see [24], [26]), the diffuse interface model yields as a part of the solution a well controlled smooth approximation of characteristic function of the phase. This fact originally observed in the form of a wave-like solution of reaction-diffusion systems (see [1], [21]) lead to the formulation of a model of solidification with additional consequences in understanding physics of phase transitions ([16], [19]). The model equations consist of the heat equation with nearly singular heat source coupled to a semilinear or quasilinear parabolic equation for the order parameter known as the Allen-Cahn equation or equation of phase. The equations in various setting were studied in, e.g. [12], [13], and applied in simulation of physical phenomena ([27], [3], [5]). The application of models based on phase-field theory rose several quantitative questions questions concerning relation to the sharp-interface analogue ([5]). Problems of choice of the small parameter versus mesh size, and problems with interface stability lead to various modifications mainly in the Allen-Cahn equation (see [10], [14], [4], [7]). Quantitative comparison, performed especially in case of curve motion (or hypersurface motion) driven by mean curvature (see [9]) showed a satisfactory agreement of numerical computations with analytical solution (where it was possible) or with results obtained by numerical solution of other models and rised a question about how the anisotropy can be incorporated into the Allen-Cahn equation without loosing a possibility of work with weak formulation. This requires a second-order space differential operator in the divergence form. As shown in [2], a natural way of introducing the anisotropy into the model is the use of Finsler geometry. This has been done e.g. in [22] for the case of mean-curvature flow, and in [6] for the full phase-field model. The viscosity solution concept allowed to treat even a fully anisotropic (i.e. the case when the kinetic term is also direction-dependent) Allen-Cahn equation not coupled to the heat equation (i.e. the case when the kinetic term is also direction-dependent) – [15].

The paper extends the scope of [6], where the anisotropic model has been presented:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u + L\chi'(p)\frac{\partial p}{\partial t}, \\ \alpha\xi\frac{\partial p}{\partial t} &= \xi\nabla\cdot T^0(\nabla p) + \frac{1}{\xi}f_0(p) + F(u)\xi\Phi^1(\nabla p), \end{aligned}$$

with initial conditions

$$u|_{t=0} = u_0\ , \quad p|_{t=0} = p_0,$$

and with boundary conditions of Dirichlet type

$$u|_{\partial\Omega} = 0\ , \quad p|_{\partial\Omega} = 0.$$

Here, $\xi > 0$ is the "small" parameter, and f_0 derivative of a double-well potential. The coupling function $F(u)$ is bounded and continuous, or even Lipschitz-continuous. The anisotropy is included using the concept of the Finsler geometry, where two – possibly different Finsler metrics Φ^0 and Φ^1 describe the surface and kinetic anisotropy, T^0 is corresponding gradient operator (see below). We consider $f_0(p) = ap(1-p)(p-\frac{1}{2})$ with $a > 0$. The enthalpy is given by $\mathcal{H}(u) = u - L\chi(p)$, where the coupling function χ is monotone with bounded, Lipschitz-continuous derivative: $\chi(0) = 0$, $\chi(0.5) = 0.5$, $\chi(1) = 1$, $supp(\chi') \subset \langle 0, 1\rangle$. For the sake of simplicity, $n = 2$, Ω is a bounded domain in $\mathbb{R}^n$ with a C^2 boundary, and boundary conditions are homogeneous. Obviously, the extension to higher dimensions, and to other boundary conditions is possible.

The analysis presented in this article has been motivated by interesting numerical studies obtained by the model both for the case of curve dynamics in the plane (see [6], and Figure 1.1), dynamics of hypersufaces in 3D ([23]), and for the case of microstructure growth in solidification (see [6]). The model works with an anisotropy rigorously implemented into the equations. In addition, the

