Radial Implicative Fuzzy Inference Systems

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Abstrakt

The paper surveys the basic knowledge about the special class of fuzzy inference systems which is the class of the radial implicative fuzzy inference systems. It is presented their definition together with several important properties of them such as coherence and universal approximation property.

1. Radial fuzzy inference systems

The concept of fuzzy inference system (FIS) is well known over thirty years, e.g., [1]. The architecture of standard FIS is given by four building blocks, a fuzzifier, a rule base, an inference engine and a deffuzifier. The input signal flows from fuzzifier, through inference engine, which cooperates with a rule base, to the deffuzifier. Mathematically, a FIS (in MISO configuration) performs a function from $\mathcal{R}^n$ to $\mathcal{R}$.

A "knowledge" of a FIS is stored in the rule base. This is tradionally given by a set of m IF-THEN rules. An IF-THEN rule has the canonical form

$$\text{IF } x_1 \; is \; A_{j1} \; \; and \; \; x_2 \; is \; A_{j2} \; \; and \; \ldots \; and \; \; x_n \; is \; A_{jn} \; \; \text{THEN} \; \; y \; is \; B_j, \qquad (4)\text{span}$$

where $A_{ji}, B_j, i = 1, \ldots, n, j = 1, \ldots, m$ are fuzzy sets defined on respective universal sets $X_1, \ldots, X_n$, $X_i \subseteq \mathcal{R}$, $Y \subseteq \mathcal{R}$. The linguistic connective and is represented by a t-norm (associative, commutative, monotone and conjunction-like operation from $[0,1]^2$ to $[0,1]$, see [1, 2] for exact definition).

Considering $n > 1$ we have antecedent of a rule representing a fuzzy relation on $X_1 \times X_2 \times \cdots \times X_n$ given as

$$A_j(\boldsymbol{x}) = A_{j1}(x_1) \star A_{j2}(x_2) \star \cdots \star A_{jn}(x_n), \qquad (5)\text{span}$$

where $\star$ symbol represents a t-norm. Having particular fuzzy sets given as Gaussians and t-norm as product we have above as

$$A_j(\boldsymbol{x}) = \exp\left[-\frac{(x_1 - a_{j1})^2}{b_{j1}^2}\right] \cdot \exp\left[-\frac{(x_2 - a_{j2})^2}{b_{j2}^2}\right] \cdot \cdots \cdot \exp\left[-\frac{(x_n - a_{jn})^2}{b_{jn}^2}\right]. \qquad (6)\text{span}$$

which is on base of Gaussians properties

$$A_j(\boldsymbol{x}) = \exp\left[-\sum_{i=1}^{n}\frac{(x_i - a_{ji})^2}{b_{ji}^2}\right] = \exp\left[-||\boldsymbol{x}-\boldsymbol{a}_j||^2_{E_{\boldsymbol{b}}}\right], \qquad (7)\text{span}$$

where $\boldsymbol{a}_j = (a_{j1},\ldots,a_{jn})$, $\boldsymbol{a}_j \in \mathcal{R}^n$, is the vector of centers of particular fuzzy sets A_{ji}, $\boldsymbol{b}_j = (b_{j1},\ldots,b_{jn})$, $\boldsymbol{b} \in \mathcal{R}^n_+$ is the vector of theirs width parameters and $||\cdot||_{E_{\boldsymbol{b}}}$ is the scaled Euclidean norm defined as

$$||\boldsymbol{u}||_{E_{\boldsymbol{b}}} = \sqrt{\sum_{i=1}^{n}\frac{u_i^2}{b_i^2}}. \qquad (8)\text{span}$$

Comparing the form of antecedent and succedent,

$$A_j(\boldsymbol{x}) = \exp\left[-||\boldsymbol{x}-\boldsymbol{a}_j||^2_{\boldsymbol{b}}\right] \qquad B_j(y) = \exp\left[-(y-c)^2/d^2\right], \qquad (9)\text{span}$$

we see that they have the same form. That is, their computation is given, up to dimension, by the same radial basis function. Actually, this property gives the formal definition of radial FIS.

