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Abstract:

Analogy is used to treat the system of non-interacting integrate-and-fire neurons as the ideal Fermi gas. It allows one to obtain the nonlinear gain curve in the form of sigmoid in agreement with biological findings. As the byproduct the gain-threshold mechanism in neurons is presented. Surprisingly enough, this is in agreement with new biological findings, too. Besides, the application of this mechanism to the dynamics of neurons leads to the non-monotone transfer function.

Keywords:

Nonlinear gain curve, gain-threshold dependence, non-monotone transfer function

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1 Introduction

There have been intensive attempts to study and understand more deeply some basic problems in neuronal systems, during the last years. But one can say a progress seems to be slow and not adequate to intensity of our interests put in. So far we have not been able to formulate rigorously some general universal laws like, e.g., in physics also in the field of neuroscience. On the other hand the book of Nature is written in the language of mathematics, as Galileo said. It follows one needs to apply some general ideas on basic problems in this field, too. But to do this the method of analogies seems to be very efficient. In words of great mathematician S. Banach: The good mathematicians see analogies between theorems or theories; the very best see analogies between analogies. (S. Ulam in the Foreword to Gamov's autobiography.)

In the paper we follow this strategy in applying a general and very powerful method of statistical mechanics, especially the Boltzmann treatment of entropy, to the system of non-interacting neurons. Then the analogy between integrate-and-fire neurons as fermions will be exploited in a most general mathematical sense. It corresponds to the rule of "correct Boltzmann counting" and does not correspond to any physical, resp. biological properties of particles resp. units in the system. It is just a rule that defines the mathematical model [1]. But as we shall see the output of the game will be mostly rewarding. It allows one to obtain the nonlinear gain curve in the form of sigmoid in very agreement with biological findings [2]. Even more, as a by product we get the gain-threshold functional dependency as the novel quality of biological neurons. Surprisingly enough this is in agreement with new biological findings, too [3].

2 System of non-interacting neurons as an analogy to the ideal Fermi gas

Let us have N free neurons in a sense we do not suppose any learning so far. Then for a case of integrate-and-fire types of neurons one can proceed in a standard way. If $x_j, j = 1, \dots, N$ are inputs (activities) to the i -th neuron and Θ_i is the bias or threshold of the i -th neuron, one can introduce the effective field h_i in the form

$$h_i = \sum_j w_{ij} x_j \quad (2.1)$$

and the local field ξ_i in the form

$$\xi_i = h_i - \Theta_i \quad (2.2)$$

where $w_{ij}, i, j = 1, \dots, N$ are called weights between the j -th and i -th neurons. Now in analogy with Hopfield [4] one can write down the Hopfield energy-like function

$$H = -\frac{1}{2} \sum_{i,j \neq i} w_{ij} x_j y_i - \sum_i \Theta_i y_i \quad (2.3)$$

where $y_i, i = 1, \dots, N$ is the output of i -th neuron. But in our case as the neurons can only fire or to be quiet we have $y_i = \{0, 1\}$. By other words neurons can be only in two states, active then $y_i = 1$, or quiet and then $y_i = 0$.

Now one can look on such a system of free neurons as on the ideal neuronal gas in the equilibrium. With the analogy of statistical mechanics a state of such system is characterized by a set of occupation numbers $\{n_i\}$, where n_i is the number of neurons firing with the given value of ξ_i in the state under consideration. If we put $n_i = \langle y_i \rangle$, i.e., the average of y_i through the system we have for a total energy-like quantity E

$$E = \sum_i n_i \epsilon_i \quad (2.4)$$

and the total numbers of neurons is

$$N = \sum_i n_i \quad (2.5)$$

where in the analogy to Hopfield

$$\epsilon_i = -\xi_i \quad (2.6)$$

But for integrate-and-fire neurons one can have only $n_i = \{0, 1\}$ in analogy with Fermi ideal gas of statistical mechanics. Then also Pauli exclusive principle holds and the number of states corresponding to $\{n_i\}$ is $g\{n_i\} = 1$. As we mentioned before every neuron can fire or be quiet.

