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Abstract:

In the paper the quasi-coupled semi-coercive contact problem in thermoelasticity is investigated. The FEM analysis of the problem investigated is also analysed.

Keywords:
Semi-coercive contact problem, variational inequality, thermo-elasticity, finite element method

1 Introduction

In this paper we shall deal with the solvability of a generalized semi-coercive contact problem in linear thermo-elasticity. The problem studied is formulated as the primary variational inequality problem [2], [3], [4], [8], [10], [14], [16], [17], i.e. in term of displacements, arising from the variational formulation of the contact problem with friction in thermo-elasticity. We will assume the generalized case of bodies of arbitrary shapes which are in mutual contacts. On one part of boundary the bodies are loaded and on the second one they are fixed and therefore, as a result, some of the bodies can shift and rotate. Numerical approximation of the studied problem represents a nondifferentiable optimization problem [11].

2 Formulation of the Problem

2.1 The equilibrium and heat equations

Let $0, x_1, \ldots, x_N$ be the orthogonal Cartesian coordinate system, where N is the space dimension and let $\mathbf{x} = (x_1, \ldots, x_N)$ be a point in this Cartesian system. Let the body, being in an initial stress-strain state and created by a system of elastic anisotropic or isotropic bodies, occupy a region Ω. Let Ω be the region in $\mathbb{R}^N$, $N = 2, 3$, with a Lipschitz boundary $\partial\Omega$. Moreover, we shall assume that $\Omega = \bigcup_{\iota=1}^{s} \Omega^\iota$. Let the boundary $\partial\Omega$ be divided into disjoint parts Γ_τ, Γ_u, Γ_c and Γ_0 such that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c \cup \Gamma_0 \cup \mathcal{R}$, where the surface measure of $\mathcal{R}$ is zero and the parts Γ_τ, Γ_u, Γ_c and Γ_0 are open sets in $\partial\Omega$. Assume that Lamé coefficients λ and μ as well as anisotropic elastic coefficients c_{ijkl} and thermal conductivity coefficients κ, κ_{ij} are bounded functions. Moreover, we will assume that $\lambda^\iota, \mu^\iota, c^\iota_{ijkl}, \kappa^\iota, \kappa^\iota_{ij} \in C^1(\overline{\Omega}^\iota)$. The heat equation and the equilibrium equations for every subdomain Ω^ι of Ω read as follows:

$$\frac{\partial}{\partial x_i}\left(\kappa^\iota_{ij}\frac{\partial T^\iota}{\partial x_j}\right) + Q^\iota = 0, \quad \frac{\partial \tau_{ij}(\mathbf{u}^\iota)}{\partial x_j} + F^\iota_i = 0, \quad i,j = 1,\ldots,N,\ \iota = 1,\ldots,s \ \text{ in } \Omega^\iota , \tag{2.1}$$

where $\mathbf{F}^\iota$ are body forces and $Q^\iota = W^\iota + \rho^\iota \beta^\iota_{ij} T^\iota_0 e_{ij}(\mathbf{u}^{\iota\prime})$, where W^ι are thermal sources, and the term $\rho^\iota \beta^\iota_{ij} T^\iota_0 e_{ij}(\mathbf{u}^{\iota\prime})$ represents the energy dissipated in the bodies occupying Ω^ι in the form of heat and generated by their deformation, $e_{ij}(\mathbf{u}^{\iota\prime})$ is the strain rate tensor, where we denote $\mathbf{u}' = \partial\mathbf{u}/\partial t$. A repeated index implies summation from 1 to N.

The relation between the displacement vector $\mathbf{u} = (u_i)$, $i = 1, \ldots, N$, and the small strain tensor e_{ij} is defined by

$$e_{ij} = e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,\ldots,N . \tag{2.2}$$

The relation between the stress and strain tensors is defined by the Hooke's law (in thermo-elasticity also known as the Duhamel-Neumann's law)

$$\tau^\iota_{ij} = c^\iota_{ijkl} e_{kl}(\mathbf{u}^\iota) - \beta^\iota_{ij}(T^\iota - T^\iota_0), \quad i,j,k,l = 1,\ldots,N,\ \iota = 1,\ldots,s , \tag{2.3}$$

in the anisotropic case, whereas in the isotropic case

$$c^\iota_{ijkl} = \mu^\iota(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda^\iota \delta_{ij}\delta_{kl}, \beta^\iota_{ij} - \gamma^\iota \delta_{ij} , \tag{2.4}$$

where λ^ι, μ^ι represent the Lamé coefficients, and β^ι_{ij}, γ^ι coefficients of thermal expansion, T^ι_0 is the initial temperature. The coefficients c^ι_{ijkl} form a matrix of the type $(N \times N \times N \times N)$ and satisfy the symmetry conditions

$$c^\iota_{ijkl} = c^\iota_{jikl} = c^\iota_{klij} = c^\iota_{ijlk} \tag{2.5}$$

and

$$0 < a_0^\iota \leq c_{ijkl}^\iota(\mathbf{x})\xi_{ij}\xi_{kl} \mid \xi \mid^{-2} \leq A_0^\iota < +\infty \text{ for a.e. } \mathbf{x} \in \Omega^\iota, \quad \xi \in \mathbb{R}^{N^2}, \ \xi_{ij} = \xi_{ji}, \tag{2.6}$$

where a_0^ι, A_0^ι are constants independent of $\mathbf{x} \in \Omega^\iota$ and for the isotropic case $a_0^\iota = 2\min\{\mu^\iota(\mathbf{x}); \mathbf{x} \in \Omega^\iota\}$ and $A_0^\iota = \max\{2\mu^\iota(\mathbf{x}) + 3\lambda^\iota(\mathbf{x}); \mathbf{x} \in \Omega^\iota\}$. The coefficients of thermal expansion satisfy the symmetry conditions

$$\beta_{ij}^\iota = \beta_{ji}^\iota, \quad i,j = 1,\ldots,N, \ \iota = 1,\ldots,s. \tag{2.7}$$

The thermal conductivity coefficients κ_{ij}^ι satisfy the symmetry condition

$$\kappa_{ij}^\iota = \kappa_{ji}^\iota \ \text{ on } \Omega^\iota, \ \ i,j = 1,\ldots,N, \ \iota = 1,\ldots,s \tag{2.8}$$

and

$$0 < k_0^\iota \leq \kappa_{ij}^\iota(\mathbf{x})\zeta_i\zeta_j \mid \zeta \mid^{-2} \leq k_1^\iota < +\infty \ \text{ for a.e. } \mathbf{x} \in \Omega^\iota, \ \zeta \in \mathbb{R}^N, \tag{2.9}$$

where k_0^ι, k_1^ι are constants independent of $\mathbf{x} \in \Omega^\iota$.

2.2 Boundary conditions

We shall consider the condition of heat flux and of loading on the part Γ_τ of the boundary in the form

$$\kappa_{ij}\frac{\partial T}{\partial x_i}n_i = q \ (=0), \ \ \tau_{ij}n_j = P_i \ \text{ on } \Gamma_\tau. \tag{2.10}$$

Let the boundary $\Gamma_u = {}^1\Gamma_u \cup {}^2\Gamma_u$. Let us assume that a portion of the examined body is fixed at a certain boundary which will be denoted by ${}^1\Gamma_u$. Let us denote the outward unit normal to the boundary $\partial\Omega$ by $\mathbf{n} = (n_i)$. We thus have e.g. the conditions

$$\kappa_{ij}\frac{\partial T}{\partial x_i}n_i = q \ (=0), \ \ u_i = {}^1 u_{0i} \ (=0) \ \text{ on } {}^1\Gamma_u. \tag{2.11}$$

Furthermore, we can assume that the temperature and the displacement vector are given at the boundary ${}^2\Gamma_u$

$$T = T_1, u_i = {}^2u_{0i} \ (\neq 0), \quad i = 1,\ldots,N, \ \text{ on } {}^2\Gamma_u. \tag{2.12}$$

From Eqs (2.1), (2.3) and (2.10) it follows that the effect of a change of temperature due to the deformation is equivalent to the replacement of mass forces by forces $F_i - \frac{\partial}{\partial x_j}(\beta_{ij}(T - T_0))$ and of surface loading by surface forces $P_i + \beta_{ij}(T - T_0)n_j$.

Assume that the elastic body occupying the region Ω consists of s bodies Ω^ι, $\iota = 1,\ldots,s$, so that $\Omega = \bigcup_{\iota=1}^{s} \Omega^\iota$, and let several neighbouring bodies, say Ω^k and Ω^l, are in a mutual contact. Denote the common contact boundary between both bodies Ω^k and Ω^l before deformation by Γ_c^{kl}. Let $\mathbf{t} = (t_i)$ be the unit tangential vector to the contact boundary Γ_c^{kl}. Further, denote by $\tau_i = \tau_{ij}n_j$ the stress vector, its normal and tangential components by $\tau_n = \tau_i n_i = \tau_{ij}n_in_j$, $\boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n\mathbf{n}$, and the normal and tangential components of displacement vector by $u_n = u_in_i$ and $\mathbf{u}_t = \mathbf{u} - u_n\mathbf{n}$. Denote by $\mathbf{u}^k$, $\mathbf{u}^l$, T^k, T^l (indices k, l correspond with the neighbouring bodies in contact) the displacements and the temperatures in the neighbouring bodies. All these quantities are functions of spatial coordinates. Then on the contact boundaries Γ_c^{kl} the condition of non-penetration

$$u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) \leq 0 \ \text{ on } \Gamma_c^{kl} \tag{2.13}$$

holds.