A FIS is called radial if there exists a non-increasing function $act : [0,+\infty) \to [0,1]$, $act(0) = 1$, $\lim_{s\to+\infty} act(s) = 0$, and a strictly increasing function $g : [0,+\infty) \to [0,+\infty)$, $g(0) = 0$, both continuous, such that antecedent $A_j(\boldsymbol{x})$ and succedent $B_j(y)$ of the jth rule, $j = 1,\ldots,m$, can be written as

$$A_j(\boldsymbol{x}) = act(\ g(||\boldsymbol{x}-\boldsymbol{a}_j||_{\boldsymbol{b}_j})\ ) \quad \text{and} \quad B_j(y) = act\,(\ g\,(|y-c_j|/d_j)\,)\,. \qquad (10)\text{span}$$

On base of this definition we see that in above example the act function is given as $act(s) = \exp(-s)$ and g function as $g(s) = s^2$. There is a question if there can be defined another radial FISs on base of other well known t-norms such as Lukasiewicz, defined as $x\star y = \max\{0, x+y-1\}$ and minimum $x\star y = \min\{x,y\}$ and other shapes of fuzzy sets such as triangular ones. We have these two lemmas.

Lemma 1.1 *Let A_{ji}, B_j be triangular fuzzy sets, i.e.,*

$$A_{ji} = \max\left\{0, 1-\left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right\}, \qquad B_j = \max\left\{0, 1-\left|\frac{y-c_j}{d_j}\right|\right\}, \qquad (11)\text{span}$$

and t-norm be chosen as minimum then resulting FIS *is radial.*

Proof: Considering antecedent of particular rule we have

$$\begin{aligned} A_j(\boldsymbol{x}) &= \min_i\left\{\max\left\{0, 1-\left|\frac{x_i-a_{ji}}{b_{ji}}\right|\right\}\right\}, & (12)\\ A_j(\boldsymbol{x}) &= \max\left\{0, \min_i\left\{1-\left|\frac{x_i-a_{ji}}{b_{ji}}\right|\right\}\right\}, & (13)\\ A_j(\boldsymbol{x}) &= \max\left\{0, 1-\max_i\left\{\left|\frac{x_i-a_{ji}}{b_{ji}}\right|\right\}\right\}. & (14)\end{aligned}$$

Hence it is

$$A_j(\boldsymbol{x}) = \max\{0, 1-||\boldsymbol{x}-\boldsymbol{a}_j||_{C_{\boldsymbol{b}_j}}\}, \qquad (15)\text{span}$$

where $||\cdot||_{C_{\boldsymbol{b}_j}}$ is the scaled cubic norm given as

$$||\boldsymbol{u}||_{C_{\boldsymbol{b}_j}} = \max\left\{\left|\frac{u_1}{b_1}\right|,\ldots,\left|\frac{u_n}{b_n}\right|\right\}, \qquad (16)\text{span}$$

for $\boldsymbol{b} = (b_1,\ldots,b_n)$, $\boldsymbol{b}\in\mathcal{R}^n_+$.

Setting $act(s) = \max\{0, 1 - s\}$ for $s \in [0, +\infty)$ and $g(s) = s$, we have

$$\begin{aligned} A_j(\boldsymbol{x}) &= act(\, g(||\boldsymbol{x} - \boldsymbol{a}||_{C_{\boldsymbol{b}_j}})\,), & (17)\\ B_j(y) &= \max\left\{0, 1 - \left|\frac{y - c_j}{d_j}\right|\right\} = act\left(g\left(\left|\frac{y - c_j}{d_j}\right|\right)\right). & (18) \end{aligned}$$