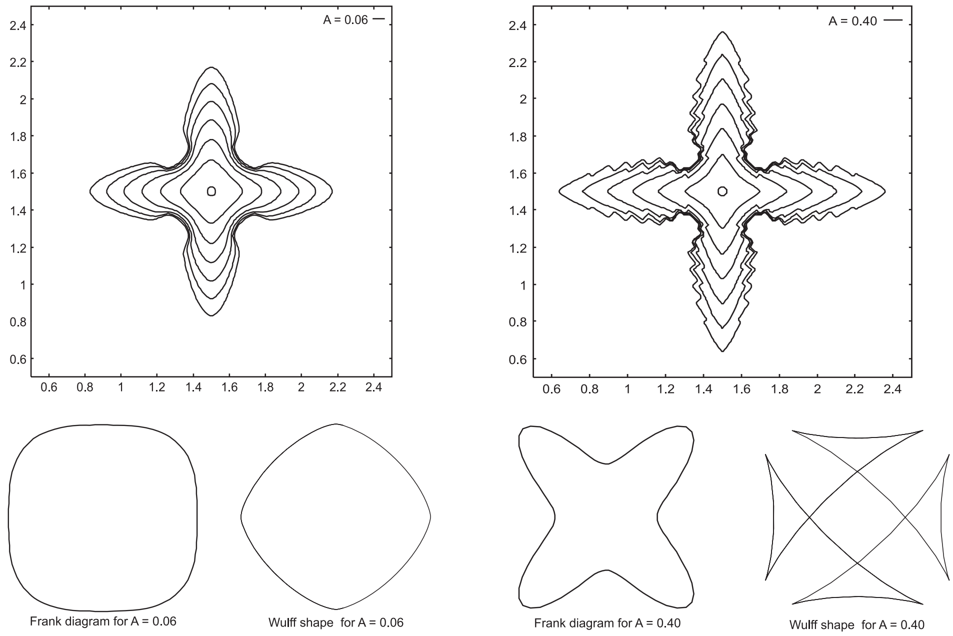

Figure 1.1: Qualitative study of dendritic growth for convex ($A = 0.06$) and non-convex anisotropy ($m = 4$, $A = 0.40$). Other parameters are: $r = 2$, $\xi = 0.02$, $u^* = 1.0$, $u_0 = 0.0$, $L = 2.0$, $\beta = 300$, $a = 4.0$, $\alpha = 3$, $L_1 = L_2 = 3.0$, $N_1 = N_2 = 200$, $\Delta t = 0.008$, initial radius $= 0.025$. The curve evolution above is accompanied by the Frank diagram and Wulff shape of each anisotropy type.

mentioned setting allows a conversion from the double-well potential to the double-obstacle potential in the Allen-Cahn equation ([10]). Finally, the model gives reasonable results even in case of non-convex anisotropies, when the presented theory is not applied. Our aim is to deliver details of the existence and uniqueness proof as well as to recover the sharp-interface relations.

2 Preliminaries

We give a brief summary of the Finsler-geometry concept, which seems to be a natural way of introducing anisotropy into the model in question. We stress out that details about this approach can be found in [2] and in references therein.

A nonnegative function $\Phi : \mathbb{R}^n \to \mathbb{R}_0^+$ which is smooth, strictly convex, $\mathcal{C}^2(\mathbb{R}^n - \{\Theta\})$ and satisfies:

$$\Phi(t\eta) = |t|\Phi(\eta), \quad t \in \mathbb{R}, \eta \in \mathbb{R}^n, \tag{2.1}$$
$$\lambda|\eta| \leq \Phi(\eta) \leq \Lambda|\eta|, \tag{2.2}$$

where $\lambda, \Lambda > 0$, is called Finsler metric. The function given by

$$\Phi^0(\eta^*) = \sup\{\eta^* \cdot \eta \mid \Phi(\eta) \leq 1\},$$

is called dual Finsler metric. They satisfy the following relations

$$\Phi^0_\eta(t\eta^*) = \frac{t}{|t|}\Phi^0_\eta(\eta^*) \quad , \quad \Phi^0_{\eta\eta}(t\eta^*) = \frac{1}{|t|}\Phi^0_{\eta\eta}(\eta^*), \quad t \in \mathbb{R} - \{0\},$$
$$\Phi(\eta) = \Phi_\eta(\eta) \cdot \eta \quad , \quad \Phi^0(\eta^*) = \Phi^0_\eta(\eta^*) \cdot \eta^*, \quad \eta, \eta^* \in \mathbb{R}^n,$$

where the index η means derivative with respect to. We define the map $T^0 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$T^0(\eta^*) := \Phi^0(\eta^*)\Phi^0_\eta(\eta^*) \text{ for } \eta^* \neq 0,$$
$$T^0(0) := 0.$$

It allows to define the Φ-gradient of a smooth function u:

$$\nabla_\Phi u := T^0(\nabla u) = \Phi^0(\nabla u)\Phi^0_\eta(\nabla u). \tag{2.3}$$

The Φ-normal vector (the Cahn-Hoffmann vector) and velocity of a levelset

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^n \mid p(t, \mathbf{x}) = const.\} :$$

given by a field p are

$$\mathbf{n}_{\Gamma,\Phi} = -\frac{\nabla_\Phi p}{\Phi^0(\nabla p)} = -\frac{T^0(\nabla p)}{\Phi^0(\nabla p)}, \quad v_{\Gamma,\Phi} = \frac{p_t}{\Phi^0(\nabla p)}.$$

The anisotropic curvature is give by the formula

$$\kappa_{\Gamma,\Phi} = div(\mathbf{n}_{\Gamma,\Phi}).$$

Compared to [2], we do not consider an explicit dependence of Φ on space, for the sake of simplicity.

In analogy to the isotropic case [8], we can investigate an anisotropic motion by mean curvature

$$\alpha v_{\Gamma,\Phi} = -\kappa_{\Gamma,\Phi} + F,$$

in the direction of $\mathbf{n}_{\Gamma,\Phi}$. Manifold described as

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^n \mid p(t, \mathbf{x}) = 0.5\},$$

with convention

$$\Omega_s(t) = \{\mathbf{x} \in \mathbb{R}^n \mid p(t, \mathbf{x}) > 0.5\}$$

induces the Hamilton-Jacobi equation

$$\alpha\frac{\partial p}{\partial t} = \Phi^0(\nabla p)\nabla \cdot (\frac{\nabla_\Phi p}{\Phi^0(\nabla p)}) + \Phi^0(\nabla p)F,$$

compared to isotropic case

$$\alpha\frac{\partial p}{\partial t} = |\nabla p|\nabla \cdot (\frac{\nabla p}{|\nabla p|}) + |\nabla p|F.$$

Similarly, we derive a modified (see [6]) anisotropic Allen-Cahn equation for curve dynamics in plane

$$\alpha\xi^2\frac{\partial p}{\partial t} = \xi^2\nabla \cdot T^0(\nabla p) + f_0(p) + F\xi^2\Phi^0(\nabla p).$$

The complete model containing the above equation is formulated as follows

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u + L\chi'(p)\frac{\partial p}{\partial t}, \\ \alpha\xi\frac{\partial p}{\partial t} &= \xi\nabla \cdot T^0(\nabla p) + \frac{1}{\xi}f_0(p) + F(u)\xi\Phi^0(\nabla p), \end{aligned} \tag{2.4}$$

with boundary and initial conditions. We notice that the forcing term can include another type of anisotropy given by different dual Finsler metric: $F(u)\xi\Phi^1(\nabla p)$, as indicated by experiment – see a remark in [6].