For the contact forces, due to the law of action and reaction, we find

$$\tau_n^k(\mathbf{x}) = -\tau_n^l(\mathbf{x}) \equiv \tau_n^{kl}(\mathbf{x}), \boldsymbol{\tau}_t^k(\mathbf{x}) = -\boldsymbol{\tau}_t^l(\mathbf{x}) \equiv \boldsymbol{\tau}_t^{kl}(\mathbf{x}). \tag{2.14}$$

Since the normal components of contact forces cannot be positive, i.e. cannot be tensile forces, then

$$\tau_n^k(\mathbf{x}) = -\tau_n^l(\mathbf{x}) \equiv \tau_n^{kl}(\mathbf{x}) \leq 0 \ \text{ on } \Gamma_c^{kl} \,. \tag{2.15}$$

During the deformation of the bodies they are in contact or they are not in contact. If they are not in contact, then $u_n^k - u_n^l < 0$, and the contact forces are equal to zero, i.e. $\tau_n^k = -\tau_n^l \equiv \tau_n^{kl} = 0$. If the bodies are in contact, i.e. $u_n^k - u_n^l = 0$, then there may exist non zero contact forces $\tau_n^k = -\tau_n^l \equiv \tau_n^{kl} \leq 0$. These cases are included in the following condition

$$(u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}))\tau_n^{kl}(\mathbf{x}) = 0 \ \text{ on } \Gamma_c^{kl} \,. \tag{2.16}$$

Further, if both bodies are in contact, then on the contact boundary the Coulombian type of friction acts. The frictional forces g_c^{kl} acting on the contact boundary Γ_c^{kl} are, in their absolute value, proportional to the normal stress component, where the coefficient of proportionality is the coefficient of Coulombian friction $\mathcal{F}_c^{kl}(\mathbf{x})$, i.e.

$$g_c^{kl}(\mathbf{x}) = \mathcal{F}_c^{kl}(\mathbf{x}) \mid \tau_n^{kl}(\mathbf{x}) \mid \,. \tag{2.17}$$

Due to the acting and frictional forces we have the following cases:

If the absolute value of tangential forces $\boldsymbol{\tau}_t^{kl}(\mathbf{x})$ is less than the frictional forces g_c^{kl}, then the frictional forces preclude the mutual shifts of both bodies being in contact. If the tangential forces $\boldsymbol{\tau}_t^{kl}$ are equal in their absolute value to the frictional forces, so that are no forces which can preclude the mutual, i.e. bilateral, motion of both elastic bodies. Thus the contact points change their position in the direction opposite to that in which the tangential stress component acts. These conditions are described by the following conditions:

$$\text{if} \qquad u_n^k - u_n^l = 0 \ \text{ then } \mid \boldsymbol{\tau}_t^{kl}(\mathbf{x}) \mid \leq g_c^{kl}(\mathbf{x}) \tag{2.18}$$

$$\text{if} \qquad \mid \boldsymbol{\tau}_t^{kl}(\mathbf{x}) \mid < g_c^{kl}(\mathbf{x}) \ \text{ then } \mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) = 0 \,, \tag{2.19}$$

which means that the friction forces are sufficient to preclude the mutual shifting between the assumed bodies and

$$\text{if} \quad \mid \boldsymbol{\tau}_t^{kl}(\mathbf{x}) \mid = g_c^{kl} \ \text{ then there exists a function } \vartheta \geq 0 \text{ such that } \ \mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) = -\vartheta \boldsymbol{\tau}_t^{kl}(\mathbf{x}) \,, \tag{2.20}$$

which means that the friction forces are not sufficient to preclude the mutual-bilateral shifting of both assumed elastic bodies. This shift acts in an opposite direction to the acting tangential forces.

On the contact boundary between the elastic bodies we shall assume that the temperatures and heat flow are continuous, i.e.

$$T^k = T^l , \ \ \kappa_{ij} \frac{\partial T}{\partial x_j} n_{i|(k)} = \kappa_{ij} \frac{\partial T}{\partial x_j} n_{i|(l)} \ \text{ on } \Gamma_{c.}^{kl} \,. \tag{2.21}$$

When $\mathcal{F}_c^{kl} = 0$ then $g_c^{kl} = 0$ and then $\boldsymbol{\tau}_t^k = -\boldsymbol{\tau}_t^l = 0$, and we speak about the case of contact problems without friction. In the case if $s = 1$, i.e. if the second body is approximated by an absolutely rigid material and the frictional forces are equal to zero, then Eqs (2.13), (2.15), (2.16) reduce to

$$u_n \leq 0, \ \ \tau_n \leq 0, \ \ u_n \tau_n = 0 \,. \tag{2.22}$$

In some problems the conditions of symmetry

$$u_n = 0, \ \ \tau_{tj} = 0, \ \ j = 1, \ldots, N \tag{2.23}$$

can be used on the axis (or plane) of symmetry Γ_0.

The amplitude of the Coulombian coefficient of friction is not known, but for the existence of a solution it can be estimated (see [4], [8], [9], [12], for the elastic case and [14], [16] for thermo-elastic case) e.g. for the isotropic case by

$$\| \mathcal{F}_c^{kl} \|_\infty < (\mu/(\lambda+2\mu))^{\frac{1}{4}}\,, \tag{2.24}$$

where λ and μ are the Lamé coefficients. We see that the coefficient of friction depends on the material properties only.

The problem investigated will be coercive if $\Gamma_u^\iota \neq \emptyset$ for all $\iota = 1,\ldots,s$ and semi-coercive if at least one part of $\Gamma_u = \bigcup_{\iota=1}^{s} \Gamma_u^\iota$, say Γ_u^j, is empty. The problem is coupled due to the coupling terms $\frac{\partial}{\partial x_j}(\beta_{ij}^\iota(T^\iota - T_0^\iota))$ following from Eqs (2.1),(2.3) and $\rho^\iota \beta_{ij}^\iota T_0^\iota e_{ij}(\mathbf{u}^{\iota\prime})$ in Eq. (2.1). In the case if the term $\rho^\iota \beta_{ij}^\iota T_0^\iota e_{ij}(\mathbf{u}^{\iota\prime})$ is omitted, then we speak about the **quasi-coupled** model problem.

3 Classical Formulation of the Model Problem

Let $\Omega = \bigcup_{\iota=1}^{s} \Omega^\iota \subset \mathbb{R}^N$, $N = 2,3$, be a union of domains with Lipschitz boundaries $\partial\Omega$, occupied by bodies about which we assume to be elastic. Let the boundary $\partial\Omega$ consist of parts $\Gamma_\tau, \Gamma_u, \Gamma_c, \Gamma_0, \partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c \cup \Gamma_0 \cup \mathcal{R}$, where $\Gamma_c = \bigcup_{k,l} \Gamma_c^{kl}$ represents the boundary between bodies being in contact, and $\mathcal{R}$ is a set of zero surface measure. Moreover, we denote the displacement vector by $\mathbf{u} = (u_1,\ldots,u_N)$. Let $\mathbf{F} \in [L^2(\Omega)]^N$, $\mathbf{P} \in [L^2(\Gamma_\tau)]^N$, and $\mathbf{u}_0 \in [C(\Gamma_u)]^N$. Next we shall deal with the following problem:

Problem $(\mathcal{P})$: Find a pair of function $(T, \mathbf{u})$, a scalar function T and a vector function $\mathbf{u}$, satisfying

$$\frac{\partial}{\partial x_i}\left(\kappa_{ij}^\iota \frac{\partial T^\iota}{\partial x_j}\right) + W^\iota + \rho^\iota \beta_{ij}^\iota T_0^\iota e_{ij}(\mathbf{u}^{\iota\prime}) = 0, \quad \frac{\partial \tau_{ij}(\mathbf{u}^\iota)}{\partial x_j} + F_i^\iota = 0$$

$$i,j = 1,\ldots,N, \quad \iota = 1,\ldots,s \ \text{ in } \Omega^\iota\,, \tag{3.1}$$

$$\tau_{ij}^\iota = c_{ijkl}^\iota e_{kl}(\mathbf{u}^\iota) - \beta_{ij}^\iota (T^\iota - T_0^\iota), \quad i,j,k,l = 1,\ldots,N, \ \iota = 1,\ldots,s\,, \tag{3.2}$$

$$\kappa_{ij}\frac{\partial T}{\partial x_i} n_i = q\ (=0), \quad \tau_{ij}n_j = P_i, \quad i,j = 1,\ldots,N \ \text{ on } \Gamma\,. \tag{3.3}$$

$$\kappa_{ij}\frac{\partial T}{\partial x_i} n_i = q\ (=0), \quad u_i =^1 u_{0i}\ (=0) \ \ i,j = 1,\ldots,N \ \text{ on } {}^1\Gamma_u, \tag{3.4}$$