Hence Mamdani I-FIS is radial. □

Lemma 1.2 *Let A_{ji}, B_j be triangular fuzzy sets, i.e.,*

$$A_{ji} = \max\left\{0, 1 - \left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right\}, \qquad B_j = \max\left\{0, 1 - \left|\frac{y - c_j}{d_j}\right|\right\}, \tag*{(19)span}$$

and t-norm be chosen as Lukasiewicz one then resulting FIS *is radial.*

Proof: By induction. Let $n = 2$ then we have for $r_1 = (x - a_{j1})/b_{j1}$, $r_2 = (x - a_{j2})/b_{j2}$,

$$\begin{aligned} A_j(\boldsymbol{x}) &= \max\{0, \max\{0, 1 - |r_1|\} + \max\{0, 1 - |r_2|\} - 1\}, & (20)\\ A_j(\boldsymbol{x}) &= \max\{0, 1 - (|r_1| + |r_2|)\}. & (21) \end{aligned}$$

By the same manipulation we prove that for $n > 2$ we have

$$A_j(\boldsymbol{x}) = \max\left\{0, 1 - \sum_{i=1}^{n} |r_i|\right\}. \tag*{(22)span}$$

Considering the scaled octaedric norm in $\mathcal{R}^n$ defined as

$$||\boldsymbol{u}||_{O_{\boldsymbol{b}}} = \sum_{i=1}^{n} \left|\frac{u_i}{b_i}\right|, \tag*{(23)span}$$

we have

$$A_j(\boldsymbol{x}) = \max\{0, 1 - ||\boldsymbol{x} - \boldsymbol{a}_j||_{O_{\boldsymbol{b}}}\}. \tag*{(24)span}$$

Setting again $act(s) = \max\{0, 1 - s\}$ for $s \in [0, +\infty)$ and $g(s) = s$ we obtain a radial FIS. □

Note that considering l_p norms in $\mathcal{R}^n$ defined for $p \geq 1$ by

$$l_p(\boldsymbol{u}) = (|u_1|^p + \ldots |u_n|^p)^{1/p} \tag*{(25)span}$$

then their scaled counterparts for $\boldsymbol{b} \in \mathcal{R}^n_+$

$$l_{p_{\boldsymbol{b}}}(\boldsymbol{u}) = \left(\left|\frac{u_1}{b_1}\right|^p + \cdots + \left|\frac{u_n}{b_n}\right|^p\right)^{1/p} \tag*{(26)span}$$

are norms in $\mathcal{R}^n$ as well. Further it is

$$l_{(p=1)_{\boldsymbol{b}}}(\boldsymbol{u}) = \left|\frac{u_1}{b_1}\right| + \cdots + \left|\frac{u_n}{b_n}\right| = ||\boldsymbol{u}||_{O_{\boldsymbol{b}}}, \tag*{(27)span}$$

$$\lim_{p \to +\infty} l_{p_{\boldsymbol{b}}}(\boldsymbol{u}) = \max\left\{\left|\frac{u_1}{b_1}\right| + \cdots + \left|\frac{u_n}{b_n}\right|\right\} = ||\boldsymbol{u}||_{C_{\boldsymbol{b}}}. \tag*{(28)span}$$

Thus we see that for the most important t-norms (any t-norm can be build from Lukasiewicz, product and minimum ones [2]) and usual shapes of fuzzy sets (triangular, Gaussians) there exists corresponding radial FIS.

2. Radial implicative fuzzy inference systems

A particular IF-THEN rule of a rule base is mathematically represented by a fuzzy relation. Its form depends on the shapes of fuzzy sets, used t-norm and interpretation of IF-THEN structure of a rule. Actually, there are two approaches known in the literature. It is the conjunctive approach and the implicative approach [1]. On the base of conjunctive one so called conjunctive FISs are defined and on base of the implicative one the implicative FISs (I-FISs) are defined.

In an implicative FIS antecedent and succedent are combined by a proper fuzzy implication given as the residuum of a t-norm used for and connective representation. A residuum $\to$ of a t-norm is generally given as (it is an operation from $[0,1]^2$ to [0,1], see [1, 2])

$$x \to y = \sup_{z}\{x \star z \leq y\}. \qquad \text{(29)span}$$

For the three basic norms it is given as $x \to y$=1 iff $x \leq y$ (this is a general property of all residua) and for $x > y$ as

- Lukasiewicz t-norm: $x \to y = 1 - x + y$
- product t-norm: $x \to y = y/x$
- minimum t-norm: $x \to y = y$