Example. We typically use the Finsler dual metric set as

$$\Phi^0(\eta^*) = \varrho\Psi(\Theta),$$

where $[\varrho, \Theta]$ are polar coordinates of η^*. Our choice is $\Psi(\Theta) = 1 + A\sin(m\Theta)$, where A is the anisotropy strength, and m the order of symmetry. The convexity condition reads as $A \le \frac{1}{1-m^2}$.

Remark. The (strong) monotonicity of the operator T^0 is equivalent to the (strict) convexity of the functional

$$\int_\Omega \Phi^0(\nabla p)^2 dx.$$

3 Existence of the Solution

We introduce the following notations:

$$(u,v) = \int_\Omega u(x)v(x)\ dx,\ \|u\| = \sqrt{\int_\Omega u(x)^2 dx} \text{ for } u,v \in \mathrm{L}_2(\Omega),$$

$$(\nabla u, \nabla v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx,\ \|\nabla u\| = \sqrt{\int_\Omega |\nabla u(x)|^2 dx} \text{ for } u,v \in \mathrm{H}^1_0(\Omega).$$

We also notice that the assumptions on χ imply that there are constants $C_\chi, L_\chi > 0$ such that $|\chi'(s)| \le C_\chi, |\chi'(s_1) - \chi'(s_2)| \le L_\chi |s_1 - s_2|$ for all $s, s_1, s_2 \in \mathbb{R}$. Similarly, the assumptions on F imply that there are constants $C_F, L_F > 0$ such that $|F(s)| \le C_F, |F(s_1) - F(s_2)| \le L_F|s_1 - s_2|$ for all $s, s_1, s_2 \in \mathbb{R}$. Our existence and uniqueness result is contained in the following theorem.

Theorem 1 *Consider the problem (2.4) in a bounded domain $\Omega \subset \mathbb{R}^2$ with a C^2 boundary, with T^0 monotone, and with F being a bounded continuous function, χ a function with $\chi(0) = 0, \chi(1) = 1, \chi(0.5) = 0.5$, χ' bounded, Lipschitz continuous with the support in $\langle 0,1\rangle$. Assume that $\xi > 0$ is fixed, and*

$$u_0, p_0 \in \mathrm{H}^1(\Omega). \tag{3.1}$$

Then, there is a solution of the problem

$$\begin{aligned} \frac{d}{dt}(u - L\chi(p), v) + (\nabla u, \nabla v) &= 0 \textit{ a.e. in } (0,T), \qquad (3.2)\\ u|_{t=0} &= u_0, \\ \alpha\xi^2 \frac{d}{dt}(p,q) + \xi^2(T^0(\nabla p), \nabla q) &= (f_0(p), q) + \xi^2(F(u)\Phi^1(\nabla p), q) \textit{ a.e. in } (0,T), \\ p|_{t=0} &= p_0. \end{aligned}$$

for each $v, q \in \mathrm{H}^1_0(\Omega)$, satisfying

$$u, p \in \mathrm{L}_\infty(0,T;\mathrm{H}^1_0(\Omega)),\ p \in \mathrm{L}_\infty(0,T;\mathrm{L}_\infty(\Omega)),$$

$$\frac{\partial u}{\partial t}, \frac{\partial p}{\partial t} \in \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega)).$$

Additionally, if F is Lipschitz-continuous, $\chi' \equiv 1$, and T^0 is strictly monotone, the solution is unique.

Proof. We follow the proof for the isotropic case given in [8] and we stress out details concerning the anisotropy. We derive a sequence of approximate solutions to the original problem. Assume that there is an orthonormal basis of the Hilbert space $\mathrm{L}_2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$ denoted as $\{v_i\}_{i\in\mathbb{N}}$ where $(\forall i \in \mathbb{N})(v_i \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar\Omega))$ with corresponding eigenvalues denoted as $\{\lambda_i\}_{i\in\mathbb{N}}$. Let $V_m = span\{v_i\}_{i\in\mathbb{N}_m}$ be a finite-dimensional subspace ($\mathbb{N}_m = \{1,\dots,m\}$); $\mathcal{P}_m : \mathrm{L}_2(\Omega) \to V_m$ be the projection operator (coinciding with the H^1-projection). We seek for a solution of an auxiliary problem:

$$\begin{aligned} \frac{d}{dt}(u^m - L\chi(p^m), v_i) + (\nabla u^m, \nabla v_i) &= 0 \text{ a.e. in } (0,T),\ \forall i = 1,\dots,m, \qquad (3.3)\\ u^m(0) &= \mathcal{P}_m u_0, \\ \alpha\xi^2\frac{d}{dt}(p^m, v_j) + \xi^2(T^0(\nabla p^m), \nabla v_j) &= (f_0(p^m), v_j) + \xi^2(F(u^m)\Phi^1(\nabla p^m), v_j) \qquad (3.4)\\ &\quad \text{a.e. in } (0,T),\ \forall j = 1,\dots,m, \\ p^m(0) &= \mathcal{P}_m p_0. \end{aligned}$$

