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# Solution of a Semi-Coercive Contact Problem in a Non-Linear Thermo-Elastic Rheology

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Abstract:

In the paper a contact problem in non-linear thermo-elastic rheology is studied. A problem of unilateral contact between bodies in non-linear thermo-elasticity firstly leads to a generalization of non-linear stress-strain relation. The stress-strain relation is derived from a positive definite strain energy density function. The weak solution is defined on the basis of a variational inequality. Then the secant modules method is used. We prove the convergence of the secant modules method to the exact solution. The problem analysed corresponds with model problems of mechanics, geomechanics, biomechanics and technology.

Keywords:
Non-linear thermo-elasticity, semi-coercive contact problem, secant modules method, finite element method

# 1 Introduction

In mechanics, geomechanics, biomechanics as well as in technological practice there is a variety of variational formulations. Thus variational inequalities physically describe the principle of virtual work in its inequality form. Such problems are represented by contact problems with or without friction in linear elasticity, thermo-elasticity, plasticity and thermo-plasticity, see e.g. [4], [5], [6], [8], [10], [11], [20] in elasticity and plasticity and [13], [14], [15], [16], [17] in thermo-elasticity and thermo-plasticity and [19] in non-linear elasticity. In [9] we analyse a generalized semi-coercive contact problem with friction arising in static linear thermo-elasticity for the case that bodies of arbitrary shapes, being in a mutual contact, are loaded by external forces.

In the present paper a semi-coercive contact problem in non-linear thermo-elastic rheology is formulated and then analysed. We develop the N-dimensional stress-strain relation ($N = 2,3$) derived from a positive definite strain energy density function $W$ of the form $W = A^{\lambda}(e_{ij})$, where $A$ is a scalar-valued function of the strains $e_{ij}$ and $\lambda$ is a positive parameter. The parameter $\lambda$ determines the degree of non-linearity for the strain dependent anisotropic elastic coefficients, which are functions of the displacement vector $\mathbf{u}$. For $0 < \lambda < 1$ the parameter $\lambda$ has the effect of producing a softening stress-strain curve, for $\lambda > 1$ has the effect of producing a hardening strain curve. For $\lambda = 1$ it produces the strain-less elastic curves, where the non-linearity stems from the anisotropic coefficients depending on the displacement $\mathbf{u}$ only.

# 2 Formulation of the Problem

To develop the $N$-dimensional stress-strain relation ($N = 2,3$) a positive definite strain energy density function will be used. Let the strain energy density function $W$ be defined by

$$W = A^{\lambda}(e_{ij})\,, \tag{2.1}$$

where $A$ is a scalar-valued function of the strains $e_{ij}$ and $\lambda$ is a positive parameter [1], [2], [19]. In this paper we shall assume that the function $A$ is defined as

$$A = c_{ijkl}(\mathbf{u})e_{ij}(\mathbf{u})e_{kl}(\mathbf{u}), \quad e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j,k,l = 1,\ldots N\,, \tag{2.2}$$

where $c_{ijkl}$ are elastic coefficients generally depending on the displacement vector $\mathbf{u} =(u_i)$ and $e_{ij}$ are the components of the small strain tensor. Assume that

$$c_{ijkl} = c_{klij} = c_{jikl}\,.$$

A repeated index implies summation from 1 to $N$. According to the theory of continuum mechanics and thermodynamics the stress tensor components are defined by the well-known relation

$$\tau_{ij} = \frac{\partial W}{\partial e_{ij}} \tag{2.3}$$

Hence and using (2.2)

$$\tau_{ij} = \lambda[A(e_{ij})]^{\lambda-1}\frac{\partial A(e_{ij})}{\partial e_{ij}} = 2\lambda[A(e_{ij})]^{\lambda-1}c_{ijkl}(\mathbf{u})e_{kl} = c^{*}_{ijkl}(\mathbf{u})e_{kl}\,. \tag{2.4}$$

