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Abstract:

The validity/failure of various formulations of compactness, Löwenheim-Skolem theorem and Linström theorem in systems of fuzzy predicate calculus is studied.

Keywords:
fuzzy logic, model theory, compactness, Löwenheim-Skolem theorem, Lindström theorem

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1 Introduction

This note is inspired by my discussion with J. Väänänen and should serve for joint research. Recall the predicate logic $BL\forall$ (basic fuzzy predicate logic) and there stronger logics $Ł\forall, G\Pi, \Pi\forall$ (Łukasiewicz, Gödel and product logic). The corresponding varieties of algebras of truth functions are those of BL-algebras, MV-algebras, G-algebras and product algebras. One works with *linearly ordered* BL-algebras (etc.), i.e. BL-chains, MV-chains etc. The truth value of a formula φ is denoted by $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$, where $\mathbf{L}$ is a BL-algebra, $\mathbf{M}$ an $\mathbf{L}$-interpretation of the language and v is as evaluation of free variables of φ.[2] A theory T (over one of our logics) is *consistent* if $T \nvdash \bar{0}$; is *complete* if for each pair φ, ψ of formulas, $T \vdash \varphi \to \psi$ of $T \vdash \psi \to \varphi$; is *Henkin* if for each sentence $(\forall x)\varphi$ unprovable in T there is a constant c such that $T \nvdash \varphi(c)$. This is used for a (traditional) proof of completeness: if $\mathcal{C}$ is one of our logics, T a theory and φ a formula then the following are equivalent:

(i) $T \vdash_{\mathcal{C}} \varphi$

(ii) for each $\mathcal{C}$-chain $\mathbf{L}$ (i.e. BL-chain, MV-chain, G-chain and Π-chain respectively) and for each $\mathbf{L}$-model of T, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$. (An $\mathbf{L}$-model being a safe $\mathbf{L}$-interpretation in which all axioms of T have value 1.)

Standard algebras are algebras on $[0,1]$ given by continuous t-norms: $Ł, G, \Pi$ have a unique standard algebra each given by the respective t-norm; standard BL-algebras are just all algebras given by continuous t-norms. Note that $G\forall$ has standard completeness. ($T \vdash_{G\forall} \varphi$ iff φ is true in all $[0,1]_G$-models of T); the other logics not and their set of t-tautologies is not Σ_1 (details are known).

2 Witnessing constants

Theorem. For $\mathcal{C} = BL\forall, Ł\forall, \Pi\forall, G\forall$, let T be a theory over $\mathcal{C}$ such that $T \vdash_{\mathcal{C}} (\exists x \varphi(x)$; let $\hat{T} = T \cup \{\varphi(c)\}$ where c is a new constant. Then $\hat{T}$ is a conservative extension of T.

For a proof see [1] 5.4.17; that theorem is formulated for $Ł\forall$ but the proof works for any of our logics.

Theorem. For $\mathcal{C}$ being $Ł$ or Π, let T be any theory, a c new constant and $\hat{T} = T \cup \{(\exists x)\varphi(x) \to \varphi(c)\}$. Then $\hat{T}$ is a conservative extension T. For BL and G this does not hold.

Hint: Modify the proof of the previous theorem; then you have to use are implication of the form $(\nu \to (\exists x)\alpha) \to (\exists x)(\nu \to \alpha)$ (ν not containing x freely), which is a tautology of $Ł\forall$ and $\Pi\forall$, but not of $G\forall, BL\forall$. See [1] 5.4.31. The counterexample in 5.3.6 for $G\forall$ is easily transformed to a model over $[0,1]_G$ violating the above statement for $G\forall$ (and hence for BL).

Theorem. Let T be a theory over $Ł\forall$, $\hat{T} = T \cup \{\varphi(d) \to (\forall x)\varphi(x)\}$, where d is a new constant. Then $\hat{T}$ is a conservative extension of T. For G, Π, BL this does not hold.

Hint: As in the previous theorem (now needing $((\forall x)\alpha \to \nu) \to (\exists x)(\alpha \to \nu)$. To find counterexamples for $\Pi\forall$ and $G\forall$ use the examples from 5.4.31 and again from 5.3.6.

3 Compactness

Consider the statement: *Let T be a theory. If each finite subtheory has a model then T has a model.* Call it First compactness theorem. Below saying "a model" we mean "an $\mathbf{L}$-model for an algebra $\mathbf{L}$" (BL-algebra, MV-algebra,...)