We use basic functions of V_m to express the solution of (3.3) as

$$u^m(t) = \sum_{i\in\mathbb{N}_m} \beta_i^m(t) v_i, \quad p^m(t) = \sum_{i\in\mathbb{N}_m} \gamma_i^m(t) v_i,$$

and to obtain a system of ordinary differential equations for the unknown functions of time: β_i^m, γ_i^m using (3.3-3.4) . We follow the procedure of the compactness method (e.g., see [25]), show that the solution of (3.3-3.4) is defined on $(0,T)$ for $T>0$ and show an appropriate convergence of the couple $[u^m, p^m]$. For this purpose, we prove an *a priori* estimate by multiplying (3.3) by $\frac{d\beta_j^m}{dt}$, (3.4) by $\frac{d\gamma_j^m}{dt}$, and summing for $j \in \mathbb{N}_m$:

$$\begin{aligned} \|\frac{\partial u^m}{\partial t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m\|^2 &= L(\frac{\partial \chi(p^m)}{\partial t}, \frac{\partial u^m}{\partial t}), \\ \alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\Phi^0(\nabla p^m)^2 + \frac{d}{dt}(w_0(p^m),1) &= \xi^2(F(u^m)\Phi^1(\nabla p^m), \frac{\partial p^m}{\partial t}). \end{aligned}$$

where we have used (2.3), and notation $w_0' = -f_0$. Using the Schwarz and Young inequalities, we get

$$\begin{aligned} \|\frac{\partial u^m}{\partial t}\|^2 + \frac{d}{dt}\|\nabla u^m\|^2 &\le L^2 C_\chi^2 \|\frac{\partial p^m}{\partial t}\|^2, \\ \frac{1}{2}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\Phi^0(\nabla p^m)^2 + \frac{d}{dt}(w_0(p^m),1) &\le \frac{C_F^2}{2\alpha}\xi^2\Phi^1(\nabla p^m). \end{aligned}$$

Combining these estimates, we have

$$\begin{aligned} &\frac{1}{4}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\frac{\partial u^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\frac{d}{dt}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\Phi^0(\nabla p^m)^2 \\ &+ \frac{d}{dt}(w_0(p^m),1) \le \frac{C_F^2}{2\alpha}\xi^2\Phi^1(\nabla p^m)^2, \end{aligned}$$

It remains to use the equivalence of the two Finsler metrics

$$\bar{\lambda}\Phi^0(\eta) \le \Phi^1(\eta) \le \bar{\Lambda}\Phi^0(\eta), \tag{3.5}$$

following from (2.2), and add non-negative terms on the right-hand side,

$$\begin{aligned} &\frac{1}{4}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\frac{\partial u^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\frac{d}{dt}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\Phi^0(\nabla p^m)^2 + \frac{d}{dt}(w_0(p^m),1) \\ &\le \frac{\bar{\Lambda}^2 C_F^2}{\alpha}\left(\frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right). \end{aligned} \tag{3.6}$$

We integrate over $(0,t)$,

$$\begin{aligned} &\left(\frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + (w_0(p^m),1)\right)(t) \\ &\le \left(\frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + (w_0(p^m),1)\right)(0)\exp\left(\frac{\bar{\Lambda}C_F^2}{\alpha}t\right). \end{aligned} \tag{3.7}$$

The assumption of the theorem together with the coincidence of projectors in L_2 and H^1 imply that $\nabla\mathcal{P}_m p_0, \nabla\mathcal{P}_m u_0 \in \mathrm{L}_2(\Omega)$ and $\mathcal{P}_m p_0$ in $\mathrm{L}_4(\Omega)$ are bounded independently of m (due to the continuous imbedding of H^1 into L_4). Consequently, the inequality (2.2) implies that, independently of m, $\nabla u^m, \nabla p^m$ are bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_2(\Omega))$, and p^m are bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_6(\Omega))$ for each finite time $T>0$.

We are able to show **an additional estimate** by testing (3.4) by $(p^m)^l$ for $l \in \mathbb{N}$ being arbitrary, but odd. We follow [20].

$$\begin{aligned} &\frac{\alpha\xi^2}{l+1}\frac{d}{dt}((p^m)^{l+1},1) + \xi^2 l(T^0(\nabla p^m),(p^m)^{l-1}\nabla p^m) \\ &= (f_0(p^m),(p^m)^l) + \xi^2(F(u^m)\Phi^1(\nabla p^m),(p^m)^l). \end{aligned}$$

Using the fact that $T^0(\nabla p^m)\nabla p^m = \Phi^0(\nabla p^m)^2$, we have

$$\frac{\alpha\xi^2}{l+1}\frac{d}{dt}((p^m)^{l+1},1) + \xi^2 l(\Phi^0(\nabla p^m)^2,(p^m)^{l-1})$$
$$= (f_0(p^m),(p^m)^l) + \xi^2(F(u^m)\Phi^1(\nabla p^m),(p^m)^l).$$