The scalar coefficient $2\lambda[A(e_{ij})]^{\lambda-1}$ depends upon the state of strain and can simulate hardening and softing behaviours of materials and

$$c^{*}_{ijkl}(\mathbf{u}) = 2\lambda[A(e_{ij})]^{\lambda-1}c_{ijkl}(\mathbf{u}) \tag{2.5}$$

are strain dependent non-linear elastic coefficients. The parameter $\lambda$ determines the degree of non-linearity for the strain dependent elastic coefficients $c^{*}_{ijkl}(\mathbf{u})$. If $\lambda < 1$ then the parameter $\lambda$ has the effect of producing a softening stress-strain curve, if $\lambda = 1$ it produces the strain-less elastic curve, where the non-linearity of the coefficients $c^*_{ijkl}(\mathbf{u})$ depends on the displacements $\mathbf{u}$ only, while for $\lambda > 1$ it has the effect of producing a hardening stress-strain curve.

Let $\Omega \subset R^N$ be a region occupied by a system of elastic bodies $\Omega^\iota$, so that $\Omega = \cup_{\iota=1}^s \Omega^\iota$. Let $\Omega^\iota$ have Lipschitz boundaries $\partial\Omega^\iota$ and assume that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c \cup \mathcal{R}$, where the disjoint parts $\Gamma_\tau, \Gamma_u$ and $\Gamma_c$ are open subsets and the surface measure of $\mathcal{R}$ is zero. Moreover, let $\Gamma_u = {}^1\Gamma_u \cup {}^2\Gamma_u$ and $\Gamma_c^{kl} = \partial\Omega^k \cap \partial\Omega^l$, $k \neq l$, $\overline{\Gamma}_c = \cup_{k,l}\Gamma_c^{kl}$.

Let $u_n = u_i n_i, \mathbf{u}_t = \mathbf{u} - u_n \mathbf{n}, \tau_n = \tau_{ij} n_j n_i, \boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n \mathbf{n}$ be normal and tangential components of displacement and stress vectors $\mathbf{u} = (u_i)$, $\boldsymbol{\tau} = (\tau_i)$, $\tau_i = \tau_{ij} n_j$, $i,j = 1,..,N, \mathbf{n} = (n_i)$ is the unit outward normal vector to $\partial\Omega$.

Let body forces $\mathbf{F} \in [L^2(\Omega)]^N$, surface forces $\mathbf{P} \in [L^2(\Gamma_\tau)]^N$ and heat sources $W \in L^2(\Omega)$ be given. Then the model to be solved represents a contact problem in non-linear thermo-elasticity.

## 3 Weak Solution of the Non-Linear Problem

In the next we shall investigate the following model problem:

**Problem** $(\mathcal{P})$: Let $N = 2(3), s \geq 2$. Find a pair of functions $(T, \mathbf{u})$, a scalar function $T$ and a vector function $\mathbf{u}$, satisfying