[2] I also mention $RPL\forall$-rational Pavelka logic, i.e. extension of $Ł\forall$ by constants for rational truth values. This system (or a variant of it) has been extensively studied by Vilém Novák et al., who is developing some model theory of it, including Herbrand theorem, see [2]

Theorem. The first compactness theorem is true for all logics $BL\forall, Ł\forall, G\forall, \Pi\forall$.

Hint: This is because having a model is equivalent to being consistent (by [1] 5.2.8, 5.2.9, proving even more).

First *standard* compactness theorem is as above, with "standard model" instead of "model" (i.e. model over a standard algebra of truth values.)

Theorem. The first standard compactness theorem holds for $Ł\forall, G\forall$ but not for $\Pi\forall, BL\forall$.

Proof: The positive part follows from the fact that each consistent theory over $Ł\forall$ or $G\forall$ has a standard model over this logic – see [1] 5.3.1 for $G\forall$ and 5.4.24 for $Ł\forall$.

To show that the first standard compactness theorem fails for $\Pi\forall$, consider the theory T with one binary predicate $\prec$ and one unary predicate P. The axioms say:

$\prec$ is a crisp linear order without a largest element
(crispness expressed by $(\forall x, y)(x \prec y \vee \neg(x \prec y))$),
$\neg(\forall x)P(x), \neg(\exists x)\neg P(x)$ (this says that all values
of $P(x)$ are positive and their infimum is 0),
$x \prec y \to (P(x) \to P^n(y))$ for $n = 1, 2 \ldots$
($P^n(y)$ is $P(y)\& \ldots \& P(y)$ – n times)

Clearly this theory has no standard model: assume $\mathbf{M}$ is such a model, and take $a, b \in M$ with $\|b \prec a\| = 1$ and $0 < \|P(a)\| < \|P(b)\| < 1$. Then take an n such that $\|P(b)\|^n < \|P(a)\|$ – we see that for this n the axiom schema is not 1-true.

But T does have a non-standard model over a product chain given by a non-archimedean group. Let G be the group whose elements are infinite sequences of positive rationals equal to 1 for all but finitely many members with coordinatewise multiplication and inverse lexicographic order ($\{a_n\}_n < \{b_n\}_n$ iff for the last n where they differ, $a_n < b_n$).[3] The unit $\{1\}$ has all element equal 1; G_- consists of $\{a_n\}_n$ whose last element differing from 1 is less than 1. Let $\mathbf{A}$ be the product chain with the field $G_- \cup \{0\}$ where 0 is the least element and the zero element. Now let M be the set of negative integers, let $<$ be its standard order and for $-m \in M$ (m positive integer) let $\|P(-m)\| = (1^{m-1}, \frac{1}{2}, 1^\infty)$ (the m-th element is $\frac{1}{2}$, all other are 1). Now if $-m_1 < -m_2$, thus $m_1 > m_2$ and each $n > 0$, $(1^{m_1-1}, \frac{1}{2}, 1^\infty) < (1^{m_2-1}, \frac{1}{2^n}, 1^\infty)$, thus $\|P(-m_1) \to P^n(-m_2)\| = 1$. And for each $g \in G_-$ there is an m such that $(1^{m-1}, \frac{1}{2}, 1^\infty) < g$, thus in $\mathbf{A}$, $\inf \|P(-m)\| = 0$. □

The *second compactness theorem says:* If φ is true in all models of T then for some finite $T_0 \subseteq T$, φ is true in all models of T_0. Analogously, we formulate the *second standard compactness theorem.*

Theorem. The second compactness theorem is true for all our logics. The second standard compactness theorem is true for $G\forall$, fails for $Ł\forall$ as well as for $\Pi\forall$.

Proof: The former statement is obvious by (general) completeness theorem. Validity of the standard version for $G\forall$ follows from standard completeness of $G\forall$; for the failure of the standard version for $Ł\forall$ cf. [1] 3.2.14. (This example is for propositional logic but is trivially made to an example in predicate logic understanding each propositional variable p as an atomic formula $P(c)$, p unary, c a constant.)