We proceed by considering the following inequalities resulting from the Young inequality, Hölder inequality and from equivalence of Finsler metrics Φ^0 and Φ^1 via (2.2) and (3.5):

$$\frac{d_0}{2}s^{l+3} - d_1 \le -f_0(s)s^l \le \frac{3d_0}{2}s^{l+3} + d_1 \text{ for } s \in \mathbb{R},$$

$$\begin{aligned} |(F(u^m)\Phi^1(\nabla p^m),(p^m)^l)| &\le C_F\bar{\Lambda}\sqrt{(\Phi^0(\nabla p^m)^2,(p^m)^{l-1})}\sqrt{((p^m)^{l+1},1)} \\ &\le l(\Phi^0(\nabla p^m)^2,(p^m)^{l-1}) + \frac{C_F^2\bar{\Lambda}^2}{4l}((p^m)^{l+1},1). \end{aligned}$$

Then,

$$\frac{\alpha\xi^2}{l+1}\frac{d}{dt}((p^m)^{l+1},1) - d_1|\Omega| \le \xi^2\frac{C_F^2\bar{\Lambda}^2}{4l}((p^m)^{l+1},1),$$

and consequently

$$\frac{d}{dt}((p^m)^{l+1},1) \le \frac{C_F^2\bar{\Lambda}^2(l+1)}{4\alpha l}((p^m)^{l+1},1) + \frac{d_0(l+1)}{\alpha}|\Omega|.$$

The Gronwall lemma implies

$$((p^m(t))^{l+1},1) \le ((p^m(0))^{l+1},1)\exp\frac{C_F^2\bar{\Lambda}^2(l+1)}{4\alpha l}t + \frac{4d_0 l}{\alpha C_F^2\bar{\Lambda}^2}|\Omega|\left(\exp\frac{C_F^2\bar{\Lambda}^2(l+1)}{4\alpha l}t - 1\right),$$

a relation valid for $t \in (0,T)$, for all $l \in \mathbb{N}$ odd. Passing to the limit $l \to +\infty$ (see [17]), we obtain

$$\|p^m(t)\|_\infty \le \|p^m(0)\|_\infty + 1 \tag{3.8}$$

uniformly for each $m \in \mathbb{N}$, independently on ξ.

Integrating (3.7) over $(0,T)$, we get

$$\begin{aligned} &\int_0^T \left(\frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + (w_0(p^m),1)\right)(t)dt \\ &\le \left(\frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + (w_0(p^m),1)\right)(0)\frac{\alpha}{C_F^2}\left(\exp(\frac{\bar{\Lambda}C_F^2}{\alpha}T) - 1\right). \end{aligned}$$

We use this estimate for the integration of the relation (3.6), and we see that

$$\begin{aligned} &\int_0^T \left(\frac{1}{4}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\frac{\partial u^m}{\partial t}\|^2\right)(t)dt \\ &\quad + \left(\frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right)(T) \\ &\le \left(\frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right)(0) \\ &\quad + \frac{\bar{\Lambda}C_F^2}{\alpha}\int_0^T \left(\frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + \frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right)(t)dt \\ &\le \left(\frac{\alpha\xi^2}{4L^2C_\chi^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\Phi^0(\nabla p^m)^2 + (w_0(p^m),1)\right)(0)\exp(\frac{\bar{\Lambda}C_F^2}{\alpha}T). \end{aligned}$$

Passing to a subsequence m', we have $u^{m'} \rightharpoonup u$ and $p^{m'} \rightharpoonup p$ in $\mathrm{L}_2(0,T;\mathrm{H}^1_0(\Omega))$. The non-linear terms in (2.4) require stronger convergence result. Using the the compact-imbedding theorem as in ([8]) with the setting

$$\{u^m\}_{m=1}^{\infty} \text{ bounded in } \mathrm{L}_2(0,T;\mathrm{H}^1_0(\Omega)), \{\frac{\partial u^m}{\partial t}\}_{m=1}^{\infty} \text{ bounded in } \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega)),$$

$$\{p^m\}_{m=1}^{\infty} \text{ bounded in } \mathrm{L}_6(0,T;\mathrm{H}^1_0(\Omega)), \{\frac{\partial p^m}{\partial t}\}_{m=1}^{\infty} \text{ bounded in } \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega)),$$

we see that $\{u^{m'}\}_{m'=1}^{\infty}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, and $\{p^{m'}\}_{m'=1}^{\infty}$ converges strongly in $\mathrm{L}_6(0,T;\mathrm{L}_6(\Omega))$. The polynomial form of f_0 implies the existence of the strong limit of $f_0(p^{m'})$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ being equal to $f_0(p)$. We also observe that the term $F(u^m)\Phi^1(\nabla p^m)$ is bounded in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ due to (2.2), and, therefore, the subsequence converges weakly to $F(u)\Phi^1(\nabla p)$ in this space, as shown in the Lemma 2 and 3 of [8]. Convergence of $\chi(p^m)$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ via subsequence is guarranteed by boundedness of χ'. Finally, the term $\chi'(p^m)\frac{\partial p^m}{\partial t}$ is bounded in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ which implies the convergence of subsequence to a function $\tilde{\chi}$ in this space which equals to $\frac{\partial \chi'(p)}{\partial t}$ via definition of time derivative in the sense of distributions.

