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Hájek, Petr
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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: (+4202) 6605 1111, fax: (+4202) 858 57 89,
e-mail:hajek@cs.cas.cz

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Abstract:

The basic fuzzy logic BL is extended by two unary connectives L, U (lower, upper) whose standard semantics is, given a continuous t-norm, the function assigning to each $x \in [0, 1]$ the biggest idempotent $\leq x$ (least idempotent $\geq x$). An axiom system is presented and shown complete with respect to the corresponding class of algebras. But the set of tautologies for a fixed continuous t-norm may have an arbitrarily high degree of unsolvability.

Keywords:
fuzzy logic, hedges, continuous t-norms

1 Introduction.

The reader is assumed to know the basic fuzzy propositional logic BL (see [4]). It is strongly complete w. r. t. all BL-algebras: a theory T proves φ over BL iff $e_{\mathbf{L}}(\varphi) = 1_{\mathbf{L}}$ for each (linearly ordered) BL-algebra $\mathbf{L}$ and each $\mathbf{L}$-evaluation e of propositional variables which is an $\mathbf{L}$-model of T (i.e. $e_{\mathbf{L}}(\alpha) = 1_{\mathbf{L}}$ for each axiom $\alpha \in T$). Moreover, it is standardly complete (see [2]): $BL \vdash \varphi$ iff $e_*(\varphi) = 1$ for each (BL-algebra on $[0,1]$ given by a) continuous t-norm $*$ (φ is a $*$-tautology). For particular continuous t-norm $*$ (Łukasiewicz, Gödel, product t-norm) we have standard completeness w. r. t. a corresponding extension of BL (see [4]). Continuous t-norms non-isomorphic to any of the three just named are non-trivial ordinal sums of copies of them (Mostert-Shields, see again [4] or elsewhere). Each such t-norm determines its set of tautologies; for some of them we know a complete axiomatization. Haniková [9] has a complete axiomatization of $Ł \oplus G$; Agliano and Montagna [1] have results immediately implying that tautologies of $*$ are completely axiomatized by BL (without any additional axiom) iff $*$ is an ordered sum of infinitely many summands among which Ł occurs infinitely many times and also as the *least* summand. Haniková [9] also shows that if $m \neq n$ then any sum of n summands Π has a set of tautologies different from the set of tautologies of any sum of m summands Π. In more details, she shows that there is a formula which is a $*$-tautology iff $*$ is a sum of $\leq k$ summands Π ($k \geq 1$) but there is no formula which would be a $*$ tautology iff $*$ is a sum of $\geq k$ summands Π). Several problems remain open, in particular: is there a continuous t-norm whose set of tautologies is not decidable (recursive)? We can only give a positive answer (and say much more) for a language extended by two new natural unary connectives.

A *hedge* is a mapping by $[0,1]$ into itself; it can be taken to be a truth function of a new unary connective. Clearly, negation is given by a hedge. We shall work with non-decreasing hedges (truth modifiers). One such famous hedge is Baaz's Δ where $\Delta 1 = 1$ and $\Delta x = 0$ for $x < 1$ ($\Delta\varphi$ may be read "φ is absolutely true"). BL_Δ is the extension of BL by axioms $\Delta 1 - \Delta 5$ (see [4]); it is strongly complete w. r. t. BL_Δ-algebras (linearly ordered BL_Δ-algebras). Checking [2] it is easy to show that BL_Δ is standardly complete w. r. t. continuous t-norms with the above-described semantics of Δ ($\Delta 1 = 1, \Delta x = 0$ o. w.). Recall also a hedge for Gödel logic studied in [7] and hedges for "very true" are studied (over BL) in [6].

Let $*$ be a given continuous t-norm. Recall that $x \in [0,1]$ is an idempotent (of $*$) if $x * x = x$. Trivial idempotents are 0 and 1. Thanks to continuity, for each $x \in [0,1]$ there is a uniquely determined element $l(x) \in [0,1]$ which is the largest idempotent $\leq x$ and a uniquely determined element $u(x) \in [0,1]$ which is the least idempotent $\geq x$. (If x is itself an idempotent then obviously $l(x) = x = u(x)$.) Now u and l are non-decreasing hedges; the corresponding connectives will be denoted by L, U. Clearly, $L\varphi$ may be read as "φ is very true" and $U\varphi$ as "φ more or less true"; the reader may find it more or less natural. We present a logic $BL^!_{lu}$, show its completeness and standard completeness and show that for each $X \subseteq N$ there is a continuous t-norm T_X whose set of tautologies $Taut(T_X)$ (in the language of BL_{lu} is least as much unsolvable as X is; hence $Taut(T_X)$ can have arbitrarily high degree of unsolvability.

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2 The logics $BL^!_{lu}$.