$$
\begin{aligned}
-\frac{\partial}{\partial x_i}\left(\kappa^\iota_{ij}\frac{\partial T^\iota}{\partial x_j}\right) &\equiv W^\iota,\ \frac{\partial \tau_{ij}(\mathbf{u}^\iota)}{\partial x_j} + F^\iota_i = 0,\ \ i,j = 1,\dots,N,\ \iota = 1,\dots,s \text{ in } \Omega^\iota\,, && (3.1)\\
\tau^\iota_{ij} &= c^{*\iota}_{ijkl}(\mathbf{u}^\iota) e_{kl}(\mathbf{u}^\iota) - \beta^\iota_{ij}(T^\iota - T^\iota_0),\ \ i,j,k,l = 1,\dots,N,\ \iota = 1,\dots,s\,, && (3.2)\\
\kappa_{ij}\frac{\partial T}{\partial x_j} n_i &= 0,\ \ \tau_{ij} n_j = P_i,\ \ i,j = 1,\dots,N \text{ on } \Gamma_\tau\,, && (3.3)\\
\kappa_{ij}\frac{\partial T}{\partial x_j} n_i &= 0,\ \ u_i = {}^1u_{0i},\ \ i,j = 1,\dots,N \text{ on } {}^1\Gamma_u\,, && (3.4)\\
T &= T_1, u_i = {}^2u_{0i},\ \ i = 1,\dots,N, \text{ on } {}^2\Gamma_u\,, && (3.5)\\
T^k &= T^l,\ \ \kappa_{ij}\frac{\partial T}{\partial x_i} n_{i|(k)} = \kappa_{ij}\frac{\partial T}{\partial x_i} n_{i|(l)} \text{ on } \Gamma_c^{kl}\,, && (3.6)\\
u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) &\leq 0, \tau_n^k(\mathbf{x}) = -\tau_n^l(\mathbf{x}) \equiv \tau_n^{kl}(\mathbf{x}) \leq 0\,, && (3.7)\\
& \quad (u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}))\tau_n^{kl}(\mathbf{x}) = 0 \text{ on } \Gamma_c^{kl}\,,\\
\boldsymbol{\tau}_t^{kl}(\mathbf{x}) &= 0 \text{ on } \Gamma_c^{kl}\,. && (3.8)
\end{aligned}
$$

Both the problems in thermics and non-linear elasticity can be solved separately and the coupling term $\frac{\partial}{\partial x_j}(\beta^\iota_{ij}(T^\iota - T^\iota_0))$has a meaning of the body forces. Since we assume that $\beta^\iota_{ij} \in C^1(\overline{\Omega}^\iota)$, $T^\iota, T^\iota_0 \in H^1(\Omega^\iota)$, then $\beta^\iota_{ij}(T^\iota - T^\iota_0) \in H^1(\Omega^\iota)$ and therefore $\frac{\partial}{\partial x_j}(\beta^\iota_{ij}(T^\iota - T^\iota_0)) \in L^2(\Omega^\iota)$. In what follows, we set $\tilde{F}^\iota_i = F^\iota_i - \frac{\partial}{\partial x_j}(\beta^\iota_{ij}(T^\iota - T^\iota_0))$.

Let us introduce the sets of virtual temperatures and displacements and the set of admissible displacements by

$$
\begin{aligned}
{}^1V &= \{z \mid z \in {}^1W \equiv H^1(\Omega^1) \times \cdots \times H^1(\Omega^s), z = T_1 \text{ on } \cup\ {}^2\Gamma_\tau\}\,,\\
V_0 &= \{\mathbf{v} \mid \mathbf{v} \in [H^1(\Omega^1)]^N \times \cdots \times [H^1(\Omega^s)]^N, \mathbf{v} = 0 \text{ on } \Gamma_u\}\,,\\
K &= \{\mathbf{v} \mid \mathbf{v} \in V_0, v_n^k - v_n^l \leq 0 \text{ on } \cup_{k,l}\Gamma_c^{kl}\}\,,
\end{aligned}
$$

and the set of all displacements and rotations

$$R^\iota = \{\mathbf{v} \mid \mathbf{v} \in [H^1(\Omega^\iota)]^N, e_{ij}(\mathbf{v}) = 0 \text{ a.e. }\},\ R = \sqcap_{\iota=1}^s R^\iota\,.$$

For $N = 3$ $R^\iota = \{\mathbf{v} \mid \mathbf{v} \in [H^1(\Omega^\iota)]^3, \mathbf{v} = \mathbf{a}^\iota + \mathbf{b}^\iota \times \mathbf{x}\}$,
for $N = 2$ $R^\iota = \{\mathbf{v} \mid \mathbf{v} \in [H^1(\Omega^\iota)]^2, v_1 = a^\iota_1 - b^\iota x_2, v_2 = a^\iota_2 + b^\iota x_1\}$,
where $\mathbf{a}^\iota, \mathbf{b}^\iota$ are arbitrary real vectors for $N = 3$ or $b^\iota$ any real scalar for $N = 2$.