$$T = T_1\ (=0), \quad u_i =^2 u_{0i}\ (\neq 0), \quad i = 1,\ldots,N, \ \text{ on } {}^2\Gamma_u\,, \tag{3.5}$$

$$T^k = T^l, \quad \kappa_{ij}\frac{\partial T}{\partial x_i} n_{i|(k)} = \kappa_{ij}\frac{\partial T}{\partial x_i} n_{i|(l)} \ \ i,j = 1,\ldots,N \ \text{ on } \Gamma_c^{kl} \tag{3.6}$$

$$u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) \le 0, \quad \tau_n^k(\mathbf{x}) = -\tau_n^l(\mathbf{x}) \equiv \tau_n^{kl}(\mathbf{x}) \le 0,$$

$$(u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}))\tau_n^{kl}(\mathbf{x}) = 0 \ \text{ on } \Gamma_c^{kl} \tag{3.7}$$

$$\text{if} \quad u_n^k - u_n^l = 0 \text{ then } \ |\boldsymbol{\tau}_t^{kl}(\mathbf{x})| \le g_c^{kl}(\mathbf{x}) \ \text{ on } \Gamma_c^{kl}\,, \tag{3.8}$$

$$\text{if} \quad |\boldsymbol{\tau}_t^{kl}(\mathbf{x})| < g_c^{kl}(\mathbf{x}) \ \text{ then } \mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) = 0\,, \tag{3.9}$$

$$\text{if} \quad |\boldsymbol{\tau}_t^{kl}(\mathbf{x})| = g_c^{kl} \text{ then there exists a function } \vartheta \ge 0 \text{ such that}$$

$$\mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) = -\vartheta \boldsymbol{\tau}_t^{kl}(\mathbf{x})\,, \tag{3.10}$$

$$u_n = 0, \quad \tau_{tj} = 0, \quad j = 1,\ldots,N \ \text{ on } \Gamma_0 \tag{3.11}$$

where the last condition (i.e. Eq. (3.11)) is suitable for numerical computation as it represents the condition of symmetry. As we see above the problem is coupled due to the coupling terms $\frac{\partial}{\partial x_j}(\beta_{ij}^\iota(T^\iota - T_0^\iota))$ following from Eqs (3.1), (3.2) and $\rho^\iota \beta_{ij}^\iota T_0^\iota e_{ij}(\mathbf{u}^{\iota\prime})$ in Eq. (3.1) and is quasi-coupled if the term $\rho^\iota \beta_{ij}^\iota T_0^\iota e_{ij}(\mathbf{u}^{\iota\prime})$ in Eq. (3.1) is neglected. The problem is coercive if all ${}^j\Gamma_u^\iota \neq \emptyset$ and semi-coercive for $\Gamma_u = \emptyset$ or if at least one part of $\Gamma_u = \bigcup_{j=1}^{2} \bigcup_{\iota=1}^{s} {}^j\Gamma_u^\iota$, say ${}^2\Gamma_u^m$, is empty.

Definition 3.1 *A pair of functions $(T^\iota, \mathbf{u}^\iota)$ is called a classical solution of the problem $(\mathcal{P})$, if $T^\iota \in C^2(\Omega^\iota) \cap C^1(\Omega^\iota)$, $\mathbf{u}^\iota \in [C^2(\Omega^\iota)]^N \cap [C^1(\Omega^\iota)]^N$ and satisfy Eqs (3.1), (3.2) in every point of Ω, boundary conditions (3.3)-(3.6) in every point of Γ_τ or ${}^1\Gamma_u$ or ${}^2\Gamma_u$, respectively, contact conditions and conditions of the Coulombian law of friction (3.7)-(3.10) on $\cup_{k,l}\Gamma_c^{kl}$ and conditions (3.11) on Γ_0.*

4 Variational (Weak) Solution of the Problem

In the following we shall consider the 2D semi-coercive case with the conditions (3.7)-(3.10) on $\cup\Gamma_c^{kl}$, $k \neq l$ and (3.11) on Γ_0. The generalization to the 3D cases is possible and not so difficult.

Let $\Omega \subset \mathbb{R}^N$, $N = 2$ be a union of domains, occupied by a body, with a Lipschitz boundaries $\partial\Omega^\iota$, consisting of four parts Γ_τ, Γ_u, Γ_c, Γ_0, $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c \cup \Gamma_0 \cup \mathcal{R}$, all defined above. Let $\mathbf{x} = (x_i)$, $i = 1, 2$, be the Cartesian co-ordinates and let $\mathbf{n} = (n_i)$, $\mathbf{t} = (t_i) = (-n_2, n_1)$ be the outward normal and tangential vectors to $\partial\Omega$. Let us look for the temperature $T \in H^1(\Omega)$ and the displacement vector $\mathbf{u} = (u_i) \in \mathbf{W}(\Omega) = [H^1(\Omega)]^2$, where $H^1(\Omega)$ is the Sobolev space in the usual sense. Let $e_{ij}(\mathbf{u})$, $\tau_{ij}(\mathbf{u})$ be the small strain tensor and the stress tensor, respectively, $T_0 = T_0(\mathbf{x})$ the initial temperature at which the bodies are in an initial strain and stress state, $\beta_{ij}^\iota(\mathbf{x}) \in C^1(\overline{\Omega}^\iota)$ the coefficient of thermal expansion, satisfying (2.7) and $\rho^\iota = \rho^\iota(\mathbf{x}) \in C(\overline{\Omega}^\iota)$ the density. Let $W \in L^2(\Omega)$, $F_i \in L^2(\Omega)$ be the heat sources and components of the body forces and $c_{ijkl}^\iota, \kappa_{ij}^\iota \in C^1(\overline{\Omega}^\iota)$ the coefficients of elasticity and of heat conductivity, respectively, satisfying the usual symmetry conditions and the usual ellipticity and continuity conditions, i.e. Eqs (2.5), (2.6), (2.8) and (2.9). Let $g_c^{kl} \in L^2(\Gamma_c^{kl})$ be given slip limits.

Since the problem $(\mathcal{P})$ is static the dissipative term $\rho^\iota \beta_{ij}^\iota T_0^\iota e_{ij}(\mathbf{u}^{\iota\prime})$ can be omitted, so that we shall investigate the 2D quasi-coupled problem. We shall assume that (we adopt the same notation) $F_i^\iota = F_i^\iota - \frac{\partial}{\partial x_j}(\beta_{ij}^\iota(T^\iota - T_0^\iota)) \in L^2(\Omega^\iota)$, $Q^\iota \in L^2(\Omega^\iota)$, $q_0 \in L^2(\Gamma_\tau)$ is the heat flow, $P_i \in L^2(\Gamma_\tau)$, $u_{0i} \in H^{\frac{1}{2}}(\Gamma_u)$, $T_1 \in L^2(\Gamma_\tau)$, $T_0 \in H^1(\Omega)$. Let us denote by $(.,.)$ the scalar product in $[L^2(\Omega)]^2$, by $\langle .,. \rangle$ the scalar product in $[L^2(\Gamma_c)]^2$, by $\| \, . \, \|_k$ the norm in $[H^k(\Omega)]^2$, k being an integer, where $H^k(\Omega)$ denotes the Sobolev space in the usual sense. Let us denote by

$$
\begin{aligned}
{}^1V_0 &= \{z \mid z \in {}^1W \equiv H^1(\Omega^1) \times \cdots \times H^1(\Omega^s),\ z = 0 \text{ on } \cup\ {}^2\Gamma_u\},\\
{}^1V &= \{z \mid z \in {}^1W, z = T_1 \text{ on } \cup\ {}^2\Gamma_u\},\\
V_0 &= \{\mathbf{v} \mid \mathbf{v} \in W \equiv [H^1(\Omega^1)]^2 \times \cdots \times [H^1(\Omega^s)]^2,\ \mathbf{v} = \mathbf{0} \text{ on } {}^1\Gamma_u \cup {}^2\Gamma_u,\ v_n = 0 \text{ on } \Gamma_0\}\\
V &= \{\mathbf{v} \mid \mathbf{v} \in \mathbf{W}, \mathbf{v} = \mathbf{u}_0 \text{ on } {}^1\Gamma_u \cup {}^2\Gamma_u,\ v_n = 0 \text{ on } \Gamma_0\}
\end{aligned}
$$

the spaces and sets of virtual temperatures and virtual displacements, respectively, and by

$$
K = \left\{ \mathbf{v} \mid \mathbf{v} \in V,\ v_n^k - v_n^l \leq 0 \text{ on } \bigcup_{k,l} \Gamma_c^{kl} \right\}
$$

the set of all admissible displacements, which for $\mathbf{u}_0 = 0$ is a convex cone with vertex at the origin.

As our quasi-coupled problem investigated is indeed not coupled, then both the problems in thermics and elasticity can be solved separately (see [14]) and the additional term $\frac{\partial}{\partial x_j}(\beta_{ij}^\iota(T^\iota - T_0^\iota))$ has a meaning of body forces. Since we assume that $\beta_{ij}^\iota \in C^1(\overline{\Omega}^\iota)$, T^ι, $T_0^\iota \in H^1(\Omega^\iota)$, then $\beta_{ij}^\iota(T^\iota - T_0^\iota) \in H^1(\Omega^\iota)$ and therefore $\frac{\partial}{\partial x_j}(\beta_{ij}^\iota(T^\iota - T_0^\iota)) \in L^2(\Omega^\iota)$.

