Compactness of Various Fuzzy Logics

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Abstrakt

Compactness is an important property of classical logic. It states that simultaneous satisfiability of an infinite set of formulas is equivalent to the satisfiability of all its finite subsets. In fuzzy logics, we have different degrees of satisfiability, hence the questions of compactness become more complicated. Here we give an overview of recent results on compactness and we extend them to various fuzzy logics.

This paper is a joint work with Mirko Navara from the Czech Technical University (cf. [6]).

1. Introduction

Dealing with vague or uncertain information, we often replace the two classical truth degrees $0, 1$ by a continuous scale $[0, 1]$. Then we have also various possibilities how to define the interpretation of the basic logical connectives. Depending on this choice, we obtain various fuzzy logics. We refer to [2, 9, 10] for a detailed description of the most frequent approaches.

In classical logic, a set of formulas is satisfiable if there is an evaluation which evaluates them all by 1. The compactness theorem holds for classical logic: A set of formulas is satisfiable if and only if all its finite subsets are satisfiable. It is natural to ask in which fuzzy logics analogues of this theorem hold. As the first step, we have to generalize the notion of satisfiability. We may again require all formulas to be evaluated by 1, but this is not the only possibility. Sometimes other alternatives are well-motivated, hence, following [2], we work with K-satisfiability, where K can be an arbitrary subset of $[0, 1]$. We say that a set of formulas is K-satisfiable if there is an evaluation which evaluates them all by values in K. (In particular, we get the former case if we choose $K = \{1\}$.) Using K-satisfiability, we may formulate various types of compactness. Here we present new results about validity of compactness in the most frequently used fuzzy propositional logics.

In particular, we prove that product and Gödel logic do not satisfy the compactness property, but we present also partial positive results. Then we extend these observations to logics with Baaz $\triangle$ operator, logics with involutive negations, and LΠ logics which form a common extension of Lukasiewicz and product logic. We prove that most of these stronger logics do not satisfy the compactness property and even more—they are not K-compact for almost all forms of the set K.

2. Basic fuzzy logical operations

Following [9, 10], we deal here with logics which have the real interval $[0, 1]$ as the set of truth values and the following basic connectives:

- nullary false statement **0**, interpreted by 0,
- binary conjunction $\wedge$, interpreted by a *t-norm* $T\colon [0,1]^2 \to [0,1]$, i.e., a commutative, associative, non-decreasing operation with a neutral element 1,
- binary implication $\to$, interpreted by the *residuum* R of T, i.e.,

$$R(x, y) = \sup \{z \in [0, 1] : T(x, z) \le y\} .$$

We start from a nonempty countable set A of *atomic symbols* (*atoms*) and we define the class of *well-formed formulas* in a fuzzy logic (*formulas* for short) in the standard way. Each function which assigns truth values to atomic formulas is uniquely extended (using the interpretations of connectives) to an *evaluation* $e\colon \mathcal{F}_{\mathcal{P}} \to [0, 1]$.

In this approach, the semantics of the logic is fully determined by the choice of the t-norm T. The three basic triangular norms lead to the following three main examples of fuzzy logics:

- For the minimum $T_{\mathbf{G}}(x, y) = \min(x, y)$ we obtain *Gödel logic* G.
- For the triangular norm $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ we obtain *Lukasiewicz logic* L.
- For the algebraic product $T_{\mathbf{P}}(x, y) = x \cdot y$ we obtain *product logic* Π.

The respective residua in these logics are:

$$R_{\mathbf{G}}(x, y) = \begin{cases} 1 & \text{if } x \le y \ , \\ y & \text{otherwise} \ , \end{cases}$$

$$R_{\mathbf{L}}(x, y) = \begin{cases} 1 & \text{if } x \le y \ , \\ 1 - x + y & \text{otherwise} \ . \end{cases}$$

$$R_{\mathbf{P}}(x, y) = \begin{cases} 1 & \text{if } x \le y \ , \\ \dfrac{y}{x} & \text{otherwise} \ . \end{cases}$$

The residuum $R_{\mathbf{L}}$ is continuous, $R_{\mathbf{G}}$ is not continuous in the points (x, x), $0 \le x < 1$, and $R_{\mathbf{P}}$ has a discontinuity in $(0, 0)$.