We see that $K$ is a convex cone and it can be shown that $K$ is a closed convex subset of $V_0$. Let $P_V = V_0 \cap R$ and let $V_0 = P_V \oplus Q_0$ be the orthogonal decomposition of $V_0$.

To formulate the variational (weak ) formulation of the above problem $(\mathcal{P})$, we multiply (3.1a) by $z - T$ and (3.1b) by $v_i - u_i$, integrate over $\Omega$ and use boundary conditions. Then after some modifications we obtain the following variational problem $(\mathcal{P})_v$ :

Find a pair of functions $(T, \mathbf{u}), T \in {}^1V, \mathbf{u} \in K$, satisfying

$$
\begin{aligned}
b(T, z - T) &\geq s(z - T) \;\; \forall z \in {}^1V_h \,, && (3.9)\\
a(\mathbf{u}; \mathbf{u}, \mathbf{v} - \mathbf{u}) &\geq (\mathbf{f}, \mathbf{v} - \mathbf{u}) \;\; \forall \mathbf{v} \in K \,, && (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
b(T, z) &= \sum_{\iota=1}^{s} b^{\iota}(T^{\iota}, z^{\iota}) = \int_{\Omega} \kappa_{ij}(\mathbf{x}) \frac{\partial T}{\partial x_j} \frac{\partial z}{\partial x_i} d\mathbf{x} \,, \\
s(z) &= \sum_{\iota=1}^{s} s^{\iota}(z^{\iota}) = \int_{\Omega} Q z d\mathbf{x} + \int_{\Gamma_{\tau}} q_0 z ds \,, \\
l(z) &= \frac{1}{2} b(z, z) - s(z) \,, \\
a(\mathbf{w}; \mathbf{u}, \mathbf{v}) &= \sum_{\iota=1}^{s} a(\mathbf{w}^{\iota}; \mathbf{u}^{\iota}, \mathbf{v}) = 2 \int_{\Omega} \lambda [A(e_{ij}(\mathbf{w}))]^{\lambda - 1} c_{ijkl}(\mathbf{w}) e_{ij}(\mathbf{u}) e_{kl}(\mathbf{v}) d\mathbf{x} \,, \\
(\mathbf{f}, \mathbf{v}) &= \sum_{\iota=1}^{s} (\mathbf{f}^{\iota}, \mathbf{v}) = \int_{\Omega} F_i^{\iota} v_i d\mathbf{x} + \int_{\Gamma_{\tau}} P_i v_i ds \,.
\end{aligned}
$$

For the existence of a potential energy functional $L(\mathbf{v})$, such that its Gâteaux differential

$$
DL(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}; \mathbf{u}, \mathbf{v}) - (\mathbf{f}, \mathbf{v}) \,,
$$

the following condition is necessary (see e.g. Gajewski et al. (1974) [3]):

$$
\int_{\Omega} Dc^{*}_{ijkl}(\mathbf{u}, \mathbf{w}) e_{kl}(\mathbf{u}) e_{ij}(\mathbf{v}) d\mathbf{x} = \int_{\Omega} Dc^{*}_{ijkl}(\mathbf{u}, \mathbf{v}) e_{kl}(\mathbf{u}) e_{ij}(\mathbf{w}) d\mathbf{x} \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \,.
$$

Then we have

$$
L(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \int_{0}^{1} a(t\mathbf{v}; t\mathbf{v}, \mathbf{v}) dt d\mathbf{x} - (\mathbf{f}, \mathbf{v})
$$

or in an equivalent form

$$
\begin{aligned}
& T \in {}^1V, Dl(T, z - T) \geq 0 \;\; \forall z \in {}^1V \,, && (3.11)\\
& \mathbf{u} \in K, DL(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq 0 \;\; \forall \mathbf{v} \in K \,, && (3.12)
\end{aligned}
$$

where $l(z)$ and $L(\mathbf{v})$ are defined above.

