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**Institute of Computer Science**
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Abstract:

The basic fuzzy logic $BL$ is extended by two unary connectives $L, U$ (lower, upper) whose standard semantics is, given a continuous $t$-norm, the function assigning to each $x \in [0, 1]$ the biggest idempotent $\leq x$ (least idempotent $\geq x$). An axiom system is presented and shown complete with respect to the corresponding class of algebras. But the set of tautologies for a fixed continuous $t$-norm may have an arbitrarily high degree of unsolvability.

Keywords:
fuzzy logic, hedges, continuous $t$-norms

# 1 Introduction.

The reader is assumed to know the basic fuzzy propositional logic $BL$ (see [4]). It is strongly complete w. r. t. all $BL$-algebras: a theory $T$ proves $\varphi$ over $BL$ iff $e_{\mathbf{L}}(\varphi) = 1_{\mathbf{L}}$ for each (linearly ordered) $BL$-algebra $\mathbf{L}$ and each $\mathbf{L}$-evaluation $e$ of propositional variables which is an $\mathbf{L}$-model of $T$ (i.e. $e_{\mathbf{L}}(\alpha) = 1_{\mathbf{L}}$ for each axiom $\alpha \in T$). Moreover, it is standardly complete (see [2]): $BL \vdash \varphi$ iff $e_*(\varphi) = 1$ for each ($BL$-algebra on $[0,1]$ given by a) continuous $t$-norm $*$ ($\varphi$ is a $*$-tautology). For particular continuous $t$-norm $*$ (Łukasiewicz, Gödel, product $t$-norm) we have standard completeness w. r. t. a corresponding extension of $BL$ (see [4]). Continuous $t$-norms non-isomorphic to any of the three just named are non-trivial ordinal sums of copies of them (Mostert-Shields, see again [4] or elsewhere). Each such $t$-norm determines its set of tautologies; for some of them we know a complete axiomatization. Haniková [9] has a complete axiomatization of $Ł \oplus G$; Agliano and Montagna [1] have results immediately implying that tautologies of $*$ are completely axiomatized by $BL$ (without any additional axiom) iff $*$ is an ordered sum of infinitely many summands among which Ł occurs infinitely many times and also as the *least* summand. Haniková [9] also shows that if $m \neq n$ then any sum of $n$ summands $\Pi$ has a set of tautologies different from the set of tautologies of any sum of $m$ summands $\Pi$. In more details, she shows that there is a formula which is a $*$-tautology iff $*$ is a sum of $\leq k$ summands $\Pi$ ($k \geq 1$) but there is no formula which would be a $*$ tautology iff $*$ is a sum of $\geq k$ summands $\Pi$). Several problems remain open, in particular: is there a continuous $t$-norm whose set of tautologies is not decidable (recursive)? We can only give a positive answer (and say much more) for a language extended by two new natural unary connectives.

A *hedge* is a mapping by $[0,1]$ into itself; it can be taken to be a truth function of a new unary connective. Clearly, negation is given by a hedge. We shall work with non-decreasing hedges (truth modifiers). One such famous hedge is Baaz's $\Delta$ where $\Delta 1 = 1$ and $\Delta x = 0$ for $x < 1$ ($\Delta\varphi$ may be read "$\varphi$ is absolutely true"). $BL_\Delta$ is the extension of $BL$ by axioms $\Delta 1 - \Delta 5$ (see [4]); it is strongly complete w. r. t. $BL_\Delta$-algebras (linearly ordered $BL_\Delta$-algebras). Checking [2] it is easy to show that $BL_\Delta$ is standardly complete w. r. t. continuous $t$-norms with the above-described semantics of $\Delta$ ($\Delta 1 = 1, \Delta x = 0$ o. w.). Recall also a hedge for Gödel logic studied in [7] and hedges for "very true" are studied (over $BL$) in [6].