At this point one can proceed in two different but equivalent ways. The first is to exploit the "correct Boltzmann counting" [1], which is just the rule that defines the mathematical model, so also well applicable to our neuronal system. By calculating the Boltzmann entropy and maximizing the number of states of our system $\{\bar{n}_i\}$, corresponding to the set of occupational numbers $\{n_i\}$ for the case of Fermi-like gas, one gets directly the formula for \bar{n}_i . This is naturally done under conditions (2.4) and (2.5). The second way is to use the grand partition function to calculate the mean value of n_i for a Fermi-like neuronal gas. As in what follows we will need the partition function we exploit the second technique of handling this matter here.

The grand partition function for the Fermi-like gas can be written in the form [1].

$$\Xi = \sum_{N=0}^{\infty} z^N \sum_{\{n_i\}} e^{-\beta \sum_{i=1}^N \epsilon_i n_i} \quad (2.7)$$

where conditions (2.4) and (2.5) hold and $\{n_i\} = \{0, 1\}$. Here

$$z = e^{\beta\mu} \quad (2.8)$$

is so called fugacity and μ is the chemical potential. Besides for the ideal gas $\beta = \frac{1}{kT}$, where k is Boltzmann constant and T is an absolute temperature. We will specify the analogical meaning of these parameters for our neuronal gas later on. Now, the summation of N can be done as summation of n_i independently, so (2.7) can be written in the form

$$\Xi = \prod_i \sum_{n_i} [ze^{-\beta\epsilon_i}]^{n_i} = \prod_i (1 + ze^{-\beta\epsilon_i}) \quad (2.9)$$

So we are prepared to get a first interesting result. Under above conditions and along lines of our analogy we have for a mean value of occupation numbers, or the number of firing neurons [1]

$$\bar{n}_i = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_i} \ln \Xi = \frac{ze^{-\beta\epsilon_i}}{1 + ze^{-\beta\epsilon_i}} = \frac{1}{1 + e^{+\beta(\epsilon_i - \mu)}} \quad (2.10)$$

By a comparison with the experimental biological findings

$$\langle y_i \rangle = \frac{1}{1 + e^{-\lambda\xi_i}} = \frac{1}{1 + e^{-\lambda(h_i - \Theta_i)}} \quad (2.11)$$

we find the analogical meaning of parameters as follows

$$\begin{aligned} \lambda &= \beta \\ h_i &= -\epsilon_i \\ \Theta_i &= -\mu \end{aligned} \quad (2.12)$$

With this in mind one can look on (2.10) as on sigmoidal transfer function.

3 The equation of state and the gain-threshold dependency derivation for neurons

To proceed further one can use the equation of state for the Fermi-like neuronal gas [1]

$$\frac{pV}{kT} = \ln \Xi = \sum_i \ln(1 + ze^{-\beta\epsilon_i}) \quad (3.1)$$

where variables p, V, T have a meaning of analogy to their thermodynamic-like interpretation. It will be explain later more explicately. Besides (3.1) one has the self-consistency condition

$$N = y \frac{\partial}{\partial z} \ln \Xi = \sum_i \frac{ze^{-\beta\epsilon_i}}{1 + ze^{-\beta\epsilon_i}} \quad (3.2)$$

By introducing a specific volume per neuron

$$v = \frac{V}{N} \quad (3.3)$$

and using techniques of statistical mechanics ([1]) we get from (3.1) and (3.2)

$$\frac{p}{kT} = \frac{1}{L^3} f_{5/2}(z) \quad (3.4)$$

$$\frac{1}{v} = \frac{1}{L^3} f_{3/2}(z) \quad (3.5)$$

where L , the thermal length, is

$$L = \sqrt{\frac{2\pi\hbar^2}{mkT}} \quad (3.6)$$

and

$$f_{5/2}(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} z^j}{j^{5/2}} \quad (3.7)$$

$$f_{3/2}(z) = z \cdot \frac{\partial}{\partial z} f_{5/2}(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} z^j}{j^{3/2}} \quad (3.8)$$

Let us note that the absolute temperature T here is not factual temperature, e.g., in our brain but one can take it as an analogy of appropriate "noise" temperature which plays the important role in neuronal systems [5]. It will be specified later on.