## 4 Secant Modules Method

The thermal part of the problem can be analysed as in [9], [17]. The non-linear elastic problem can be solved by the secant modules method (see e.g. [12], [16], [17]).

The **secant modules method** consists in solving a sequence of variational inequalities of the form

$$\mathbf{u}_{n+1} \in K, a(\mathbf{u}_n; \mathbf{u}_{n+1}, \mathbf{v} - \mathbf{u}_{n+1}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}_{n+1}), \qquad n = 1, 2, \ldots \tag{4.1}$$

where $\mathbf{u}_n$ is the $n$-th approximate solution of the problem studied.

Hence the problem studied leads to the solution of a sequence of variational inequalities with variable coefficients of the semi-coercive type of the form:

Let $\mathbf{u}_n \in K$, $n = 1, 2, \ldots$ be such that

$$a(\mathbf{u}_n; \mathbf{u}_{n+1}, \mathbf{v} - \mathbf{u}_{n+1}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}_{n+1}) \quad \forall \mathbf{v} \in K\,. \tag{4.2}$$

Let us assume that the bilinear form $a(\mathbf{w}; \mathbf{u}, \mathbf{v})$ is symmetric in $\mathbf{u}, \mathbf{v}$ and such that

$$\begin{aligned} a(\mathbf{w}; \mathbf{u}, \mathbf{u}) &\geq c_0 \parallel \mathbf{u} \parallel^2, c_0 = \text{const.} > 0\,, \qquad (4.3)\\ \mid a(\mathbf{w}; \mathbf{u}, \mathbf{v}) \mid &\leq c_1 \parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel\,, \qquad (4.4) \end{aligned}$$

there exists a functional $\mathcal{L}$ such that

$$\begin{aligned} a(\mathbf{u}; \mathbf{u}, \mathbf{v}) &= D\mathcal{L}(\mathbf{u}, \mathbf{v})\,, \qquad (4.5)\\ \frac{1}{2}a(\mathbf{u}; \mathbf{v}, \mathbf{v}) - \frac{1}{2}a(\mathbf{u}; \mathbf{u}, \mathbf{u}) - \mathcal{L}(\mathbf{v}) + \mathcal{L}(\mathbf{u}) &\geq 0 \qquad (4.6) \end{aligned}$$

and that $\mathcal{L}$ has the second Gâteaux differential $D^2\mathcal{L}(\mathbf{w}; \mathbf{u}, \mathbf{v})$, the mapping $\mathbf{w} \longmapsto D^2\mathcal{L}(\mathbf{w}; \mathbf{u}, \mathbf{v})$ is continuous on every segment and that

$$D^2\mathcal{L}(\mathbf{w}; \mathbf{u}, \mathbf{u}) \geq c \parallel \mathbf{u} \parallel^2, c = \text{const.} > 0 \tag{4.7}$$

holds. Assume that $\mathcal{L}(\mathbf{v}), D\mathcal{L}(\mathbf{v}, \mathbf{w}), D^2\mathcal{L}(\mathbf{v}; \mathbf{w}, \mathbf{y})$ and $a(\mathbf{w}; \mathbf{v}, \mathbf{v})$ are independent of an addition of $\mathbf{p} \in R$ in all variables. Let

$$a(\mathbf{w}_n; \mathbf{v}, \mathbf{z}) \to a(\mathbf{w}; \mathbf{v}, \mathbf{z}) \;\; \forall \mathbf{v}, \mathbf{z} \in V_0 \text{ if } \mathbf{w}_n \to \mathbf{w}\,.$$

Moreover, for $\mathbf{f} \in V_0'$

$$(\mathbf{f}, \mathbf{p}) < 0 \; \forall \mathbf{p} \in R \cap K \backslash \{0\} \;\text{ and }\; (\mathbf{f}, \mathbf{p}) \neq 0 \;\; \forall \mathbf{p} \in R \cap V_0 \backslash \{0\} \tag{4.8}$$

hold, where $(\mathbf{f}, \mathbf{p})$ is defined above. Assume that the only element $\mathbf{p} \in R \cap K$ such that also $-\mathbf{p} \in R \cap K$ is $\mathbf{p} = 0$.