The language of BL_{lu} is that of BL extended by three modalities Δ, L, U. Standard semantics, given by any continuous t-norm, was described above. First we find some tautologies.

Lemma. For each continuous t-norm, the following formulas are tautologies:

$(LU1)$	$L\varphi \to \varphi, \varphi \to U\varphi$
$(LU2)$	$L\varphi \equiv (L\varphi \& L\varphi), U\varphi \equiv (U\varphi \& U\varphi)$
$(LU3)$	$\Delta((\varphi \to \psi)\&(\psi \equiv (\psi\&\psi))\&(\psi \to U\varphi)) \to (\psi \equiv U\varphi)$
	$\Delta(L\varphi \to \psi)\&(\psi \equiv (\psi\&\psi))\&(\psi \to \varphi)) \to (\psi \equiv L\varphi)$
(K)	$L(\varphi \to \psi) \to (L\varphi \to L\psi), U(\varphi \to \psi) \to (U\varphi \to U\psi)$
$(LU4)$	$L(\varphi\&\psi) \equiv (L\varphi\&L\psi), U(\varphi\&\psi) \equiv (U\varphi\&U\psi)$
$(LU5)$	$L(\varphi \wedge \psi) \equiv (L\varphi \wedge L\psi), U(\varphi \wedge \psi) \equiv (U\varphi \wedge U\psi)$
$(LU6)$	$L(\varphi \vee \psi) \equiv (L\varphi \vee L\psi), U(\varphi \vee \psi) \equiv (U\varphi \vee U\psi)$

Proof by easy checking. Note that $LU1 - LU3$ just say that for $x = e(\varphi), l(x)$ is the biggest idempotent $\leq x$ and $u(x)$ is the least idempotent $\geq x$.

(K): If $lx \leq ly$ then $(lx \Rightarrow ly) = 1$ and hence $l(x \Rightarrow y) \Rightarrow (lx \Rightarrow ly) = 1$. Assume $lx > ly$; then $x > y$, $lx \Rightarrow ly = ly$ (lx, ly being idempotents, see [5],) and $x \Rightarrow y = y$ (x, y being separated by an idempotent), thus $l(x \Rightarrow y) = ly$ and $e(x \Rightarrow y) \Rightarrow (lx \Rightarrow ly) = 1$.

(LU4): Assume $x \leq y$ without loss of generality. Then $lx \leq ly, lx * ly = lx$, $x * y \geq x * x \geq lx * lx = lx \geq l(x * y)$, hence $l(x * y) = lx$ by (LU3), thus $l(x * y) = lx * ly$.

(LU5–6) – similar.

Definition. The logic BL_{lu} is the extension of BL_Δ by $LU1 - LU3$ (i.e. the deduction rules are modus ponens and Δ-generalization). A BL_{lu}-algebra is an expansion $\mathbf{L}$ of a BL_Δ-algebra by two unary operations l, u making $(LU1 - LU3)$ to $\mathbf{L}$-tautologies.

Strong completeness. Let T be a theory over BL_{lu} and φ a formula. Then the following are equivalent:

(i) $T \vdash_{BL_{lu}} \varphi$

(ii) For each BL_{lu}-algebra $\mathbf{L}$, φ is true in each $\mathbf{L}$-model of T.

(iii) The same for each linearly ordered BL_{lu}-algebra.

Lemma. BL_{lu} proves (K) and (LU4–LU6).

Proof. Due to the completeness ot is enough to verify that (K), (LU4–LU6) are $\mathbf{L}$-tautologies for each BL_{lu}-chain. Verify easily that the above proof for continuous t-norms works also for $BL - en$-chains.

Proof by checking the proof of strong completeness of BL_Δ.

Concerning standard completeness one has to be careful. Analysing the proof of standard completeness of BL in [2] one can show that BL_Δ has standard completeness. The following facts are relevant:
(i) If $\mathbf{L}$ is a linearly ordered BL_Δ-algebra (a BL_Δ-chain) then $\Delta 1 = 1$ and $\Delta x = 0$ otherwise (since $\Delta\varphi \vee \neg\Delta\varphi$ is a tautology).
(ii) Each BL_Δ-chain is a subalgebra of its saturation. Thus if $\mathbf{L}$ is a BL_Δ-chain such that $e_{\mathbf{L}}(\varphi) < 1$ you may assume $\mathbf{L}$ saturated and produce a continuous t-norm $*$ and an evaluation e' such that $e'_*(\varphi) < 1$ in full analogy to the proof of standard completeness of BL. Now for BL_{lu} we have (i) but we do *not* have (ii): use the well-known example of $L \oplus \Pi$ with the inner idempotent removed. Then for each non-extremal element $l(x) = 0$ and $u(x) = 1$, but this is not the case in the saturation (which is Ł $\oplus\, \Pi$). Thus we have to eliminate algebras like this, finding an axiom guaranteeing that the interval between $l(x)$ and $u(x)$ is either a MV-algebra or a product algebra. This can be done using a variant of the axiom (Ł$, \Pi, G$) of [3]:

Lemma [3]. A BL-chain is an MV-chain, G-chain or Π-chain iff the identity

$$(x \Rightarrow x * y) \Rightarrow ((x \Rightarrow 0) \cup y \cup [(x \Rightarrow x * x) \cap (y \Rightarrow y * y)]) = 1$$

is valid in it.