In the case of very small z or by other words for the high temperature (noise) limit (see (2.8)) we have from (3.5) and (3.8) to the first order of z

$$\frac{L^3}{v} \cong z. \quad (3.9)$$

After a substitution of (2.8) and (3.6) and using the analogy (2.12) we get by a simple rearrangement

$$\lambda = \frac{v^{2/3}}{A} e^{-\frac{2}{3} \frac{\Theta}{kT}} \equiv k_1 e^{-k_2 \Theta} \quad (3.10)$$

where

$$k_1 = \frac{v^{2/3}}{A}; A = \text{const} \quad (3.11)$$

and

$$k_2 = \frac{2}{3\sqrt{(\Delta V_i)^2}} \quad (3.12)$$

where [5]

$\Delta V_i = V_i - \bar{V}_i$, V_i is action potential on the i -th neuron.

$$\bar{V}_i = \sum_j \bar{V}_{ij} = p \sum_j w_{ij} x_j(t - \tau_{ij}) \quad (3.13)$$

is a mean value of action potential;

τ_{ij} - is a time delay of refractory period for a synapse between i -th and j -th neurons. Commonly is taken $p \doteq 0.6$ [5]. In this case $\overline{(\Delta V_i)^2}$ then presents local synaptic noise.

Besides this natural analogy and the meaning of synapse noise temperature in the parameter k_2 in (24) the dependency of λ upon Θ in (22) is even of more interest here. Let us try to analyze this result in more details now. On the one hand the variables λ, Θ are main characteristics of any sigmoidal transfer function for neurons. Namely, the Θ is called the bias or threshold and shifts the sigmoid along the horizontal axis and the λ is the steepness of sigmoid and can play the important role in the stability vs. instability of neuronal dynamics. On the other hand these two cardinal parameters of sigmoidal transfer function have also a biological meaning. Namely, the Θ is connected to a sensitivity and can play the role in describing the attention and motivations [6]. And the λ , the sigmoid slope is connected to the excitability, e.g., it decreases under anesthesia and increases with arousal [2]. There have been many direct or indirect indications for a possible functional dependency of λ upon Θ [6], [7], [8]. One of the authors (L. A.) was even able to show by analyzing experimental results of Miyashita [9] and their modeling by Amit [8] that the functional dependence of gain (λ) upon threshold (Θ) is of exponential form. Our independent and rigorous derivation of that dependency of λ upon Θ here is of very support of that result.

It is fascinating to mention here that the similar functional dependency between gain (λ) and threshold (Θ) of sigmoidal transfer function was harvested as the by product by one of the authors (L. A.) in deriving the conditions for chaotic dynamics in single neurons [3].

It has been known for some time the sigmoid slope can play the important role in computational capabilities, e.g., in the pattern formation and recognition in the olfactory bulb [10], in the number of local minima in the system with fixed point attractors [4], and so on [11]. So it is only natural to await that taking into account the $\lambda - \Theta$ dependency, or the gain-threshold mechanism of (3.10) will also have the profound effect upon computational capabilities of such neurons. This was demonstrated by one of authors [12] building up neural networks of such neurons. A more detailed analysis of possible consequences of gain-threshold dependency (3.10) upon computational capabilities of neurons, e.g., a generation of bursts in firing neurons as an intermittent way to chaotic dynamics of neurons, will be published elsewhere [13].

But for a concreteness in the next section we give a short account of how the exploitation of gain-threshold dependency can lead immediately to getting the non-monotone transfer function dynamics in such neurons. Details of derivation and consequences will be published by one of the authors (L. A.) elsewhere.

4 The non-monotone transfer function from the gain-threshold mechanism

In attempts to increase associative memories and to improve their dynamics the artificial construction of more complicated transfer functions instead of sigmoidal one has been used. Namely the so called non-monotone transfer function for the case of associative memory was proposed in [14]. It is then argued by numerical experiments that many problems of the conventional model with the sigmoidal transfer function are overcome by this improved dynamics. As mentioned by the Morita [14] who was the first to introduce such dynamics, to study the associative memory with non-monotone dynamics is important not only for practical purposes but also for understanding dynamical properties of associative neural networks. There have been many papers published on dynamics of neural networks

with non monotone transfer function neurons. We are not going to technical details on this literature here. Instead we sketch a derivation of such non-monotone transfer function dynamics from our gain-threshold functional dependence (3.10).