Definition 4.1 *By a variational (weak) solution of the problem $(\mathcal{P})$ we mean a pair of functions $(T, \mathbf{u}), T \in {}^1V$, $\mathbf{u} \in K$, such that*

$$
\begin{aligned}
b(T, z - T) &\geq s(z - T) \quad \forall z \in {}^1V\,, &\quad (4.1)\\
a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \langle g_c^{kl}, |\, \mathbf{v}_t^k - \mathbf{v}_t^l \,| - |\, \mathbf{u}_t^k - \mathbf{u}_t^l \,| \rangle &\geq S(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K\,, &\quad (4.2)
\end{aligned}
$$

where for $T, z \in H^1(\Omega)$, $\mathbf{u}, \mathbf{v} \in \mathbf{W}$ we put

$$
\begin{aligned}
b(T,z) &= \sum_{\iota=1}^{2} b^\iota(T^\iota,z^\iota) = \int_\Omega \kappa_{ij}(\mathbf{x})\frac{\partial T}{\partial x_j}\frac{\partial z}{\partial x_i}d\mathbf{x}\,,\\
s(z) &= \sum_{\iota=1}^{2} s^\iota(z^\iota) = \int_\Omega Qzd\mathbf{x} + \int_{\Gamma_\tau} q_0 z ds\,,\\
a(\mathbf{u},\mathbf{v}) &= \sum_{\iota=1}^{2} a^\iota(\mathbf{u},\mathbf{v}) = \int_\Omega c_{ijkl}e_{ij}(\mathbf{u})e_{kl}(\mathbf{v})d\mathbf{x}\,,\\
S(\mathbf{v}) &= \sum_{\iota=1}^{2} S^\iota(\mathbf{v}^\iota) = \int_\Omega F_i v_i d\mathbf{x} + \int_{\Gamma_\tau} P_i v_i ds\,,\\
j_{gn}(\mathbf{v}) &= \int_{\bigcup_{k,l}\Gamma_c^{kl}} g_c^{kl} \mid \mathbf{v}_t^k - \mathbf{v}_t^l \mid ds = \langle g_c^{kl}, \mid \mathbf{v}_t^k - \mathbf{v}_t^l \mid \rangle\,.
\end{aligned}
$$

The problem (4.1)-(4.2) is equivalent to the following variational formulation:
Find a pair of functions $(T,\mathbf{u})$, $T \in {}^1V$, $\mathbf{u} \in K$, such that

$$
\begin{aligned}
l(T) &\le l(z) \quad \forall z \in {}^1V\,, && (4.3)\\
L(\mathbf{u}) &\le L(\mathbf{v}) \quad \forall \mathbf{v} \in K\,, && (4.4)
\end{aligned}
$$

where $l(z)$, $L(\mathbf{v})$ are defined by

$$
l(z) = \frac{1}{2}b(z,z) - s(z),\;\; L(\mathbf{v}) = L_0(\mathbf{v}) + j_{gn}(\mathbf{v}),\;\; L_0(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - S(\mathbf{v})\,.
$$

It can be proved that any classical solution of the problem $(\mathcal{P})$ is a weak solution and conversely, if the weak solution is smooth enough, it represents a classical solution.

To prove the existence and uniqueness of the variational (weak) solution we introduce the set of all rigid displacements and rotations

$$
P = \sqcap_{\iota=1}^{s} P^\iota,\;\; P^\iota = \{\mathbf{v}^\iota \mid \mathbf{v}^\iota = (v_1^\iota, v_2^\iota), v_1^\iota = a_1^\iota - b^\iota x_2, v_2^\iota = a_2^\iota + b^\iota x_1\}\,,
$$

where $a_i^\iota \in \mathbb{R}^1$, $b^\iota \in \mathbb{R}^1$, $i = 1,2$, $\iota = 1,\dots,s$, are arbitrary and the set of bilateral admissible rigid displacements

$$
P_0 = \{\mathbf{v} \in K \cap P \mid \mathbf{v} \in P_0 \Rightarrow -\mathbf{v} \in P_0\} = \left\{ \mathbf{v} \in P_V = P \cap V \mid v_n^k - v_n^l = 0 \text{ on } \bigcup_{k,l}\Gamma_c^{kl} \right\}.
$$

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^2$, $\mathbf{u}^\iota \in [H^1(\Omega^\iota)]^2$. Then*

$$
e_{ij}(\mathbf{u}^\iota) = 0, \quad \forall i,j = 1,2 \quad \Longleftrightarrow \quad u_1^\iota = a_1^\iota - b^\iota x_2, \quad u_2^\iota = a_2^\iota + b^\iota x_1\,, \tag{4.5}
$$

where a_1^ι , a_2^ι, b^ι are real constants.

The proof is parallel to that of [13].

Lemma 4.2 *Let $\Gamma_u^\iota \neq \emptyset$, $\iota = 1,\dots,s$. Then*

$$
P_V \equiv P \cap V_0 = \{0\}\,,
$$

i.e. only the zero function lies in the intersection $P \cap V_0$.

For the proof see [13],p. 91.

Lemma 4.3 *Since* $e_{ij}(\mathbf{v})=0\ \forall \mathbf{v}\in P,\ \forall i,j,$ *then*

$$a(\mathbf{u},\mathbf{v})=0\ \ \forall \mathbf{v}\in P\,. \tag{4.6}$$

Moreover, if $\mathbf{w}\in\mathbf{W}$, $e_{ij}(\mathbf{w})=0\ \forall i,j$, *then* $\mathbf{w}\in P$.

For the proof see [5].

Lemma 4.4 *Let there exist a weak solution of the problem* $(\mathcal{P})$. *Then*

$$S(\mathbf{w})\le j_{gn}(\mathbf{w})\ \ \forall \mathbf{w}\in K\cap P,\ \text{ i.e. }\ \int_\Omega F_i w_i d\mathbf{x}+\int_{\Gamma_\tau} P_i w_i ds-\int_{\bigcup\limits_{k,l}\Gamma_c^{kl}} g_c^{kl}\mid \mathbf{w}_t^k-\mathbf{w}_t^l\mid ds\le 0\ \ \forall \mathbf{w}\in K\cap P\,. \tag{4.7}$$

Proof. Since the problem is quasi-coupled, it is sufficient to investigate the elastic part of the problem only. Since the weak solution $\mathbf{u}$ satisfies

$$a(\mathbf{u},\mathbf{v}-\mathbf{u})-S(\mathbf{v}-\mathbf{u})+j_{gn}(\mathbf{v})-j_{gn}(\mathbf{u})\ge 0\quad \forall \mathbf{v}\in K\,,$$

then putting $\mathbf{v}=\mathbf{u}+\mathbf{w}$, $\mathbf{v}-\mathbf{u}=\mathbf{w}\in K\cap P$, we find that $\mathbf{v}\in K$ and

$$a(\mathbf{u},\mathbf{u}+\mathbf{w}-\mathbf{u})-S(\mathbf{u}+\mathbf{w}-\mathbf{u})+j_{gn}(\mathbf{u}+\mathbf{w})-j_{gn}(\mathbf{u})\ge 0\,.$$

Hence

$$\begin{aligned} a(\mathbf{u},\mathbf{w})-S(\mathbf{w}) &\ge \langle g_c^{kl},\mid \mathbf{u}_t^k-\mathbf{u}_t^l\mid-\mid \mathbf{v}_t^k-\mathbf{v}_t^l\mid\rangle=\langle g_c^{kl},\mid \mathbf{u}_t^k-\mathbf{u}_t^l\mid-\mid(\mathbf{u}+\mathbf{w})_t^k-(\mathbf{u}+\mathbf{w})_t^l\mid\rangle=\\ &= \langle g_c^{kl},\mid \mathbf{u}_t^k-\mathbf{u}_t^l\mid-\mid(\mathbf{u}_t^k-\mathbf{u}_t^l)+(\mathbf{w}_t^k-\mathbf{w}_t^l)\mid\rangle\,. \end{aligned}$$

Due to Lemma 4.3 and since $\mid a+b\mid\le\mid a\mid+\mid b\mid$, i.e. $\mid a+b\mid-\mid a\mid\le\mid b\mid$, then $S(\mathbf{w})\le\langle g_c^{kl},|\mathbf{w}_t^k-\mathbf{w}_t^l|\rangle$, which completes the proof.

Remark 4.1 *The Eq. (4.7) represents the condition of the total equilibrium and is the necessary condition for the existence of the solution.*

Lemma 4.5 *Any classical solution of the problem* $(\mathcal{P})$ *is a weak solution. On the other hand, if the weak solution is smooth enough, it is a classical solution.*

The proof is a modification of that of [4], [8], [16].