Using the basic logical connectives $\wedge, \to$ and **0**, we can define derived logical connectives. Negation $\neg$ is defined as

$$\neg\varphi = \varphi \to \mathbf{0} \ .$$

Its interpretation is the fuzzy negation N given by

$$N(x) = R(x, 0) = \sup \{z \in [0, 1] : T(x, z) = 0\} \ .$$

In Lukasiewicz logic this leads to *standard negation* $N_{\mathbf{S}}(x) = 1 - x$, in Gödel and product logic we obtain *Gödel negation*

$$N_{\mathbf{G}}(x) = \begin{cases} 1 & \text{if } x = 0 \ , \\ 0 & \text{if } x > 0 \ . \end{cases}$$

For additional information on these logics, we refer to [9, 10]. Their detailed study and the proofs of completeness can be found in [10, 11].

Furthermore we examine the compactness property for logics with the set of connectives extended by an additional unary connective $\triangle$ (0-1 *projector* or *Baaz delta*) with interpretation Δ defined by

$$\Delta(x) = \begin{cases} 1 & \text{if} \quad x = 1 \ , \\ 0 & \text{if} \quad x < 1 \ . \end{cases}$$

Taking the Gödel, Lukasiewicz, or product t-norm as the interpretation of conjunction, we obtain Gödel, Lukasiewicz, or product logic with $\triangle$ denoted by $\mathrm{G}_\triangle$, $\mathrm{L}_\triangle$, $\Pi_\triangle$, respectively.

Then we examine the compactness property for Gödel and product logics with the set of connectives extended by an additional unary connective $\sim$ (*involutive negation*) interpreted so that $e(\sim\varphi) = 1 - e(\varphi)$. We call these logics Gödel involutive and product involutive logics. We will denote them by $\mathrm{G}_\sim$ and $\Pi_\sim$. They were introduced by Esteva, Godo, Hájek and Navara in [7]. (For Lukasiewicz logic, this notion does not bring anything new as it already contains an involutive negation.)

Then we investigate the compactness of $\mathrm{L}\Pi$ and $\mathrm{L}\Pi\frac{1}{2}$ propositional logics. These logics were introduced by Esteva, Godo, and Montagna in [8]. Then they were studied mainly in [4] and [5]. They unify Lukasiewicz and product logics (and many others) and have many interesting properties (such as completeness), but we shall see that they lack the compactness property and are not K-compact for an almost arbitrary set K.

The $\mathrm{L}\Pi$ logic has three basic connectives: Lukasiewicz implication and product conjunction and product implication. Furthermore, it has all other connectives of product, Gödel and Lukasiewicz logics, including the $\triangle$ connective.

$\mathrm{L}\Pi$ *logic* has the same standard semantics as product involutive logic. And the $\mathrm{L}\Pi\frac{1}{2}$ *logic* is $\mathrm{L}\Pi$ logic with an additional nullary connective **c** (a *non-trivial constant statement*) interpreted by an arbitrary element c from the open interval $(0, 1)$.

3. Satisfiability and compactness property

We present an analogue of the notion of satisfiability in fuzzy logics. It is natural to admit various degrees of simultaneous satisfiability of a set of formulas.

Definition 1 *For a set Γ of formulas in a fuzzy logic and $K \subseteq [0, 1]$, we say that Γ is K-*satisfiable *if there exists an evaluation e such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$. The set Γ is said to be* finitely K-satisfiable *if each finite subset of Γ is K-satisfiable. Formula φ is called K-satisfiable if the set $\{\varphi\}$ is K-satisfiable.*

K-satisfiability obviously implies finite K-satisfiability. The reverse implication holds in classical logic, as well as in some fuzzy logics. This property is called compactness of a logic.

Definition 2 *We say that a logic is K-*compact *if K-satisfiability is equivalent to finite K-satisfiability. A logic satisfies the* compactness property *if it is K-compact for each* closed *subset K of $[0, 1]$.*

In the latter definition, it is necessary to consider only sets K which are closed (hence also compact), as we shall show in the next section. The following two observations will simplify our study:

At first observe that if we extend the set of connectives of a logic, L, then the resulting logic, L', becomes "less compact", i.e., if L' is K-compact (resp. has the compactness property) then L is K-compact (resp. has the compactness property). This is obvious since each counterexample to K-compactness in L is also a counterexample to K-compactness in L'.