Let $*$ be a given continuous $t$-norm. Recall that $x \in [0,1]$ is an idempotent (of $*$) if $x * x = x$. Trivial idempotents are 0 and 1. Thanks to continuity, for each $x \in [0,1]$ there is a uniquely determined element $l(x) \in [0,1]$ which is the largest idempotent $\leq x$ and a uniquely determined element $u(x) \in [0,1]$ which is the least idempotent $\geq x$. (If $x$ is itself an idempotent then obviously $l(x) = x = u(x)$.) Now $u$ and $l$ are non-decreasing hedges; the corresponding connectives will be denoted by $L, U$. Clearly, $L\varphi$ may be read as "$\varphi$ is very true" and $U\varphi$ as "$\varphi$ more or less true"; the reader may find it more or less natural. We present a logic $BL^!_{lu}$, show its completeness and standard completeness and show that for each $X \subseteq N$ there is a continuous $t$-norm $T_X$ whose set of tautologies $Taut(T_X)$ (in the language of $BL_{lu}$ is least as much unsolvable as $X$ is; hence $Taut(T_X)$ can have arbitrarily high degree of unsolvability.

# 2 The logics $BL^!_{lu}$.

The language of $BL_{lu}$ is that of $BL$ extended by three modalities $\Delta, L, U$. Standard semantics, given by any continuous $t$-norm, was described above. First we find some tautologies.

*Lemma.* For each continuous $t$-norm, the following formulas are tautologies:

$(LU1)$  $L\varphi \to \varphi, \varphi \to U\varphi$

$(LU2)$  $L\varphi \equiv (L\varphi \& L\varphi), U\varphi \equiv (U\varphi \& U\varphi)$

$(LU3)$  $\Delta((\varphi \to \psi) \& (\psi \equiv (\psi \& \psi)) \& (\psi \to U\varphi)) \to (\psi \equiv U\varphi)$

$\quad\quad\quad \Delta(L\varphi \to \psi) \& (\psi \equiv (\psi \& \psi)) \& (\psi \to \varphi)) \to (\psi \equiv L\varphi)$

$(K)$  $L(\varphi \to \psi) \to (L\varphi \to L\psi), U(\varphi \to \psi) \to (U\varphi \to U\psi)$

$(LU4)$  $L(\varphi \& \psi) \equiv (L\varphi \& L\psi), U(\varphi \& \psi) \equiv (U\varphi \& U\psi)$

$(LU5)$  $L(\varphi \wedge \psi) \equiv (L\varphi \wedge L\psi), U(\varphi \wedge \psi) \equiv (U\varphi \wedge U\psi)$

$(LU6)$  $L(\varphi \vee \psi) \equiv (L\varphi \vee L\psi), U(\varphi \vee \psi) \equiv (U\varphi \vee U\psi)$

*Proof* by easy checking. Note that $LU1 - LU3$ just say that for $x = e(\varphi), l(x)$ is the biggest idempotent $\leq x$ and $u(x)$ is the least idempotent $\geq x$.

(K): If $lx \leq ly$ then $(lx \Rightarrow ly) = 1$ and hence $l(x \Rightarrow y) \Rightarrow (lx \Rightarrow ly) = 1$. Assume $lx > ly$; then $x > y$, $lx \Rightarrow ly = ly$ ($lx, ly$ being idempotents, see [5],) and $x \Rightarrow y = y$ ($x, y$ being separated by an idempotent), thus $l(x \Rightarrow y) = ly$ and $e(x \Rightarrow y) \Rightarrow (lx \Rightarrow ly) = 1$.

(LU4): Assume $x \leq y$ without loss of generality. Then $lx \leq ly, lx * ly = lx, x * y \geq x * x \geq lx * lx = lx \geq l(x * y)$, hence $l(x * y) = lx$ by (LU3), thus $l(x * y) = lx * ly$.

(LU5–6) – similar.

*Definition.* The logic $BL_{lu}$ is the extension of $BL_\Delta$ by $LU1 - LU3$ (i.e. the deduction rules are modus ponens and $\Delta$-generalization). A $BL_{lu}$-algebra is an expansion $\mathbf{L}$ of a $BL_\Delta$-algebra by two unary operations $l, u$ making $(LU1 - LU3)$ to $\mathbf{L}$-tautologies.