Passage to the limit. Choose test functions $w, q \in \mathcal{D}(\Omega)$, multiply (3.3) by (w, v_j) and (3.4) by (q, v_j), sum over $\mathbb{N}_m$. Then choose scalar functions $\varphi, \psi \in \mathcal{C}^1(\langle 0,T\rangle)$, for which $\varphi(T) = \psi(T) = 0$. Integrate both equations by parts over $(0,T)$. Knowing that

1. $\nabla p^{m'}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ to ∇p (Lemma 2 of [8]),
2. $T^0(\nabla p^{m'})$ converges weakly to $T^0(\nabla p)$ in $\mathrm{L}_2(0,T;\mathrm{H}^1(\Omega))$ due to the semicontinuity of T^0, see [18];
3. $\mathcal{P}_{m'}p_0$, $\mathcal{P}_{m'}u_0$ converge strongly to p_0, u_0 in $\mathrm{L}_2(\Omega)$,
4. $F(u^{m'})\Phi^1(\nabla p^{m'})$ converges weakly to $F(u)\Phi^1(\nabla p)$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ (Lemma 3 of [8]),
5. $\chi(p^{m'})$ converges weakly to $\chi(p)$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$,
6. $p^{m'}(0) = \mathcal{P}_{m'}p_0$, $u^{m'}(0) = \mathcal{P}_{m'}u_0$,

we are able to pass to the limit, and we obtain the following relations:

$$\begin{aligned}
&(u_0 - L\chi(p_0), w)\varphi(0) - \int_0^T (u - L\chi(p), w)\frac{d\varphi}{dt}dt + \int_0^T \varphi(\nabla u, \nabla w)dt = 0,\\
&\alpha\xi^2(p_0, q)\psi(0) - \int_0^T \alpha\xi^2(p,q)\frac{d\psi}{dt}dt \\
&\quad + \int_0^T \psi[\xi^2(T^0(\nabla p), \nabla q) - (f_0(p), q) - \xi^2(F(u)\Phi^1(\nabla p), q)]dt = 0.
\end{aligned} \tag{3.9}$$

If $\varphi, \psi \in \mathcal{D}(0,T)$, we have

$$\begin{aligned}
\frac{d}{dt}(u - L\chi(p), w) + (\nabla u, \nabla w) &= 0,\\
\alpha\xi^2\frac{d}{dt}(p,q) + \xi^2(T^0(\nabla p), \nabla q) &= (f_0(p), q) + \xi^2(F(u)\Phi^1(\nabla p), q).
\end{aligned}$$

The weak solution satisfies the initial condition. Indeed, in (3.9), by using scalar functions $\varphi, \psi \in \mathcal{C}^1(\langle 0,T\rangle)$, for which $\varphi(T) = \psi(T) = 0$, we obtain

$$\begin{aligned}
&(u(0) - L\chi(p(0)), w)\varphi(0) - \int_0^T (u - L\chi(p), w)\frac{d\varphi}{dt}dt + \int_0^T \varphi(\nabla u, \nabla w)dt = 0,\\
&\alpha\xi^2(p(0), q)\psi(0) - \int_0^T \alpha\xi^2(p,q)\frac{d\psi}{dt}dt \\
&\quad + \int_0^T \psi[\xi^2(T^0(\nabla p), \nabla q) - (f_0(p), q) - \xi^2(F(u)\Phi^1(\nabla p), q)]dt = 0.
\end{aligned}$$

Subtracting these equations from (3.9), we get

$$(u_0 - L\chi(p_0) - u(0) + L\chi(p(0)), w)\varphi(0) = 0,\ (p_0 - p(0), q)\psi(0) = 0,\ \forall w, q \in \mathcal{D}(\Omega).$$

From this we see that $u(0) = u_0$, $p(0) = p_0$ in $\mathrm{L}_2(\Omega)$.

In case when F is Lipschitz-continuous with the Lipschitz constant denoted by L_F, $\chi(p) = p$, and when T^0 is strictly continuous, we prove **uniqueness** of the solution of (3.2). We consider two solutions of the problem (3.2), denoted by $[u_1, p_1]$ and $[u_2, p_2]$. Subtracting corresponding systems of equations and denoting $[u_{12}, p_{12}] = [u_1 - u_2, p_1 - p_2]$, multiplying the first equation by $u_{12} - Lp_{12}$ and the second equation by p_{12}, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_{12} - Lp_{12}\|^2 + \|\nabla(u_{12} - Lp_{12})\|^2 + L(\nabla p_{12}, \nabla(u_{12} - Lp_{12})) &= 0 \text{ in } (0,T), \\ (u_{12} - Lp_{12})(0) &= 0, \\ \frac{1}{2}\alpha\xi^2\frac{d}{dt}\|p_{12}\|^2 + \xi^2(T^0(\nabla p_1) - T^0(\nabla p_2), \nabla p_{12}) & \\ = (f_0(p_1) - f_0(p_2), p_{12}) + \xi^2(F(u_1)\Phi^1(\nabla p_1) - F(u_2)\Phi^1(\nabla p_2), p_{12}) \text{ in } (0,T), & \\ p_{12}(0) = 0. & \end{aligned}$$