For Π we may reproduce that example using the interpretation of Łukasiewicz logic in product logic (cf. [1] 4.1.14–4.1.18). Let us give the definitions: b, p, q are propositional variables; $\neg_b\varphi$ is $\varphi \to b$, $\varphi \veebar_b \psi$ is $\neg_b(\neg_b\varphi \& \neg_b\psi)$. Axioms of T are:

$$\neg\neg b, b \to p, \neg_b p \to q, n_b p \to q,$$

where $1_b p$ is p, $(n+1)_b p$ is $n_b p \veebar_b p$. Similarly as in [1] 3.2.14, q is true in each model of T but for each finite $T_0 \subseteq T$ you find a model of T in which the value of q is less than 1. □

[3] The algebraist would say that this is just an example of a lineary ordered abelian group with infinitely many convex subgroups.

4 Löwenheim–Skolem

Löwenheim-Skolem theorem for classical logic says that if a theory has a model then it has an (at most) countable model (assumed is that the language is at most countable; we assume this throughout.) For our logic we have the following:

Theorem. Let Γ be BL or its schematic extension ($Ł, \Pi, G, \ldots$), let T be a theory. If T has a model over a Γ-algebra $\mathbf{L}$ then it has countable model over an at most countable Γ-chain $\mathbf{L}'$.

Proof: It follows from the fact that T has a model over the chain $L_{\hat{T}}$ when $\hat{T}$ is a complete Henkin extension of T and the domain of the model is formed by countably many constants of the language of $\hat{T}$. $\square$

If T has a model over an algebra $\mathbf{L}$ (uncountable), can we conclude that it has a countable model over the same algebra $\mathbf{L}$? We give a partial answer.

Theorem. Let T have a model $\mathbf{M}$ over a BL-chain $\mathbf{L}$ and assume that $\mathbf{L}$ has a countable dense subset. Then T has a countable model over $\mathbf{L}$.

Proof: First observe that we may assume that the set of values $\|\varphi\|_{\mathbf{M}}$ of all sentences is dense in $\mathbf{L}$. (If it is not, expand the language by a unary predicate P and constants d_i; and expand $\mathbf{M}$ choosing $(d_i)_{\mathbf{M}} \in M$ arbitrarily (one–one) and letting $\|P(d_i)\|_{\mathbf{M}}$ run over the dense subset.) We shall further expand $\mathbf{M}$ by adding "witnessing constants". Put $\mathbf{M}_0 = \mathbf{M}$; given $\mathbf{M}_n$, in the language L_n extending the language of $\mathbf{M}_0$ by countably many constants, add a new set of countably many constants $c_{n,0}, c_{n,1}, \ldots$ and for each sentence $(\forall x)\chi(x)$ of the language L_n with $\|(\forall x)\chi(x)\|_{\mathbf{M}_n} < 1$ find $a \in M$ with $\|\chi(a)\|_{\mathbf{M}_n} = 1$ and interpret one of the new constants by a. $\hat{\mathbf{M}}$ is the union of all the $\mathbf{M}_n$'s – an expansion of $\mathbf{M}$ by countably many constants. Now put $\hat{T} = Th(\hat{\mathbf{M}})$; $\hat{T}$ is complete and Henkin (since $\mathbf{L}$ is a chain, for each pair φ, ψ of sentences either $\|\varphi\|_{\hat{\mathbf{M}}} \leq \|\psi\|_{\hat{\mathbf{M}}}$ or $\|\psi\|_{\hat{\mathbf{M}}} \leq \|\varphi\|_{\hat{\mathbf{M}}}$ – this gives completeness).

Now let M_ω be the set of interpretations of all constants. It is a countable subset of M and the restriction of $\hat{\mathbf{M}}$ to M_ω is the derived model $\mathbf{M}_\omega$. To see this show by induction on the complexity of a sentence φ that $\|\varphi\|_{\mathbf{M}_\omega} = \|\varphi\|_{\hat{\mathbf{M}}}$, for the induction step for quantifier modifying [1] 5.2.6(2). We show $\|(\forall x)\varphi(x)\|_{\hat{\mathbf{M}}} = \inf_c \|\varphi(c)\|_{\hat{\mathbf{M}}} = \inf_c \|\varphi(c)\|_{\mathbf{M}_\omega}$.