*Strong completeness.* Let $T$ be a theory over $BL_{lu}$ and $\varphi$ a formula. Then the following are equivalent:

(i) $T \vdash_{BL_{lu}} \varphi$

(ii) For each $BL_{lu}$-algebra $\mathbf{L}$, $\varphi$ is true in each $\mathbf{L}$-model of $T$.

(iii) The same for each linearly ordered $BL_{lu}$-algebra.

*Lemma.* $BL_{lu}$ proves $(K)$ and (LU4–LU6).

*Proof.* Due to the completeness ot is enough to verify that $(K)$, (LU4–LU6) are $\mathbf{L}$-tautologies for each $BL_{lu}$-chain. Verify easily that the above proof for continuous $t$-norms works also for $BL - en$-chains.

*Proof* by checking the proof of strong completeness of $BL_\Delta$.

Concerning standard completeness one has to be careful. Analysing the proof of standard completeness of $BL$ in [2] one can show that $BL_\Delta$ has standard completeness. The following facts are relevant:

(i) If $\mathbf{L}$ is a linearly ordered $BL_\Delta$-algebra (a $BL_\Delta$-chain) then $\Delta 1 = 1$ and $\Delta x = 0$ otherwise (since $\Delta\varphi \vee \neg\Delta\varphi$ is a tautology).

(ii) Each $BL_\Delta$-chain is a subalgebra of its saturation. Thus if $\mathbf{L}$ is a $BL_\Delta$-chain such that $e_{\mathbf{L}}(\varphi) < 1$ you may assume $\mathbf{L}$ saturated and produce a continuous $t$-norm $*$ and an evaluation $e'$ such that $e'_*(\varphi) < 1$ in full analogy to the proof of standard completeness of $BL$. Now for $BL_{lu}$ we have (i) but we do *not* have (ii): use the well-known example of $L \oplus \Pi$ with the inner idempotent removed. Then for each non-extremal element $l(x) = 0$ and $u(x) = 1$, but this is not the case in the saturation (which is $\mathbf{L} \oplus \Pi$). Thus we have to eliminate algebras like this, finding an axiom guaranteeing that the interval between $l(x)$ and $u(x)$ is either a $MV$-algebra or a product algebra. This can be done using a variant of the axiom $(\mathbf{L}, \Pi, G)$ of [3]:

*Lemma* [3]. A $BL$-chain is an $MV$-chain, $G$-chain or $\Pi$-chain iff the identity

$$(x \Rightarrow x * y) \Rightarrow ((x \Rightarrow 0) \cup y \cup [(x \Rightarrow x * x) \cap (y \Rightarrow y * y)]) = 1$$

is valid in it.

This leads us to the following axiom. $(LU!)$

$[\Delta((L\varphi \equiv L\psi) \& (U\varphi \equiv U\psi)) \& (\varphi \to (\varphi \& \psi))] \to$

$\to [((\varphi \equiv L\varphi) \vee (\psi \equiv U\psi) \vee ((\varphi \equiv (\varphi \& \varphi)) \wedge \psi \equiv (\psi \& \psi))].$

Observe that if a $BL_{lu}$-chain makes $(LU!)$ to a tautology then each interval $[l(x), u(x)]$ is an $MV$-chain, $G$-chain or $\Pi$-chain, thus it is itself saturated or is a subalgebra of its saturation and the usual construction works. We have proved:

**Theorem.** Let $BL_{lu}^!$ be the extension of $BL_{lu}$ by the axiom $(LU!)$. The logic $BL_{lu}^!$ has standard completeness: a formula is provable in $BL_{lu}^!$ iff it is a tautology w. r. t. each continuous $t$-norm.

## 3 Undecidability of $*$-tautologies.

**Theorem.** There is a recursive sequence $\{\Phi_n | n \in N\}$ of formulas of $BL_{lu}$ and a system $\{T_X | X \subseteq N\}$ of continuous $t$-norms such that for each $n \in N$ and $X \subseteq N$,

$$\Phi_n \text{ is a } T_X\text{-tautology iff } n \in X.$$

Consequently, if the degree of unsolvability of $X$ is $deg(X)$ and the degree of unsolvability of $T_X$-tautologies is $deg(TAUT(T_X))$ then $deg(X) \leq degTAUT(T_X)$ (e.g. if $X$ is not arithmetical the $TAUT(T_X)$ is not arithmetical). Moreover, if $X \neq Y$ then $TAUT(T_X) \neq TAUT(T_Y)$.