Denote

$$\Psi(p_1, p_2) = \frac{f_0(p_1) - f_0(p_2)}{p_{12}}.$$

The *a priori* estimate (3.8) guarantees that there is a constant $C_f > 0$ such that

$$|\Psi(p_1, p_2)| \le C_f \text{ in } (0,T) \times \Omega,$$

(as implied by the continuous imbedding $\mathrm{H}_0^1(\Omega) \subset\!\!> \mathrm{L}_s(\Omega)$ for $s \in \langle 1, +\infty)$). Therefore,

$$|(\Psi(p_1, p_2)p_{12}, p_{12})| \le C_f\|p_{12}\|^2,$$

Using the Young and Schwarz inequalities and strong monotonicity of T^0 $((T^0(\nabla p_1) - T^0(\nabla p_2), \nabla p_{12}) \ge c_0\|\nabla p_{12}\|^2)$, we get

$$\begin{aligned} \frac{d}{dt}\|u_{12} - Lp_{12}\|^2 &\le L^2\|\nabla p_{12}\|^2, \\ \frac{1}{2}\alpha\xi^2\frac{d}{dt}\|p_{12}\|^2 + c_0\xi^2\|\nabla p_{12}\|^2 &\le C_f\|p_{12}\|^2 + \frac{\xi^2}{c_0}L_F\Lambda_1\|u_{12}\|\|\nabla p_1\|_{\mathrm{L}_4(\Omega)}\|p_{12}\|_{\mathrm{L}_4(\Omega)} \\ &\quad + \xi^2 C_F\Lambda_1\|\nabla p_{12}\|\|p_{12}\|, \end{aligned}$$

in $(0,T)$, where $\Lambda_1 = \bar{\Lambda}\Lambda$. Considering the fact that there is a constant C_p for which

$$\int_0^T \|\nabla p_1\|^2_{\mathrm{L}_4(\Omega)}dt \le C_4^2\int_0^T \|p_1\|^2_{\mathrm{H}^2(\Omega)}dt \le C_p^2, \tag{3.10}$$

where C_4 is the norm of the imbedding $\mathrm{H}_0^1(\Omega)$ into $\mathrm{L}_4(\Omega)$, and $\Delta p_1 \in \mathrm{L}_2(0,T; \mathrm{L}_2(\Omega))$, we obtain

$$\begin{aligned} \frac{d}{dt}\|u_{12} - Lp_{12}\|^2 &\le L^2\|\nabla p_{12}\|^2 \text{ in } (0,T), \\ \frac{1}{2}\alpha\xi^2\frac{d}{dt}\|p_{12}\|^2 + \frac{c_0\xi^2}{2}\|\nabla p_{12}\|^2 &\le (C_f + \frac{\xi^2}{c_0}(C_F^2\Lambda_1^2 + 2L^2C_4^2\|\nabla p_1\|^2_{\mathrm{L}_4(\Omega)}L_F^2\Lambda_1^2))\|p_{12}\|^2 \\ &\quad + 2C_4^2\|\nabla p_1\|^2_{\mathrm{L}_4(\Omega)}L_F^2\Lambda_1^2\frac{\xi^2}{c_0}\|u_{12} - Lp_{12}\|^2 \text{ in } (0,T). \end{aligned}$$

Combining these inequalities, we have in $(0,T)$:

$$\frac{d}{dt}\left(\frac{1}{2}\frac{\alpha}{c_0}\xi^2\|p_{12}\|^2 + \frac{\xi^2}{2L^2}\|u_{12} - Lp_{12}\|^2\right) \le M(t)\left(\frac{1}{2}\xi^2\|p_{12}\|^2 + \frac{\xi^2}{2L^2}\|u_{12} - Lp_{12}\|^2\right)$$

with

$$M(t) = \frac{2(C_f + \frac{\xi^2}{c_0}(C_F^2\Lambda_1^2 + 2L^2L_F^2\Lambda_1^2C_4^2\|\nabla p_1\|^2_{\mathrm{L}_4(\Omega)})}{c_0 \min(\frac{\alpha}{c_0}, 1)\xi^2}.$$

Such an inequality, together with (3.10) and with the initial conditions implies, that

$$p_{12}(t) = u_{12}(t) = 0 \text{ in } \mathrm{L}_2(\Omega),\ \forall t \in (0, T).$$

as follows from the Gronwall lemma. $\square$

4 Asymptotical Behaviour

In this section, we show that the function p_ξ tends to a stepwise function provided $\xi \to 0$. This is an extension of results described in [5], which were inspired by [11]. A priori estimate (3.7) implies that the energy functional

$$E_\xi[p_\xi](t) = \int_\Omega [\xi \frac{1}{2}\Phi^0(\nabla p_\xi)^2 + \frac{1}{\xi}w_0(p_\xi)]dx,$$

is bounded as

$$E_\xi[p_\xi](t) \leq E_\xi[p_\xi](0)\exp\{\frac{C_F^2}{2\alpha}t\} \quad t \in (0, T),$$

where p_ξ is second component of the solution of (3.2).