Then observe that if $\{0, 1\} \subseteq K$, then for any formula φ there is an evaluation e such that $e(\varphi) \in K$ (it suffices to evaluate all atomic symbols by 0, then the evaluation of each each formula becomes either 0 or 1). Thus we will restrict ourselves to sets K not containing 0 and 1 simultaneously. We will also observe that in various logics K-compactness depends on presence 1 in K. Therefore the following classes of subsets of $[0, 1]$ will be important in the study of satisfiability and compactness:

Definition 3 *A nonempty subset K of $[0, 1]$ is of type C if $0 \notin K$ or $1 \notin K$. Furthermore, if K is of type C we define other type C_1 if $1 \in K$.*

4. The results

The following tables summarize our results. The first column indicates the types of logics, the second shows whether the logic enjoys the compactness property. The third column deals with K-compactness property for the sets of type C and the last one deals with the remaining sets. The possible elements of these tables are

All	logic is K-compact for all sets of this type
None	logic is K-compact for no set of this type
C_1	logic is K-compact at least for all sets of type C_1
Compact	logic is K-compact exactly for all compact subsets
Dense	logic is K-compact at most for those sets with a dense subset
Q	logic is K-compact at most for those sets containing all rationals

We also present a diagram of logics studied in this paper ordered by the richness of their sets of connectives. Notice that the results depends on the cardinality of the set of atoms for G, $G_\triangle$, $G_\sim$ logics.

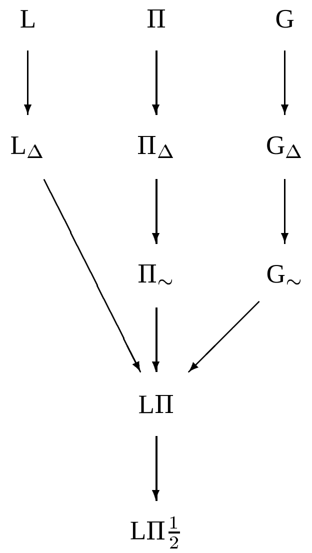
Tabulka 3: K-compactness for finite and infinite sets of atoms

K-compactness for finite sets of atoms

Logic	Compactness	C	non C
Ł	Yes	Compact	All
Π	No	C_1	All
G, $G_\triangle$, $G_\sim$	Yes	All	All
$Ł_\triangle$, $\Pi_\triangle$, $\Pi_\sim$, ŁΠ	No	None	All
ŁΠ$\frac{1}{2}$	No	None	Q

K-compactness for infinite sets of atoms

Logic	Compactness	C	non C
Ł	Yes	Compact	All
Π, G	No	C_1	All
$G_\triangle$, $G_\sim$	No	Dense	All
$Ł_\triangle$, $\Pi_\triangle$, $\Pi_\sim$, ŁΠ	No	None	All
ŁΠ$\frac{1}{2}$	No	None	Q



5. Conclusion

We studied various types of fuzzy logics—Łukasiewicz, Gödel, and product logic, logics with $\triangle$, logics with involutive negation, ŁΠ logic and ŁΠ$\frac{1}{2}$ logic. We have found out that the analogue of classical compactness property holds for Łukasiewicz logic. In some other logics at least partial positive results were obtained. In general, enriching the set of connectives we weaken the compactness of the logic. The ŁΠ$\frac{1}{2}$ logic represents the extreme case as it does not satisfy K-compactness for an almost arbitrary subset $K \subseteq [0,1]$.

There are several open questions. Is Gödel or product logic K-compact for some sets K of type C which are not of type C_1? (The answer is positive for the set $\{0\}$.) Is Gödel logic with $\triangle$ or Gödel involutive logic K-compact for all sets K containing a dense subset? Is ŁΠ$\frac{1}{2}$ logic K-compact for all sets K containing all rationals from $[0,1]$? How many atoms are really needed to prove our theorems? (We know that all results presented here can be proven with three atoms, some properties with two atoms, and sometimes only one atom is sufficient.)

On the other hand, these are the only questions unanswered in our paper—for all the other combinations of a logic and a subset $K \subseteq [0,1]$ the problem of K-compactness has been solved here.

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