As a result we have the following theorem

**Theorem 1** *Let (4.3), (4.4), (4.6), (4.7) be satisfied for* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in Q_0$, *and let (4.5), (4.8) hold. Then*

*(i) the functionals*

$$L(\mathbf{v}) = \mathcal{L}(\mathbf{v}) - (\mathbf{f}, \mathbf{v}), \omega(\mathbf{u}) = \frac{1}{2}a(\mathbf{v}; \mathbf{u}, \mathbf{u}) - (\mathbf{f}, \mathbf{u}) \tag{4.9}$$

*are coercive and weakly lower semi-continuous in* $K$.

*(ii) the problem to find* $\mathbf{u} \in K$ *such that*

$$D\mathcal{L}(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}) \;\; \forall \mathbf{v} \in K \tag{4.10}$$

*has a unique solution.*

*The problem to find* $\mathbf{u}_{n+1} \in K$

*(iii) for $\mathbf{u}_n \in K$, $n = 1, 2, \ldots$, such that*

$$a(\mathbf{u}_n; \mathbf{u}_{n+1}, \mathbf{v} - \mathbf{u}_{n+1}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}_{n+1}) \ \ \forall \mathbf{v} \in K\,, \tag{4.11}$$

*has a unique solution and*

$$\lim_{n\to\infty} \Pi_Q \mathbf{u}_n = \Pi_Q \mathbf{u}\,, \tag{4.12}$$

*where $\Pi_Q$ denotes the projection of $V_0$ onto $Q_0$.*
*If $\lim\limits_{k\to\infty} \mathbf{u}_{n_k} = \mathbf{w}$, then $\mathbf{w}$ is the solution of (4.10).*

**Proof** The coerciveness of $L$ follows from the fact that

$$L(\mathbf{v}) \geq c_0 \parallel \mathbf{v} \parallel -c_1, \ \ \forall \mathbf{v} \in K\,, \tag{4.13}$$

where $c_0, c_1$ are positive constants (see e.g. [12], Lemma 2.1). Hence

$$\lim_{\mathbf{v}\in K, \|\mathbf{v}\|\to\infty} \inf \frac{L(\mathbf{v})}{\parallel \mathbf{v} \parallel} \geq c_0 > 0\,.$$

Due to the definition of $\mathcal{L}(\mathbf{v})$, the functional $L$ is convex and G-differentiable and therefore, it is weakly lower semi-continuous. Hence the existence of the solution of (4.10), due to the coerciveness and weakly lower semi-continuity follows immediately. Similarly

$$\omega(\mathbf{u}) = \frac{1}{2} a(\mathbf{v}; \mathbf{u}, \mathbf{u}) - (\mathbf{f}, \mathbf{u}) \geq c_2 \parallel \mathbf{u} \parallel -c_3 \ \ \forall \mathbf{u} \in K \tag{4.14}$$

is valid uniformly with respect to $\mathbf{v}$. By a similar way we prove the existence of the solution of (4.11). To prove the uniqueness of the solution of (4.10) we assume that $\mathbf{u}^1$ and $\mathbf{u}^2$ are two solutions of (4.10). Then $\mathbf{u}^2 = \mathbf{u}^1 + \mathbf{p}$, $\mathbf{p} \in R$, $\mathbf{u}^1 + \mathbf{p} \in K$ and $(\mathbf{f}, \mathbf{p}) = 0$ holds. By assumption (4.8b) we obtain $\mathbf{p} = \mathbf{0}$ and the uniqueness. Similarly we prove the uniqueness of a solution of (4.11). Let us denote by $\Pi_Q$ the projector of $V_0$ onto $Q_0$. Now we shall prove the convergence of (4.12) for $n \to \infty$. According to the previous results, inserting $\mathbf{v} = 0$ in (4.11) and using (4.3), (4.14), then there exists a sequence $\{\mathbf{u}_n\}$ such that