Hence in an I-FIS a particular rule representation is given as

$$R_j(\boldsymbol{x}, y) = A_j(\boldsymbol{x}) \to B_j(y), \tag{30}$$

$$R_j(\boldsymbol{x}, y) = (A_{j1}(x_1) \star \cdots \star A_{jn}(x_n)) \to B_j(y). \tag{31}$$

On base of this representation particular rules are combined to compound relation giving a fuzzy relation representing the whole rule base. The combination is for implicative FIS given by the t-norm representing a fuzzy intersection. The t-norm is the same as used for and connective representation. Thus we have

$$RB(\boldsymbol{x}, y) = \bigcap_{j=1}^{m} R_j(\boldsymbol{x}, y) = R_1(\boldsymbol{x}, y) \star \cdots \star R_n(\boldsymbol{x}, y). \qquad \text{(32)span}$$

Now we can state the definition of radial I-FIS. A FIS is radial implicative one if it is radial according to definition 1 and it has the implicative representation of rule base.

3. Computation of radial I-FIS

A computation of standard FIS is given by the compositional rule of inference [1]. On base of this rule output (fuzzy set B') of inference engine, given as a response on input $\boldsymbol{x}^*$, has the form

$$B'(y) = \sup_{\boldsymbol{x}}\{A'_{\boldsymbol{x}^*}(\boldsymbol{x}) \star RB(\boldsymbol{x}, y)\}, \qquad \text{(33)span}$$

where $A'_{\boldsymbol{x}^*}(\boldsymbol{x})$ is the fuzzy set given by a fuzzifier as a response on the crisp input $\boldsymbol{x}^* \in \mathcal{R}$ and $\star$ is the t-norm used for and connective representation. Using singleton fuzzifier, which is the most common choice in practice, transforming a crisp input on fuzzy singleton, i.e.,

$$fuzz(\boldsymbol{x}^*) = A'_{\boldsymbol{x}^*}(\boldsymbol{x}) = \begin{cases} 1 & \text{for } \boldsymbol{x} = \boldsymbol{x}^* \\ 0 & \text{for } \boldsymbol{x} \neq \boldsymbol{x}^* \end{cases}, \qquad \text{(34)span}$$

and employing the fact that for any t-norm $0 \star x = x \star 0 = 0$ we have above CRI rule (33) in the simpler form of

$$B'(y) = RB(\boldsymbol{x}^*, y). \qquad \text{(35)span}$$

Further considering the computation of CRI rule in a more restrictive form, so called Δ-CRI rule, we have above

$$B'(y) = \Delta(RB(\boldsymbol{x}^*, y)), \tag{36}$$

where Δ is an operation from [0,1] to $\{0,1\}$ defined as

$$\Delta(x) = \begin{cases} 1 & \text{for } x = 1, \\ 0 & \text{for } x \in [0,1). \end{cases} \tag{37}$$

Since for any t-norm it is $x \star y = 1$ if and only if $x = 1$ and $y = 1$ we can (36) write as

$$B'(y) = \Delta(R_1(\boldsymbol{x}^*, y)) \star \cdots \star \Delta(R_n(\boldsymbol{x}^*, y)). \tag{38}$$

So $B'(y) = 1$ if and only if $R_j(\boldsymbol{x}^*, y) = 1$ simultaneously for all $j = 1, \ldots, m$. Otherwise $B'(y) = 0$. Hence B' is a crisp set.

Having particular rule represented in an implicative way it is $R_j(\boldsymbol{x}^*, y) = 1$ if and only if it is $A_j(\boldsymbol{x}^*) \leq B_j(y)$, which is given by the properties of residua. Considering an implicative FIS to be radial we can state these inequalities.

$$\begin{aligned} d_j \cdot ||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} &\geq |y - c_j|, && (39)\\ ||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} &\geq |y - c_j|/d_j, && (40)\\ g(||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j}) &\geq g(|y - c_j|/d_j), && (41)\\ act(g(||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j})) &\leq act(g(|y - c_j|/d_j)), && (42)\\ A_j(\boldsymbol{x}^*) &\leq B_j(y). && (43) \end{aligned}$$