Theorem 4.1 *Assume that*

$$P_0=\{0\},\ P\cap K\ne\{0\} \tag{4.8}$$

and

$$S(\mathbf{w})<j_{gn}(\mathbf{w})\quad \mathbf{w}\in P\cap K\backslash\{0\}\,. \tag{4.9}$$

Then L *is coercive on* K *and there exists a weak solution of the problem* $(\mathcal{P})$. *If*

$$\mid S(\mathbf{w})\mid> j_{gn}(\mathbf{w})\quad \forall \mathbf{w}\in P_V\backslash\{0\} \tag{4.10}$$

then the solution is unique. If

$$\mid S(\mathbf{w})\mid\le j_{gn}(\mathbf{w})\quad \forall \mathbf{w}\in P_V$$

then for any two solutions $\mathbf{u}$, $\mathbf{u}^*$

$$\mathbf{w}\equiv\mathbf{u}^*-\mathbf{u}\in P_V,\quad S(\mathbf{w})=j_{gn}(\mathbf{u}^*)-j_{gn}(\mathbf{u}) \tag{4.11}$$

holds.

Proof. It suffices to consider the elastic part of the problem $(\mathcal{P})$. We shall use a slight generalization of Theorem 1.4 of [13], Chapter 13.

Let $|\,\mathbf{u}\,|$ be a seminorm in a Hilbert space H with a norm $\|\mathbf{u}\|$. Let us define a subspace

$$\mathfrak{R} = \{\mathbf{u} \in H \mid |\,\mathbf{u}\,| = 0\}\,.$$

Assume that $\dim \mathfrak{R} < \infty$ and

$$c_1\|\mathbf{u}\| \leq |\,\mathbf{u}\,| + \|\Pi_{\mathfrak{R}}\mathbf{u}\| \leq c_2\|\mathbf{u}\| \quad \forall \mathbf{u} \in H\,, \tag{4.12}$$

where $\Pi_{\mathfrak{R}}$ denotes the orthogonal projection to $\mathfrak{R}$.

Let K be a closed convex subset of H containing the origin and $K \cap \mathfrak{R} \neq \{0\}$. Let $\beta : H \to \mathbb{R}^1$ be a penalty functional whose Gâteaux differential satisfies

$$D\beta(t\mathbf{u}, \mathbf{v}) = tD\beta(\mathbf{u}, \mathbf{v}) \quad \forall t \in \mathbb{R}^1\,, \quad t > 0\,, \quad \mathbf{u}, \mathbf{v} \in H\,,$$

and let $\beta(\mathbf{u}) \leq 0 \; \forall \mathbf{u} \in H$,

$$\beta(\mathbf{u}) = 0 \Longleftrightarrow \mathbf{u} \in K\,,$$

Let f be a linear continuous functional on H and $j : H \to \mathbb{R}^1$ a continuous functional such that

$$j(\mathbf{v}) \geq 0,\; j(t\mathbf{v}) = tj(\mathbf{v}) \quad \forall t \in \mathbb{R}^1,\; t > 0,\; \mathbf{v} \in H\,.$$

Assume that

$$f(\mathbf{w}) < j(\mathbf{w}) \quad \forall \mathbf{w} \in K \cap \mathfrak{R} \backslash \{0\}\,.$$

Then there exist positive constants c_1, c_2 such that

$$|\,\mathbf{u}\,|^2 + \beta(\mathbf{u}) + j(\mathbf{u}) - f(\mathbf{u}) \geq c_1\|\mathbf{u}\| - \mathbf{c}_2 \quad \forall \mathbf{u} \in H\,. \blacksquare \tag{4.13}$$

Let us set $H = V$, $\mathfrak{R} = P_V = P \cap V_0$,

$$|\,\mathbf{v}\,| = \left[\frac{1}{2}a(\mathbf{v}, \mathbf{v})\right]^{\frac{1}{2}}\,,\; j = j_{gn}\,,\; f = S\,,\; \beta(\mathbf{v}) = \frac{1}{2}\sum_{k,l}\int_{\Gamma_c^{kl}} [(v_n^k - v_n^l)^+]^2 ds\,.$$

In order to verify (4.12), we use the Korn's inequality and the decomposition

$$V = Q \oplus P_V\,.$$

Thus we obtain for all $\mathbf{v} \in V$ the inequality

$$\|\mathbf{v}\|^2 = \|\Pi_Q\mathbf{v}\|^2 + \|\Pi_{\mathfrak{R}}\|^2 \leq c\,|\,\Pi_Q\mathbf{v}\,|^2 + \|\Pi_{P_V}\|^2 = c\,|\,\mathbf{v}\,|^2 + \|\Pi_{P_V}\|^2\,,$$

where also Lemma 4.3 has been used. The right-hand side of (4.12) is obvious.

Now (4.13) implies

$$\frac{1}{2}a(\mathbf{v}, \mathbf{v}) + j_{gn}(\mathbf{v}) - S(\mathbf{v}) \geq c_1\|\mathbf{v}\| - c_2 \quad \forall \mathbf{v} \in K\,,$$

so that the functional L is coercive on the set K. Since L is convex and continuous on H, it is weakly lower semi-continuous and the existence of a solution follows.

Next, let $\mathbf{u}, \mathbf{u}^*$ be two solutions of the variational inequality. We may write

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}^* - \mathbf{u}) + j_{gn}(\mathbf{u}^*) - j_{gn}(\mathbf{u}) &\geq S(\mathbf{u}^* - \mathbf{u})\,, \\ a(\mathbf{u}^*, \mathbf{u} - \mathbf{u}^*) + j_{gn}(\mathbf{u}) - j_{gn}(\mathbf{u}^*) &\geq S(\mathbf{u} - \mathbf{u}^*)\,. \end{aligned}$$

Summing these inequalities, we obtain

$$a(\mathbf{u}-\mathbf{u}^*,\mathbf{u}^*-\mathbf{u})\geq 0\,,$$

so that denoting $\mathbf{w}=\mathbf{u}^*-\mathbf{u}$, we have $a(\mathbf{w},\mathbf{w})\leq 0$. As a consequence $\mathbf{w}\in P_V$ follows.

Next, we have $\mathbf{u}^*=\mathbf{u}+\mathbf{w}$,

$$L(\mathbf{u})=L(\mathbf{u}+\mathbf{w})$$

and

$$j_{gn}(\mathbf{u})=-S(\mathbf{w})+j_{gn}(\mathbf{u}+\mathbf{w}) \tag{4.14}$$

follows by inserting $a(\mathbf{u}+\mathbf{w},\mathbf{u}+\mathbf{w})=a(\mathbf{u},\mathbf{u})$. Arguing like in the proof of Lemma 4.4, we obtain

$$|\, j_{gn}(\mathbf{u}+\mathbf{w})-j_{gn}(\mathbf{u})\,|\leq j_{gn}(\mathbf{w})\,.$$

Inserting from (4.14), we arrive at

$$|\, S(\mathbf{w})\,|\leq j_{gn}(\mathbf{w})\,. \tag{4.15}$$

Now (4.10) yields that $\mathbf{w}=0$ and the uniqueness follows. The rest of the theorem is a consequence of (4.14).

Remark 4.2 *An example, satisfying $P_0=\{0\}$ and $P\cap K\neq\{0\}$ is on Fig. 1, where $s=2$. Let Γ_0 be parallel with the x_1-axis. (Note that $u_n=0$ is prescribed on Γ_0). Then*

$$P_V=P\cap V=\{\mathbf{v}=(v_1,v_2)\mid \mathbf{v}^1=0,\, v_1^2=a,\, v_2^2=0\}\,,$$

where $a\in R^1$ is arbitrary. If

$$\mathcal{V}_1^2=\int_{\Omega^2}F_1^2d\mathbf{x}+\int_{\Gamma_\tau}P_1^2ds$$

denotes the component of a resultant of external forces, then

$$S(\mathbf{w})=a\mathcal{V}_1^2,\qquad j_{gn}=|\,a\,|\,G\,,$$

where $\mathbf{w}\in P_V$ and $G=\int_{\Gamma_c^{12}}g_c^{12}\mid n_2\mid ds$.

The condition (4.9) is equivalent to $\mathcal{V}_1^2>-G$, since $P\cap K\subset P_V$ with arbitrary $a\leq 0$ and (4.10) is equivalent to $|\,\mathcal{V}_1^2\,|>G$. Combining these conditions, we conclude that if $\mathcal{V}_1^2>G$, there exists a unique solution. If $-G<\mathcal{V}_1^2\leq G$, there exists at least one solution. Any two solutions $\mathbf{u}$ and $\mathbf{u}^*$ differ by a "shift" $\mathbf{w}=\mathbf{u}^*-\mathbf{u}\in P_V$ and its "value" is

$$a=(\mathcal{V}_1^2)^{-1}\int_{\Gamma_c^{12}}g_c^{12}(|\,\mathbf{u}_t^{*1}-\mathbf{u}_t^{*2}\,|-|\,\mathbf{u}_t^1-\mathbf{u}_t^2\,|)ds\,,$$

provided $\mathcal{V}_1^2\neq 0$. If $\mathcal{V}_1^2=0$, the "value" of the "shift" is arbitrary.