Additionally, there is an estimate for the time derivative (see (3.6)):

$$\begin{aligned} \frac{1}{2}\alpha\xi \int_0^T \|\dot{p}\|^2 dt + E_\xi[p](T) &\leq E_\xi[p](0) + \frac{C_F^2}{2\alpha}\xi \int_0^T \Phi^0(\nabla p)^2 dt \\ \leq\ & E_\xi[p](0) + \frac{C_F^2}{\alpha}\int_0^T E_\xi[p](t)dt \\ \leq\ & E_\xi[p](0) + \frac{C_F^2}{\alpha}\int_0^T E_\xi[p](0)\exp\{\frac{C_F^2}{2\alpha}t\}dt. \end{aligned}$$

Consequently, there is a constant C_T such that

$$\frac{1}{2}\alpha\xi \int_0^T \parallel \dot{p} \parallel^2 dt + E_\xi[p](T) \leq C_T E_\xi[p](0).$$

These estimates allow to use the method proposed in [11]. Define the following monotone function

$$G(s) = \int_0^s |1 - (1 - 2r)^2|dr.$$

Such a choice is given by the form of the double-well potential

$$w_0(p) = \frac{a}{16}\left(1 - (1 - 2p)^2\right)^2,$$

as we see that

$$G'(s) = \sqrt{w_0(s)}.$$

Then, we prove the lemma

Lemma 1 *Be p_ξ the solution of (3.2) where $E_\xi[p_\xi](0) \leq M_0$ independently on ξ. Then there are constants $M > 0$ and $M_1 > 0$ such that*

$$sup\{\int_\Omega \Phi^0(\nabla G(p_\xi))dx \ \mid\ t \in < 0, T >\} \leq M \tag{4.1}$$

and, for $0 \leq t_1 < t_2$,

$$\int_{t_1}^{t_2}\int_\Omega |\partial_t G(p_\xi)|dxdt \leq M_1(t_2 - t_1)^{0.5}. \tag{4.2}$$

Proof. We have shown that

$$E_\xi[p_\xi](t) \le C_T M_0,$$

on $< 0, T >$. We write

$$\begin{aligned} E_\xi[p](t) &= \int_\Omega \left(\frac{\xi}{2}\Phi^0(\nabla p)^2 + \frac{1}{\xi} w_0(p) \right) dx \\ &\ge \int_\Omega \sqrt{2}\Phi^0(\nabla p_\xi)\sqrt{w_0(p_\xi)} dx = \sqrt{2}\int_\Omega \Phi^0(\nabla G(p_\xi)) dx, \end{aligned}$$

which shows (4.1) by setting $M = \frac{1}{\sqrt{2}} M_0 C_T$. Furthermore, if

$$\int_{t_1}^{t_2} dt \int_\Omega dx |\partial_t G(p_\xi)| = \int_{t_1}^{t_2} dt \int_\Omega dx |\partial_t p_\xi||G'(p_\xi)|$$

$$\le \left(\int_{t_1}^{t_2} dt \int_\Omega dx |\partial_t p_\xi|^2 \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} dt \int_\Omega dx |G'(p_\xi)|^2 \right)^{\frac{1}{2}} \le$$

$$\le (\frac{2}{\alpha} C_T^2 M_0^2)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}},$$

then (4.2) is shown, if setting $M_1 = \sqrt{\frac{2}{\alpha}} C_T M_0$. □

The previous statement leads to the existence of a step function as expected.

Theorem 2 *Let $[u_\xi, p_\xi]$ is the solution of (3.2) with the initial data satisfying $E_\xi[p_\xi](0) < M_0$ independently on ξ, and let*

$$\int_\Omega |p_\xi(0, \mathbf{x}) - v_0(\mathbf{x})| d\mathbf{x} \to 0,$$

as $\xi \to 0$, for a function $v_0 \in \mathrm{L}_1(\Omega)$. Then for any sequence ξ_n tending to 0 there is a subsequence $\xi_{n'}$ such that

$$\lim_{\xi_{n'} \to 0} p_{\xi_{n'}}(t, \mathbf{x}) = v(t, \mathbf{x}),$$

is defined a.e. in $(0, T) \times \Omega$. The function v reaches values 0 and 1, and satisfies

$$\int_\Omega |v(t_1, \mathbf{x}) - v(t_2, \mathbf{x})| d\mathbf{x} \le C|t_2 - t_1|^{\frac{1}{2}},$$

where $C > 0$ is a constant, and

$$\sup_{t \in <0,T>} \int_\Omega |\nabla v| d\mathbf{x} \le C_1,$$

in the sense of $BV(\Omega)$, where $C_1 > 0$ is a constant. The initial condition is

$$\lim_{t \to 0_+} v(t, \mathbf{x}) = v_0(\mathbf{x}),$$

a.e.

Proof. We find that

$$\begin{aligned} G(s) &= 2s^2 - \frac{4}{3}s^3 \quad \text{for} \quad s \in < 0, 1 > , \\ G(s) &= \frac{4}{3}s^3 - 2s^2 + \frac{4}{3} \quad \text{for} \quad s \in (1, +\infty). \end{aligned}$$

Consequently, a direct computation justifies that

$$|G(s)| \le \frac{4}{3} + [1 - (1 - 2s)^2]^2.$$

When using the properties of the Finsler metric (2.1-2.2), we can apply the proof presented in [5] and [11]. □

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