Hence the set of those y satisfying $A_j(\boldsymbol{x}^*) \leq B_j(y)$contains at least the closed interval

$$I_j = [c_j - d_j \cdot ||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j} \ , \ c_j + d_j \cdot ||\boldsymbol{x}^* - \boldsymbol{a}_j||_{\boldsymbol{b}_j}]. \tag{44}$$

Considering by definition interval I_j such a set of those y for which $R_j(\boldsymbol{x}^*, y) = 1$ we see that the set of y for which $B'(y) = 1$ is given by the intersection of I_j for $j = 1, \ldots, m$. Formally, we can write computation of an inference engine using Δ-CRI rule in the case of radial I-FIS as

$$B' = \bigcap_{j=1}^{m} I_j. \tag{45}$$

Since an intersection of intervals is an interval as well (we consider that it is non empty) it is straithforward to consider as the final deffuzified output of a radial I-FIS the middle point of B', i.e.,

$$y^* = \frac{L(I_{B'}) + R(I_{B'})}{2}, \tag{46}$$

where $L(I_{B'})$, $R(I_{B'})$ are the left or the right limit point of B', respectively.

Having stated the computation of a radial I-FIS we can investigate some of its properties such as coherence and universal approximation property. In the following sections we state important theorems regarding these properties without proofs.

4. Coherence

The question of coherence is the question if for any input is the output of radial I-FIS always defined. That is, for any $\boldsymbol{x}^* \in \mathcal{R}^n$ it is $\bigcap I_j \neq \emptyset$. The answer on this question is given by the following theorem.

Theorem 4.1 *Let be w_{kl} for $k, l = 1, \ldots, m$ given as*

$$w_{kl} = \begin{cases} \dfrac{|c_k - c_l|}{||\boldsymbol{a}_k - \boldsymbol{a}_l||_E} & \text{for } \ k \neq l \\ 0 & \text{for } \ k = l \end{cases}, \tag{47}$$

Then radial $\mathrm{I-FIS}$ *is coherent if and only if for all elements* w_{kl} *it is*

$$w_{kl} \leq \min\{d_k\alpha_k, d_l\alpha_l\}, \qquad (48)\text{span}$$

where α_j, $j = 1, \ldots, m$, *are positive numbers such that*

$$\alpha_j \cdot ||\boldsymbol{u}||_E \leq ||\boldsymbol{u}||_{\boldsymbol{b}_j} \qquad (49)\text{span}$$

for all $\boldsymbol{u} \in \mathcal{R}^n$.

5. Universal approximation property

The universal approximation property is an important property which justifies the employment of radial I-FISs as controllers or other approximation tools. We are able to prove this property in the following form.

A system of functions $\mathcal{G}$ defined on hypercube $H = [p_1, q_1] \times \cdots \times [p_n, q_n]$, i.e., $H \subseteq R^n$, exhibits the universal approximation property if for given $\varepsilon > 0$ and any continuous function $f : H \rightarrow \mathcal{R}$ there exists a function $g \in \mathcal{G}$ such that for all $\boldsymbol{x} \in H$ it is

$$|f(\boldsymbol{x}) - g(\boldsymbol{x})| < \varepsilon. \qquad (50)\text{span}$$

Theorem 5.1 *Let* $\mathcal{G}$ *be the system of functions given by the computation of all coherent radial* $\mathrm{I-FIS}$s *defined on given hypercube* H, $H \subseteq \mathcal{R}^n$. *Then* $\mathcal{G}$ *has the universal approximation property in the sense of definition 5.*

6. Conclusion

It was defined the class of fuzzy inference systems which was the class of radial implicative fuzzy inference systems. It was shown that important t-norms can be combined with important shapes of fuzzy sets to obtain radial I-FISs. It was shown that the computation of radial implicative FISs is given by intersection of intervals and there was presented sufficient and necessary condition to radial I-FIS be coherent. Moreover, the class of radial implicative fuzzy inference systems exhibits the universal approximation property.

References

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[2] Hájek P., Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, 1998