5 Finite Element Solution of the Problem

Let the domain $\Omega\subset\mathbb{R}^N$ be a bounded domain and let it be approximated by a polygonal (for $N=2$) or polyhedral (for $N=3$) domain Ω_h. Let the domain Ω_h be "triangulated", i.e. the domain $\overline{\Omega}_h=\Omega_h\cup\partial\Omega_h$ is divided into a system of m triangles T_{h_i} in the 2D case and into a system of m tetrahedra T_{h_i} in the 3D case, generating a triangulation $\mathcal{T}_h$ such that $\overline{\Omega}_h=\bigcup_{i=1}^m T_{h_i}$ and such that two neighbouring triangles have only a vertex or an entire side common in the 2D case, and that two neighbouring tertrahedra have only a vertex or an entire edge or an entire face common in the 3D case. Denote by $h=\max_{1\leq i\leq m}(\text{diam } T_{h_i})$ the maximal side of the triangle T_{h_i} in the 2D case and/or the maximal edge of the tetrahedron in the 3D case in $\mathcal{T}_h$. Let ρ_{T_i} denote the radius of the maximal circle (for 2D case) or maximal ball (for 3D case), inscribed in the simplex T_{h_i}. A family of triangulation $\{\mathcal{T}_h\}$, $0 < h \leq h_0 < \infty$, is said to be regular if there exists a constant $\vartheta_0 > 0$ independent of h and such that $h/\rho_{T_i} \leq \vartheta_0$ for all $h \in (0, h_0)$. We will assume that the sets $\overline{\Gamma}_u \cap \overline{\Gamma}_\tau$, $\overline{\Gamma}_u \cap \overline{\Gamma}_c$, $\overline{\Gamma}_u \cap \overline{\Gamma}_0$, $\overline{\Gamma}_c \cap \overline{\Gamma}_\tau$, $\overline{\Gamma}_c \cap \overline{\Gamma}_0$, $\overline{\Gamma}_\tau \cap \overline{\Gamma}_0$ coincide with the vertices or edges of T_{h_i}.

Let $R_i \in \Omega_h$ be an arbitrary interior vertex of the triangulation $\mathcal{T}_h$. Generally the basis function w_h^i (where w_h^i is a scalar or vector function) is defined to be a function linear on each element $T_{h_i} \in \mathcal{T}_h$ and taking the values $w_h^i(R_j) = \delta_{ij}$ at the vertices of the triangulation, where δ_{ij} is the Kronecker symbol. The function w_h^i represents a pyramid of height 1 with its vertex above the point R_i and with its support (supp w_h^i) consisting of those triangles or tetrahedra which have the vertex R_i in common. The basis function has small support since diam(supp w_h^i) $\leq 2h$ and the parameter $h \to 0$. Further, for simplicity, we shall discuss the 2D case only.

Let us assume that $N = 2$. Let ${}^1V_h$ and V_h be the spaces of linear finite elements, i.e. the spaces of continuous scalar and/or vector functions in $\overline{\Omega}_h$, piecewise linear over $\mathcal{T}_h$, i.e

$$\begin{aligned} {}^1V_h &= \{z \in C(\overline{\Omega}_h^1) \times \cdots \times C(\overline{\Omega}_h^s) \cap {}^1V \mid z_{|T_{h_i}} \in P_1 \, \forall T_{h_i} \in \mathcal{T}_h\}, \\ V_h &= \{\mathbf{v} \in [C(\overline{\Omega}_h^1)]^2 \times \cdots \times [C(\overline{\Omega}_h^s)]^2 \cap V \mid \mathbf{v}_{|T_{h_i}} \in [P_1]^2 \, \forall T_{h_i} \in \mathcal{T}_h\} \end{aligned}$$

and

$$K_h = \{\mathbf{v} \in V_h \mid v_n^k - v_n^k \leq 0 \text{ on } \cup \Gamma_c^{kl}\} = K \cap V_h .$$

Definition 5.1 *A pair of functions* $(T_h, \mathbf{u}_h)$, $T_h \in {}^1V_h$, $\mathbf{u}_h \in K_h$, *is said to be a finite element approximation of the problem* $(\mathcal{P})$, *if*

$$l(T_h) \leq l(z) \qquad \forall z \in {}^1V_h , \tag{5.1}$$

$$L(\mathbf{u}_h) \leq L(\mathbf{v}) \qquad \forall \mathbf{v} \in K_h . \tag{5.2}$$

To find an a priori estimate for the error of the solution $(T_h - T^*, \mathbf{u}_h - \mathbf{u}^*)$, a modification of the Falk's technique (see e.g. [15]) will be used. Since the problem is quasi-coupled the method will be based on the following lemma.

Lemma 5.1 *Let* $|\,.\,|$ *be the semi-norm defined by*

$$|\, \mathbf{v} \,|^2 = \int_{\bigcup_{\iota=1}^{s} \Omega_h^\iota} e_{ij}(\mathbf{v}) e_{ij}(\mathbf{v}) d\mathbf{x} . \tag{5.3}$$

Then it holds

$$\begin{aligned} c_0 \,|\, \mathbf{u} - \mathbf{u}_h \,|^2 &\leq a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \leq \\ &\leq \{a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h - \mathbf{u}) + a(\mathbf{u}, \mathbf{v} - \mathbf{u}_h) + a(\mathbf{u}, \mathbf{v}_h - \mathbf{u}) + j_{gn}(\mathbf{v}_h) - j_{gn}(\mathbf{u}) + j_{gn}(\mathbf{v}) - \\ &\quad - j_{gn}(\mathbf{u}_h) + S(\mathbf{u} - \mathbf{v}_h) + S(\mathbf{u}_h - \mathbf{v})\} \end{aligned} \tag{5.4}$$

for any $\mathbf{v} \in K$, $\mathbf{v}_h \in K_h$, $c_0 = \text{const.} > 0$.

Proof. The proof follows from the conditions

$$\begin{aligned} a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \langle g_c^{kl}, |\, \mathbf{v}_t^k - \mathbf{v}_t^l \,| - |\, \mathbf{u}_t^k - \mathbf{u}_t^l \,| \rangle - S(\mathbf{v} - \mathbf{u}) \geq 0 \qquad & \forall \mathbf{v} \in K , \\ a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \langle g_c^{kl}, |\, (\mathbf{v}_h^k)_t - (\mathbf{v}_h^l)_t \,| - |\, (\mathbf{u}_h^k)_t - (\mathbf{u}_h^l)_t \,| \rangle - S(\mathbf{v}_h - \mathbf{u}_h) \geq 0 \qquad & \forall \mathbf{v}_h \in K_h . \end{aligned}$$

Adding these inequalities, adding and subtracting the term $a(\mathbf{u}, \mathbf{u}_h) - a(\mathbf{u}_h, \mathbf{u})$ to the resulting inequality and performing some modifications, then we obtain

$$a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h) \le a(\mathbf{u}_h-\mathbf{u},\mathbf{v}_h-\mathbf{u})+a(\mathbf{u},\mathbf{v}-\mathbf{u}_h)+a(\mathbf{u},\mathbf{v}_h-\mathbf{u})-S(\mathbf{v}-\mathbf{u}_h)-S(\mathbf{v}_h-\mathbf{u})+ +j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u})+j_{gn}(\mathbf{v})-j_{gn}(\mathbf{u}_h)\,.$$

Hence (5.4) follows immediately.

Corollary 5.1 *Let $K_h \subset K$. Then substituting $\mathbf{v}=\mathbf{u}_h$ in (5.4) we have*

$$\mid \mathbf{u}-\mathbf{u}_h \mid \le c\{a(\mathbf{u}_h-\mathbf{u},\mathbf{v}_h-\mathbf{u})+a(\mathbf{u},\mathbf{v}_h-\mathbf{u})+j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u})-S(\mathbf{v}_h-\mathbf{u})\}^{\frac{1}{2}} \ \ \forall \mathbf{v}_h \in K_h,\ c=\text{const.}>0\,. \tag{5.5}$$

Theorem 5.1 *Let $\partial\Omega$ and its parts Γ_u, Γ_τ, Γ_c, Γ_0 be piecewise polygonal. Let $T^\iota \in H^2(\Omega^\iota)$, $\mathbf{u}^\iota \in [H^2(\Omega^\iota)]^2$, $\iota=1,\dots,s$, be the solutions of (4.3)-(4.4) and such that the stress components $\tau_{ij}(\mathbf{u}^\iota) \in H^1(\Omega^\iota)$, $i,j=1,2$, $\iota=1,\dots,s$, $\tau_n^{kl} \in L^\infty(\Gamma_c^{kl})$, $u_n^{k(l)} \in H^2(\Gamma_c \cap \Gamma_{ci}^{kl})$ holds for any of the boundary $\partial\Omega$, $g_c^{kl} \in L^\infty(\Gamma_c^{kl})$. Let ${}^1V_h \subset {}^1V$, $K_h \subset K$ and let the changes $u_n^k-u_n^l<0 \to u_n^k-u_n^l=0$ and $\mathbf{u}_t^k-\mathbf{u}_t^l=0 \to \mathbf{u}_t^k-\mathbf{u}_t^l \neq 0$ occur at finately many points of $\bigcup_{k,l}\Gamma_c^{kl}$ only. Then*

$$\|T-T_h\|_1 = O(h), \qquad \mid \mathbf{u}-\mathbf{u}_h \mid = O(h)\,. \tag{5.6}$$

Proof. Since the problem is quasi-coupled we will analyze both parts of the problem separately ([14], [16]). For the second part of the problem the technique of the proof is based on a generalization of results of [4], [7], [14], [16].