*Proof.* If $i \in X$ let $\mathbf{L}_{X,i} = \Pi$, if $i \notin X$ let $\mathbf{L}_{X,i} = L$; let $T_X = \oplus_{i=0}^{\infty} \mathbf{L}_{X,i}$. We construct a formula $\Psi_k$ such that $\Psi_k$ is $T_X$-satisfiable iff $k$-th summand of $T_X$ is Łukasiewicz. Then you may take $\Phi_k$ to be $\neg\Delta\Psi_k$; $k \in X$ iff $\mathbf{L}_{X,k} = \Pi$ iff $\Psi_k$ is not $T_X$-satisfiable iff $\neg\Delta\Psi_k$ is a $T_X$-tautology.

Take the following formula for $\Psi_k$ :

$$Lq_1 \equiv \overline{0} \,\&\, \bigwedge_{i=1}^{k} \neg\Delta((Lq_i \equiv Uq_i) \,\&\, \bigwedge_{i=1}^{k-1}(Uq_i \equiv Lq_{i+1}) \,\&\, [((q_k \to Lq_k) \equiv q_k]$$

The formula is true iff the value $x$ of $q_k$ is a non-extremal element in the $k$-th summand whose negation with respect to the summand is $x$ (non-extremal fixed point of the negation).

**Remark.** (1) One can write a formula $\Phi_k$ saying that if (the value of) $q_k$ is an internal element of the $k$-th summand then its double negation with respect to the summand equals to the value of $q_k$ (thus $((q_k \to Lq_k) \to Lq_k) \equiv q_k$). It can be used instead of the $\Phi_k$ in the preceding proof.

(2) One can produce an extension of $BL_{lu}^!$ which is complete for $L \oplus \Pi$. The axioms say: there are at most two components; non-extremal elements of the first component satisfy double negation; non-extremal elements of the second satisfy $\Pi 1, \Pi 2$ relative to the component. (In this context note that Montagna's appears to offer a recursive axiomatization of each ordered sum of finitely many copies of Łukasiewicz (without any use of $\Delta$) and product just in the language of BL. His approach is uniform, even if complicated.)

(3) For each formula $\varphi$ (of $BL, BL_\Delta, BL_{lu}$) one may produce (in non-deterministic polynomial time) a quantifier-free formula $\varphi^*$ of the language of real algebra such that $\varphi$ is satisfiable over $L \oplus \Pi$ iff for at least one halting run of the non-deterministic algorithm with the input $\varphi$, $\varphi^*$ is satisfiable in the field of real numbers. Similarly for each finite ordinal sum. Thus for each finite ordinal sum the set of satisfiable formulas as well as the set of tautologies is PSPACE. (cf. [8]).

# Bibliography

[1] AGLIANO P., MONTAGNA F.: *Varieties of BL-algebras I: general properties.* Submitted.

[2] CIGNOLI R., ESTEVA F., GODO L., TORRENS A.: *Basic logic to the logic of continuous t-norms and their residua.* Soft Computing, 4 (2000), 106–112.

[3] DINOLA A., ESTEVA F., GARCIA P., GODO L., SESSA S.: *The variety of BL-algebras and its subvarieties.* Submitted.

[4] HÁJEK P.: *Metamathematics of fuzzy logic.* To appear in Fuzzy sets and systems.

[5] HÁJEK P.: *BL-logic and BL-algebras.* Soft computing.

[6] HÁJEK P.: *On very true.* To appear in Fuzzy sets ans systems.

[7] HÁJEK P., HARMANCOVÁ D.: *A hedge for Gödel fuzzy logic.* Int. J. on Uncertainty, Fuzziness and Knowledge-based systems, 10 (2000), 495–498.

[8] HÁJEK P., TULIPANI S.: *Complexity of fuzzy probabilistic logics.* To appear in Fundamenta Informaticae.

[9] HANIKOVÁ Z.: *Standard algebras for fuzzy propositional calculi.* To appear in Fuzzy sets and systems.