This leads us to the following axiom. $(LU!)$
$[\Delta((L\varphi \equiv L\psi)\&(U\varphi \equiv U\psi))\&(\varphi \to (\varphi\&\psi))] \to$
$\to [((\varphi \equiv L\varphi) \vee (\psi \equiv U\psi) \vee ((\varphi \equiv (\varphi\&\varphi)) \wedge \psi \equiv (\psi\&\psi))]$.

Observe that if a BL_{lu}-chain makes $(LU!)$ to a tautology then each interval $[l(x), u(x)]$ is an MV-chain, G-chain or Π-chain, thus it is itself saturated or is a subalgebra of its saturation and the usual construction works. We have proved:

Theorem. Let $BL^!_{lu}$ be the extension of BL_{lu} by the axiom $(LU!)$. The logic $BL^!_{lu}$ has standard completeness: a formula is provable in $BL^!_{lu}$ iff it is a tautology w. r. t. each continuous t-norm.

3 Undecidability of $*$-tautologies.

Theorem. There is a recursive sequence $\{\Phi_n | n \in N\}$ of formulas of BL_{lu} and a system $\{T_X | X \subseteq N\}$ of continuous t-norms such that for each $n \in N$ and $X \subseteq N$,

$$\Phi_n \text{ is a } T_X\text{-tautology iff } n \in X.$$

Consequently, if the degree of unsolvability of X is $deg(X)$ and the degree of unsolvability of T_X-tautologies is $deg(TAUT(T_X))$ then $deg(X) \leq degTAUT(T_X)$ (e.g. if X is not arithmetical the $TAUT(T_X)$ is not arithmetical). Moreover, if $X \neq Y$ then $TAUT(T_X) \neq TAUT(T_Y)$.

Proof. If $i \in X$ let $\mathbf{L}_{X,i} = \Pi$, if $i \notin X$ let $\mathbf{L}_{X,i} = Ł$; let $T_X = \oplus_{i=0}^{\infty} \mathbf{L}_{X,i}$. We construct a formula Ψ_k such that Ψ_k is T_X-satisfiable iff k-th summand of T_X is Łukasiewicz. Then you may take Φ_k to be $\neg\Delta\Psi_k$; $k \in X$ iff $\mathbf{L}_{X,k} = \Pi$ iff Ψ_k is not T_X-satisfiable iff $\neg\Delta\Psi_k$ is a T_X-tautology.

Take the following formula for Ψ_k :

$$Lq_1 \equiv \overline{0} \& \bigwedge_{i=1}^{k} \neg\Delta((Lq_i \equiv Uq_i) \& \bigwedge_{i=1}^{k-1} (Uq_i \equiv Lq_{i+1}) \& [((q_k \to Lq_k) \equiv q_k]$$

The formula is true iff the value x of q_k is a non-extremal element in the k-th summand whose negation with respect to the summand is x (non-extremal fixed point of the negation).

Remark. (1) One can write a formula Φ_k saying that if (the value of) q_k is an internal element of the k-th summand then its double negation with respect to the summand equals to the value of q_k (thus $((q_k \to Lq_k) \to Lq_k) \equiv q_k$). It can be used instead of the Φ_k in the preceding proof.
(2) One can produce an extension of $BL^!_{lu}$ which is complete for $L \oplus \Pi$. The axioms say: there are at most two components; non-extremal elements of the first component satisfy double negation; non-extremal elements of the second satisfy $\Pi1, \Pi2$ relative to the component. (In this context note that Montagna's appears to offer a recursive axiomatization of each ordered sum of finitely many copies of Łukasiewicz (without any use of Δ) and product just in the language of BL. His approach is uniform, even if complicated.)
(3) For each formula φ (of BL, BL_Δ, BL_{lu}) one may produce (in non-deterministic polynomial time) a quantifier-free formula φ^* of the language of real algebra such that φ is satisfiable over $Ł \oplus \Pi$ iff for at least one halting run of the non-deterministic algorithm with the input φ, φ^* is satisfiable in the field of real numbers. Similarly for each finite ordinal sum. Thus for each finite ordinal sum the set of satisfiable formulas as well as the set of tautologies is PSPACE. (cf. [8]).

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