Clearly, $\|(\forall x)\varphi(x)\|_{\hat{\mathbf{M}}} \leq \|\varphi(c)\|_{\hat{\mathbf{M}}}$ for each c. Assume it is not the greatest lower bound, i.e. for some $u \in L$, $\|(\forall x)\varphi(x)\|_{\hat{\mathbf{M}}} < u < \|\varphi(c)\|_{\hat{\mathbf{M}}}$ for all c. The element u may be taken from the dense subset, i.e. we have a sentence χ with $\|\chi\|_{\hat{\mathbf{M}}} = u$ and $\|\chi \to (\forall x)\varphi(x)\|_{\hat{\mathbf{M}}} < 1$. Hence $\|(\forall x)(\chi \to \varphi(x)\|_{\hat{\mathbf{M}}} < 1$ and consequently for some c_0, $\|\chi \to \varphi(c_0)\|_{\hat{\mathbf{M}}} < 1$, i.e. $u > \|\varphi(c_0)\|_{\hat{\mathbf{M}}}$ – a contradiction. $\square$

Corollary 1 (Standard Löwenheim-Skolem.) Let $*$ be a continuous t-norm and $[0,1]_*$ the corresponding t-algebra. If T has a model over $[0,1]_*$ then it has a countable model over $[0,1]_*$.

5 Lindström theorem

To formulate a general version of Lindström's theorem, let us agree that *abstract logic* has a set of *sentences,* class of *models* and a function saying for each sentence and each model if the sentence is true in a model or not *(truth evaluation).* Thus e.g. for $BL\forall$ sentences are closed formulas, models are pairs $(\mathbf{L}, \mathbf{M})$ where $\mathbf{L}$ is a BL-chain and $\mathbf{M}$ is an $\mathbf{L}$-safe; the truth evaluation decides if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ or $\neq 1$. Having an abstract logic, the (first) compactness theorem makes sense; and if the models of the logic have the form $(\mathbf{L}, \mathbf{M})$ where $\mathbf{L}$ is an algebra and $\mathbf{L}$ an $\mathbf{L}$-interpretation of a predicate language then Löwenheim–Skolem makes sense.

Now the *Lindström theorem* (see [3] for careful presentation of Lindström's theorem) for a logic $\mathcal{C}$ says: If an abstract logic $\mathcal{D}$ extending $\mathcal{C}$ (with the same models as $\mathcal{C}$) satisfies compactness and Löwenheim–Skolem theorem then $\mathcal{D}$ is equivalent to $\mathcal{C}$, i.e. for each sentence φ of $\mathcal{D}$ there is a sentence ψ of $\mathcal{C}$ such that for each model $\mathcal{M}$, φ is $\mathcal{D}$-true in $\mathcal{M}$ iff ψ is $\mathcal{C}$-true in $\mathcal{M}$.

Theorem. Lindström theorem fails for $\mathcal{C}$ being $BL\forall, Ł\forall, G\forall, \Pi\forall$ with their general semantics.

Proof: As we have seen $\mathcal{C}$ satisfies both forms of the conpactness theorem as well as Löwenheim-Skolem theorem. Let $\mathcal{D}$ be extension of $\mathcal{C}$ by Baaz's connective Δ, which in each chain $\mathbf{L}$ satisfies $\Delta(1) = 1, \Delta(x) = 0$ for $x < 0$. $\mathcal{D}$ has the same models $(\mathbf{L}, \mathbf{M})$ as $\mathcal{C}$ but is not a sublogic of $\mathcal{C}$. To see this consider the formula $\varphi : \neg\Delta P(g)$ where P is a unary predicate and g is a constant. Clearly, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ iff $\|P(g)\|^{\mathbf{L}}_{\mathbf{M}} < 1$ and to evaluate $P(g)$ one needs just the one-element submodel of $\mathbf{M}$ consisting of the interpretation of g. In each such one-element model each $\mathcal{C}$ formula reduces to a quantifier-free formula in a notion way; thus *if* φ were $\mathcal{C}$-definable it would be definable by a propositional formula with one propositional atom. But one can easily show that $\Delta(x)$ is not definable in the propositional logic underlying $\mathcal{C}$. (For Ł use the fact that the connectives of Ł are continuous; for G and Π use the fact that the mapping sending 0 to 0 and every positive is to 1 is a homomorphism of the algebra of truth functions.

It remains to show that the logic $\mathcal{D}$ (extension of $\mathcal{C}$ by the Δ-connective) satisfies compactness and Löwenheim-Skolem. But this needs only routine checking of the proofs of these theorems for $\mathcal{C}$ (taking into account the formulation of the deduction theorem for the logic with $\mathcal{D}$ – see [1] 2.4.14) and notice that they work also for predicate calculus. $\square$

Remark. It makes little sense to ask on validity of Lindström theorem for our logics with standard semantics since only Gödel logic $G\forall$ satisfies both standard compactness theorems. For this logic our costruction shows the failure of Lindström theorem also in the case of standard semantics.

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