(i) Thermal part of the problem: Assume that $T_1=0$. The approximate solution $T_h \in {}^1V_h$ minimizes the functional $l(z)$ over ${}^1V_h$, i.e. $l(T_h)=\min_{z\in {}^1V_h} l(z)$. Since $b(T,z)=s(z)$ for any $z \in {}^1V_h$, then $b(T,T)=s(T)$ and therefore $l(z)-l(T)=\min_{z\in {}^1V_h}\frac{1}{2}b(T-z,T-z)$. Moreover,

$$\min_{z\in {}^1V_h} l(z)-l(T) = \min_{z\in {}^1V_h}\frac{1}{2}b(T-z,T-z)\,,$$

so that

$$b(T-T_h,T-T_h) \le b(T-z,T-z) \qquad \forall z \in {}^1V_h\,.$$

But the bilinear form $b(T,z)$ is continuous on $H^1(\Omega^1)\times H^1(\Omega^1)\times\cdots\times H^1(\Omega^s)\times H^1(\Omega^s)$ and bounded, i.e. $\mid b(T^\iota,z)\mid \le M^\iota\|T^\iota\|\|z\|$ $\forall T^\iota$, $z\in H^1(\Omega^\iota)$, $M^\iota=$ const. independent of T^ι,z and for all ι, and V-elliptic, i.e. $b(z,z)\ge c\|z\|^2$ $\forall z\in {}^1V$, $c=$ const. >0, then we obtain

$$\|T-T_h\|_1 \le (M/c)^{\frac{1}{2}}\|T-z\|_1 \qquad \forall z \in {}^1V_h\,. \tag{5.7}$$

The interpolation theorem yields (see [1])

$$\|z-z_h^n\|_1 \le c_1 M_{n+1}h^n, \quad \text{if } z\in H^{n+1}(\Omega)\,, \tag{5.8}$$

where z_h^n is a piecewise polynomial function of n-th degree on every triangle of the given triangulation $\mathcal{T}_h$. Further, we shall assume that $n=1$.

Let us put $z=T_h$, $z_h^1=T_h^1\in {}^1V_h$, that is the function which on every triangle of the given triangulation is equal to the polynomial interpolating of the exact solution T. Then (5.7) and (5.8) yield

$$\|T-T_h\|_1 \le (M/c)^{\frac{1}{2}}ch = O(h)\,,$$

which completes the thermal part of the proof.

(ii) Elastic part of the problem:
Using Corollary 5.1 we will estimate (5.5). This estimate can be applied, provided the solution $\mathbf{u}$ is sufficiently regular. Green's theorem implies

$$\begin{aligned} a(\mathbf{u},\mathbf{v}_h-\mathbf{u})-S(\mathbf{v}_h-\mathbf{u})+j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u}) &= \int_\Omega -\frac{\partial}{\partial x_j}(c_{ijkl}e_{kl}(\mathbf{u}))(\mathbf{v}_h-\mathbf{u})_i d\mathbf{x} + \\ &+\int_{\partial\Omega}\tau_{ij}n_j(\mathbf{v}_h-\mathbf{u})_i ds - \int_{\Gamma_\tau} P_i(\mathbf{v}_h-\mathbf{u})_i ds - \\ &-\int_\Omega F_i(\mathbf{v}_h-\mathbf{u})_i d\mathbf{x} + j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u}) = \\ &= \int_\Omega -\frac{\partial}{\partial x_j}(c_{ijkl}e_{kl}(\mathbf{u}))(\mathbf{v}_h-\mathbf{u})_i d\mathbf{x} + \\ &+\int_{\Gamma_\tau}\tau_{ij}n_j(\mathbf{v}_h-\mathbf{u})_i ds + \int_{\Gamma_u}\tau_{ij}n_j(\mathbf{v}_h-\mathbf{u})_i ds + \\ &+\int_{\cup\Gamma_c^{kl}}\tau_{ij}n_j(\mathbf{v}_h-\mathbf{u})_i ds - \int_{\Gamma_\tau} P_i(\mathbf{v}_h-\mathbf{u})_i ds - \\ &-\int_\Omega F_i(\mathbf{v}_h-\mathbf{u})_i d\mathbf{x} + j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u}) = \\ &= \int_{\cup\Gamma_c^{kl}}\tau_{ij}n_j(\mathbf{v}_h-\mathbf{u})_i ds + j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u}) = \\ &= \int_{\cup\Gamma_c^{kl}}\tau_n(\mathbf{u})(\mathbf{v}_h-\mathbf{u})_n ds + \int_{\cup\Gamma_c^{kl}}\boldsymbol{\tau}_t(\mathbf{u})(\mathbf{v}_h-\mathbf{u})_t ds + \\ &+j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u}) \geq 0, \ \ \forall \mathbf{v}_h \in K_h\,, \end{aligned}$$

since

$$-\frac{\partial}{\partial x_j}(c_{ijkl}e_{kl}(\mathbf{u})) = F_i, \quad i=1,2, \quad \text{a.e. in } \Omega\,.$$

In virtue of Corollary 5.1 we have

$$|\,\mathbf{u}-\mathbf{u}_h\,| \leq c_0\{a(\mathbf{u}_h-\mathbf{u},\mathbf{v}_h-\mathbf{u})+a(\mathbf{u},\mathbf{v}_h-\mathbf{u})-S(\mathbf{v}_h-\mathbf{u})+j_{gn}(\mathbf{v}_h)-j_{gn}(\mathbf{u})\}^{\frac{1}{2}}, \quad \forall \mathbf{v}\in K_h\,.$$

The bilinear form $a(.,.)$ is bounded, so that

$$a(\mathbf{u}_h-\mathbf{u},\mathbf{v}_h-\mathbf{u}) \leq M\,|\,\mathbf{u}_h-\mathbf{u}\,||\,\mathbf{v}_h-\mathbf{u}\,|\ .$$

Since for every $\mathbf{v}\in[C(\overline{\Omega})]^2\cap K$ the linear interpolate over $\mathcal{T}_h$ $\mathbf{v}_{LI}\in K_h$ and since the last inequality is valid for any arbitrary $\mathbf{v}_h\in K_h$, it is valid also for $\mathbf{v}_h=\mathbf{u}_{LI}$. As $ab\leq \frac{1}{2}\varepsilon a^2+\frac{1}{2\varepsilon}b^2$, $a,b\in\mathbb{R}^1$, $\varepsilon>0$, then applying the interpolation theorem we obtain

$$a(\mathbf{u}_h-\mathbf{u},\mathbf{v}_h-\mathbf{u}) \leq \frac{1}{2}\varepsilon M\,|\,\mathbf{u}_h-\mathbf{u}\,|^2+\frac{1}{2\varepsilon}M\,|\,\mathbf{v}_h-\mathbf{u}\,|^2 \leq \frac{1}{2}\varepsilon M\,|\,\mathbf{u}_h-\mathbf{u}\,|^2+c\varepsilon^{-1}h^2\|\mathbf{u}\|_2^2\,.$$

To estimate the integrals

$$J_1\left(\bigcup_{k,l}\Gamma_c^{kl}\right) = \int_{\bigcup\limits_{k,l}\Gamma_c^{kl}} \tau_n^{kl}[(v_h^k-v_h^l)_n-(u_n^k-u_n^l)]ds$$

and

$$J_2\left(\bigcup_{k,l}\Gamma_c^{kl}\right) = \int_{\bigcup_{k,l}\Gamma_c^{kl}} \boldsymbol{\tau}_t^{kl}[(\mathbf{v}_h^k - \mathbf{v}_h^l)_t - (\mathbf{u}_t^k - \mathbf{u}_t^l)]ds\,,$$

we assume that $\Gamma_c = \bigcup_{k,l}\Gamma_c^{kl} = \bigcup_{k,l}\left(\bigcup_i \Gamma_{ci}^{kl}\right)$, where Γ_{ci}^{kl} approximate piecewise linearly the boundary Γ_c^{kl}.

Let us put $\mathbf{v}_h = \mathbf{u}_{LI}$ in what follows.

To estimate the integral $J_1\left(\cup\overline{\Gamma}_c^{kl}\right)$ we distinguish the following cases:

(i) $u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) < 0$, $\mathbf{x} \in \overline{\Gamma}_{ci}^{kl}$: since $(u_n^k - u_n^l)\tau_n^{kl} = 0$, then $J_1(\overline{\Gamma}_{ci}^{kl}) = 0$,

(ii) $u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) = 0\ \forall \mathbf{x} \in \Gamma_{ci}^{kl}$. Since then also $(u_{LI}^k)_n - (u_{LI}^l)_n = 0$, we have $J_1(\overline{\Gamma}_{ci}^{kl}) = 0$.