Let us take (11) in the form

$$f(\xi) = \frac{1}{1 + e^{-\lambda\xi}} \quad (4.1)$$

for the transfer function of given neuron, when we dropped out the index i for the i -th neuron here. Now we apply the dependence $\lambda = \lambda(\Theta)$ of (3.10). Taking into account the expression for the local field ξ in the form (2.2) we have from (4.1)

$$f(\xi) = \frac{1}{1 + e^{-k_1\xi e^{-k_2(h-\xi)}}} \quad (4.2)$$

To proceed further one can apply repeatedly the technique of Taylor series and other elementary calculations (details will be published elsewhere) after some long and dubious procedures one arrives at the expression for the first approximation form of transfer function (4.1)

$$\begin{aligned} f(\xi) &= \frac{1}{2} + \frac{A}{4}\xi \left[1 + D\xi + \frac{k_2}{2}C\xi^2 \right] = \\ &= \frac{1}{2} + \frac{A}{4}C\xi \left(\xi + \frac{D}{2C} + \sqrt{\frac{D^2}{4C^2} - C} \right) \left(\xi + \frac{D}{2C} - \sqrt{\frac{D^2}{4C^2} - C} \right) \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} C &= (B + k_2 + Bk_2)\frac{k_2}{2} \\ D &= \frac{B}{2} + k_2 \\ A &= k_1^2 e^{-2k_2h} \\ B &= k_2^2 e^{-k_2h} \end{aligned} \quad (4.4)$$

Not going to technical details here we just mention the transfer function (4.2) contains the cubic nonlinearity which is typical for non-monotone transfer functions so far constructed. See, e.g. [15].

It is worth to stress here that we get the non-monotone transfer function (4.3) not by artificial, in a sense teleological construction, as has been done so far, but as a consequence of taking into account the gain-threshold mechanism we discovered here. It means the non-monotone transfer function for neurons can have its substantiation in the biology of neuronal systems as such. We can also deduce from (4.3) that the dynamics of such neurons can be rather complex. But this would shed a new light on the very problem of neural coding just to mention one of possible consequences of gain-threshold dependency (3.10). Well, here we again meet a fine point of exploiting that singular methodology of looking for analogies among analogies as was mentioned at the beginning.

5 Conclusions

The system of non-interacting integrate-and-fire neurons was treated as the analogy to the ideal Fermi gas. By this we have been able to obtain the nonlinear gain curve in the form of sigmoid, as one supposed from biology [10]. As at by product we surprisingly arrived at the gain-threshold dependence in the form of exponential relationship (3.10) between the steepness of such sigmoidal transfer function and its threshold. But this functional dependence has also the support in biology [7], [8]. Finally, the application of such gain-threshold mechanism leads backwards to the non-monotone transfer function as the first approximation of complicated neuronal dynamics.

Let us stress here that above mentioned results were obtained for the "ideal gas" approximation of non-interacting neurons. For such the case the neuronal system is in equilibrium. So this naturally means that such system cannot learn anything. But we know that in reality, neuronal systems as parts of brain, have this distinguished property that they can learn. So the question is if it would be not possible to proceed further in our previous analogies to bring the learning into the game. Well, it seems, the answer is affirmative. Here we only shortly mention the basic idea (a detailed treatment will be published elsewhere).

For the sake of simplicity one can take thresholds Θ_i as parameters with weights w_{ij} fixed to 1. Then we have the effective fields h_i in the form of (2.1). Now the point is that one can get a time dependent dynamics via the weights w_{ij} in the h_i . But the developing of weights w_{ij} in time is the so called neuronal plasticity or by other words we call it the learning. Now one can suppose that at each stage of such learning process, one calls these processes epoches, the system is allowed to come to equilibrium. And, of course, at each step the (3.10) is satisfied. We know, for this to hold, the "noise temperature" must be sufficiently high. But it means that interactions are negligible. However, in reality, one is taking into account instantenous values of outputs $y_i, i = 1, \dots, N$ after they came to equilibrium, in computing the $h_i \rightarrow$ the y_i . So, in a sense, one can speak about an application of analogy of mean field theory in the realization of such learning process. Beside, as we have time dependent effective fields $h_i, i = 1, \dots, N$ here, one can think about the possible application of time dependent Landau-Ginzburg theory, or of a kind of generalized Landau theory. This approach seems to be very promising as it leads to the more rigorous treatment of dynamics of learning process, and so on. But this is another story we will not go to details here but it will be published soon. To this end we have demonstrated that the celebrated method "analogies of analogies" can be very fruitful even in computational neuroscience.

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