$$\parallel \mathbf{u}_n \parallel \leq c \ \ \forall n\,. \tag{4.15}$$

We have

$$c \parallel \Pi_Q \mathbf{u}_{n+1} - \Pi_Q \mathbf{u}_n \parallel \leq a(\mathbf{u}_n; \mathbf{u}_{n+1} - \mathbf{u}_n, \mathbf{u}_{n+1} - \mathbf{u}_n)\,. \tag{4.16}$$

After some modification and the convergence of $L(\mathbf{u}_n)$ we have

$$\lim_{n\to\infty} \parallel \Pi_Q \mathbf{u}_{n+1} - \Pi_Q \mathbf{u}_n \parallel = 0\,. \tag{4.17}$$

Then after some modifications and using (4.15), (4.16) we obtain

$$\begin{aligned} \frac{1}{2} c \parallel \Pi_Q \mathbf{u}_n - \Pi_Q \mathbf{u} \parallel^2 \ \ &\leq \ \ D\mathcal{L}(\mathbf{u}_n, \mathbf{u}_n - \mathbf{u}) - D\mathcal{L}(\mathbf{u}, \mathbf{u}_n - \mathbf{u}) = a(\mathbf{u}_n; \mathbf{u}_n, \mathbf{u}_n - \mathbf{u}) - D\mathcal{L}(\mathbf{u}, \mathbf{u}_n - \mathbf{u}) \leq \\ &\leq \ \ D\mathcal{L}(\mathbf{u}, \mathbf{u}_{n+1} - \mathbf{u}_n) + a(\mathbf{u}_n; \mathbf{u}_{n+1}, \mathbf{u}_n - \mathbf{u}_{n+1}) + a(\mathbf{u}_n; \mathbf{u}_n - \mathbf{u}_{n+1}, \mathbf{u}_n - \mathbf{u}) \to 0\,. \end{aligned}$$

Let there exist a subsequence $\mathbf{u}_{n_k} \to \mathbf{u}^*$, $k \to \infty$. Then

$$a(\mathbf{u}_{n_k-1}; \mathbf{u}_{n_k}, \mathbf{v} - \mathbf{u}_{n_k}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}_{n_k}) \ \ \forall \mathbf{v} \in K\,, \tag{4.18}$$

hence

$$a(\mathbf{u}_{n_k-1}; \mathbf{u}^*, \mathbf{v} - \mathbf{u}^*) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}^*) + \varepsilon_{n_k}(\mathbf{v}), \varepsilon_{n_k}(\mathbf{v}) \to 0\,.$$

Since

$$a(\mathbf{u}_{n_k-1};\mathbf{u}^*,\mathbf{v}-\mathbf{u}^*) = a(\mathbf{u}_{n_k}+\mathbf{u}_{n_k-1}-\mathbf{u}_{n_k};\mathbf{u}^*,\mathbf{v}-\mathbf{u}^*) = a(\mathbf{u}_{n_k}+\Pi_Q\mathbf{u}_{n_k-1}-\Pi_Q\mathbf{u}_{n_k};\mathbf{u}^*,\mathbf{v}-\mathbf{u}^*)$$

then using (4.17) and the fact that $a(\mathbf{w}_n;\mathbf{v},\mathbf{z}) \to a(\mathbf{w};\mathbf{v},\mathbf{z})$ $\forall \mathbf{v},\mathbf{z} \in V$, if $\mathbf{w}_n \to \mathbf{w}$, we find that

$$a(\mathbf{u}_{n_k-1};\mathbf{u}^*,\mathbf{v}-\mathbf{u}^*) \to a(\mathbf{u}^*;\mathbf{u}^*,\mathbf{v}-\mathbf{u}^*) = D\mathcal{L}(\mathbf{u}^*,\mathbf{v}-\mathbf{u}^*)\,.$$

Hence $\mathbf{u}^*$ is the solution of (4.10), and $\mathbf{u}^*=\mathbf{u}$,which completes the proof.