(iii) Γ_{ci}^{kl} contains precisely one point of the change $u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) < 0 \to u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}) = 0$. Then

$$\mid J_1(\overline{\Gamma}_{ci}^{kl}) \mid \leq \|(u_{LI}^k(\mathbf{x}))_n - (u_{LI}^l(\mathbf{x}))_n - (u_n^k(\mathbf{x}) - u_n^l(\mathbf{x}))\|_{L^\infty(\overline{\Gamma}_{ci}^{kl})} \int_{\overline{\Gamma}_{ci}^{kl}} \mid \tau_n^{kl} \mid ds \leq c_{si}h^2$$

holds by virtue of the assumptions and the interpolation theory. As the case (iii) may occur at finite number of subintervals Γ_{ci}^{kl} only, we conclude that

$$\mid J_1(\overline{\Gamma}_{ci}^{kl}) \mid \leq c_s h^2\,.$$

To estimate the integral $J_2(\cup\overline{\Gamma}_{ci}^{kl})$ we distinguish the following cases:

(i) $\mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) > \mathbf{0}$ for $\mathbf{x} \in \Gamma_{ci}^{kl}$: It holds

$$J_1\left(\bigcup_{k,l}\Gamma_c^{kl}\right) + J_2\left(\bigcup_{k,l}\Gamma_c^{kl}\right) + j_{gn}(\mathbf{v}) - j_{gn}(\mathbf{u}) \geq 0\,,$$

and $g_c^{kl} \mid \mathbf{u}_t^k - \mathbf{u}_t^l \mid + \boldsymbol{\tau}_t^{kl}(\mathbf{u}_t^k - \mathbf{u}_t^l) = 0$ a.e. on $\bigcup_{k,l}\Gamma_c^{kl}$. Hence $g_c^{kl} = -\boldsymbol{\tau}_t^{kl}$ a.e. on $\bigcup_{k,l}\overline{\Gamma}_c^{kl}$. Putting $\mathbf{v}_h = \mathbf{u}_{LI}$, then $(\mathbf{u}_{LI}^k - \mathbf{u}_{LI}^l)_t = (\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI}$, where $(\mathbf{u}_{LI}^k - \mathbf{u}_{LI}^l)_t(R_i)$, $R_i \in \cup\Gamma_{ci}^{kl}$, R_i are the points of the triangulation on $\cup_{k,l}\Gamma_c^{kl}$. Then $(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} > 0$ on $\cup\overline{\Gamma}_{ci}^{kl}$ and

$$\int_{\cup\Gamma_{ci}^{kl}} \{-g_c^{kl}[(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)] + g_c^{kl}[(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)]\}ds = 0\,.$$

(ii) $\mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma_{ci}^{kl}$: since $(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} = 0$ on $\cup\overline{\Gamma}_{ci}^{kl}$, then

$$\int_{\cup\Gamma_{ci}^{kl}} \{\boldsymbol{\tau}_t^{kl}(\mathbf{u})[(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)] + g_c^{kl}[\mid (\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} \mid + \mid \mathbf{u}_t^k - \mathbf{u}_t^l \mid]\}ds = 0\,.$$

(iii) $\mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) < 0$ for $\mathbf{x} \in \Gamma_{ci}^{kl}$: As $\mathbf{u}_t^k - \mathbf{u}_t^l < 0$ on $\in \Gamma_{ci}^{kl}$, then $\mid \mathbf{u}_t^k - \mathbf{u}_t^l \mid = -(\mathbf{u}_t^k - \mathbf{u}_t^l)$ and $(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} < 0$. Thus $g_c^{kl} = \boldsymbol{\tau}_t^{kl}$ a.e. on $\overline{\Gamma}_{ci}^{kl}$. Putting $\mathbf{v}_h = \mathbf{u}_{LI}$, then

$$\int_{\cup\overline{\Gamma}_{ci}^{kl}} \{g_c^{kl}[(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)] + g_c^{kl}[-(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} + (\mathbf{u}_t^k - \mathbf{u}_t^l)]\}ds = 0\,.$$

(iv) $\mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) = 0$ changes to $\mathbf{u}_t^k(\mathbf{x}) - \mathbf{u}_t^l(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \Gamma_c^{kl}$: Let $\mathbf{v}_h = \mathbf{u}_{LI}$, then, since $\mathbf{u} \in [W^{1,\infty}(\cup\overline{\Gamma}_{ci}^{kl})]^2$, $\boldsymbol{\tau} \in [L^\infty(\cup\overline{\Gamma}_{ci}^{kl})]^2$, $g_c^{kl} \in L^\infty(\cup\overline{\Gamma}_{ci}^{kl})$, $(\mathbf{u}_t^k - \mathbf{u}_t^l) \in [W^{1,\infty}(\cup\overline{\Gamma}_{ci}^{kl})]^2$ and $\boldsymbol{\tau}_t^{kl} \in [L^\infty(\cup\overline{\Gamma}_{ci}^{kl})]^2$. Then

$$\int_{\cup\overline{\Gamma}_{ci}^{kl}} \{\boldsymbol{\tau}_t^{kl}(\mathbf{u})[(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)] + g_c^{kl}[\mid (\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} \mid - \mid \mathbf{u}_t^k - \mathbf{u}_t^l \mid]\} ds$$
$$\leq \|(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)\|_{L^\infty(\cup\overline{\Gamma}_{ci}^{kl})} \int_{\cup\overline{\Gamma}_{ci}^{kl}} (\mid \boldsymbol{\tau}_t^{kl}(\mathbf{u}) \mid + g_c^{kl}) ds \leq c_{ri} h^2 .$$

Thus

$$\left| \int_{\cup\overline{\Gamma}_{ci}^{kl}} \{\boldsymbol{\tau}_t^{kl}(\mathbf{u})[(\mathbf{u}_t^k - \mathbf{u}_t^l)_{LI} - (\mathbf{u}_t^k - \mathbf{u}_t^l)] ds + j_{gn}(\mathbf{u}_{LI}) - j_{gn}(\mathbf{u}) \right| \leq c_r h^2 .$$

Using (5.5) and the above estimates then

$$\mid \mathbf{u} - \mathbf{u}_h \mid \leq c_0 \left\{ \frac{1}{2} \varepsilon \mid \mathbf{u} - \mathbf{u}_h \mid^2 + \varepsilon^{-1} h^2 \|\mathbf{u}\|_2^2 + c_s h^2 + c_r h^2 \right\}^{\frac{1}{2}} ,$$

and (5.6) follows, choosing ε sufficiently small.

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Bibliography

[1] Bramble,J.H., Zlámal, M. (1970). Triangular elements in the finite method. Math. Comput. 24,809-820.

[2] Brezzi, F., Hager, W.W., Raviart, P.A. (1977). Error estimates for the finite element solution of variational inequalities. Part I. Primal Theory. Numer. Math. 28, 431-443.

[3] Haslinger, J., Hlaváček, I.(1980,1981). Contact between elastic bodies. Apl.Mat. 25(1980),324-348, 26(1981)263-290, 321-344.

[4] Haslinger, J., Hlaváček, I., Nečas, J. (1996). Numerical Methods for Unilateral Problems in Solid Mechanics. In: Handbook of Numerical Analysis, Vol.IV, 313-486, Elsevier Science, Amsterdam.

[5] Hlaváček, I., Nečas, J. (1970). On inequalities of Korn's type. Arch. Rational. Mech. Anal. 36, 305-334.

[6] Hlaváček, I., Lovíšek, J. (1977). A finite element analysis for the Signorini's problem in plane elastostatics. Apl.Mat 22(3)215-228.

[7] Hlaváček, I., Lovíšek, J. (1980). Finite element analysis of the Signorini problem in semi-coercive cases. Apl.Mat. 25, 274-285.

[8] Hlaváček, I., Haslinger,J., Nečas,J.,Lovíšek, J. (1988). Solution of Variational Inequalities in Mechanics. Springer Vlg., New York.

[9] Jarušek, J. (1983). Contact problems with bounded friction. Czechoslovak Mat. J. (1983,1984), 33(1983),2,237-261, 34(1984)109, 619-629.

[10] Kikuchi, N., Oden, J.T. (1988). Contact Problems in Elasticity: A Study of Variational Inequlities and Finite Element Methods. SIAM, Philadelphia, PA.

[11] Lukšan, L., Vlček,J. (2000). NDA: Algorithms for nondifferentiable optimization. Technical Report V-797, ICS AS CR, Prague.

[12] Nečas, J., Jarušek, J., Haslinger, J. (1980). On the solution of the variational inequality to the Signorini problem with small friction. Boll.Un. Mat.Ital. (5)17-B, 796-811.

[13] Nečas, J., Hlaváček, I. (1981). Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction. Elsevier, Amsterdam.

[14] Nedoma, J. (1987). On the Signorini problem with friction in linear thermo-elasticity. The quasi-coupled 2D-case. Apl. Mat. 32(3)186-199.

[15] Nedoma, J. (1994). Finite element analysis of contact problem in thermo-elasticity. The semi-coercive case. J.Comput.Appl.Math. 50,411-423.

[16] Nedoma,J. (1998). Numerical Modelling in Applied Geodynamics. John Wiley&Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto.

[17] Panagiotopoulos, P.D. (1985). Inequality Problems in Mechanics and Applications. Convex and Non-Convex Energy Functions. Birkhäuser, Boston, Basel, Stuttgart.