## 5 Numerical Approximation of the Problem

Let the domain $\Omega \subset \mathbb{R}^N, N = 2(3)$, be triangulated. Then we divide $\overline{\Omega} = \Omega \cup \partial\Omega$ into a system of $m$ triangles $T_h$ in the 2D case and into a system of $m$ tetrahedra in the 3D case, generating a triangulation $\mathcal{T}_h$ such that $\overline{\Omega} = \cup_{i=1}^m T_{hi}$ and such that two neighbouring triangles have only a vertex or an entire side common in the 2D case, and/or that two neighbouring tetrahedra have only a vertex or an entire edge or an entire face common in the 3D case. Let $h = \max\limits_{1\le i\le m} (\mathrm{diam} T_{hi})$ and let a family of triangulation $\{\mathcal{T}_h\}, h \to 0_+$, be regular in the standard sense. We further assume that the sets $\overline{\Gamma}_u \cap \overline{\Gamma}_\tau$, $\overline{\Gamma}_u \cap \overline{\Gamma}_c$, $\overline{\Gamma}_\tau \cap \overline{\Gamma}_c$ coincide with vertices or edges of $\mathcal{T}_h$. Let

$$V_h = \{\mathbf{v} \mid \mathbf{v} \in [C(\Omega)]^N, \mathbf{v}\,|_{T_h} \in [P_1]^N, \mathbf{v} = 0 \ \text{ on } \ \Gamma_u \ \forall T_h \in \mathcal{T}_h\}\,,$$

where $P_1$ is the space of all linear polynomials, and

$$K_h = \{\mathbf{v} \mid \mathbf{v} \in V_h, v_n^k - v_n^l \le 0 \text{ on } \cup_{k,l}\, \Gamma_c^{kl}\} = K \cap V_h\,.$$

We see that $K_h$ is a convex and closed subset of $V_h$ $\forall h$. Then using the FEM-secant modules method the problem leads to a sequence of approximate problems of variational inequalities with variable coefficients of the semi-coercive type of the form:
find $\mathbf{u}_{n+1}^h \in K_h$, $n = 1, 2, \ldots$ such that

$$a(\mathbf{u}_n^h;\mathbf{u}_{n+1}^h,\mathbf{v}-\mathbf{u}_{n+1}^h) \ge (\mathbf{f}^h,\mathbf{v}-\mathbf{u}_{n+1}^h) \ \ \forall \mathbf{v} \in K_h\,. \tag{5.1}$$

The analysis of such problems is parallel to that of FEM approximation of variational inequalities in linear elasticity (see e.g. in [7], [9], [17]), as the variational inequality problem (5.1) represents a system of linear variational inequalities in the theory of elasticity, where the elastic coefficients $c_{ijkl}$ are replaced by variable coefficients $c^*_{ijkl} = c^*_{ijkl}(\mathbf{u}_n^h) = \lambda A^{\lambda-1}(e_{ij}(\mathbf{u}_n^h))c_{ijkl}(\mathbf{u}_n^h)$. Similarly as in the linear case using the well-known Falk's lemma we obtain

$$\|\, \mathbf{u}_{n+1} - \mathbf{u}_{n+1}^h \,\|_1 \to 0 \ \text{ for } \ h \to 0\,. \tag{5.2}$$

The algorithms are modifications of those used in the theory of contact problems in linear elasticity, which have been discussed e.g. in [17], [18].

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