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Abstract

A combinatorial approach developed by the author in 1959 is used to explain the connection between limit process and combinatorial properties of families of finite sets.

Keywords

Topological, space, uniform convergence, weak compactness, duality, convex mean, optimization problem, Banach space, number theory

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1 Introduction

The proofs of many important results in analysis are based on inverting the order of two limit processes. Accordingly, it seems natural to look for the basic ideas which make these results possible. This is not only a requirement of aesthetic character: the desire to find a simple and transparent proof invariably leads to the discovery of the essential principles ensuring the validity of such results.

It is not, perhaps, immediately obvious that theorems on inverting the order of limit processes may be deduced from results about the combinatorial structure of families of finite sets. This fact was pointed out, in 1959, in a short note by the author [11]. The combinatorial approach has been developed further in a series of papers [12], [13], [14], [15], [2], [17] and has found its way into monographs [6], [7] and even elementary textbooks [9]. However, no systematic account has yet been published. It is the purpose of the present note to discuss the main ideas in an elementary manner in order to explain the (possibly some what unexpected) connection between limit processes and combinatorial properties of families of finite sets.

We begin by reformulating the classical definitions of different forms of convergence in a form in which the connection with combinatorial notions becomes more evident. We limit ourselves - in this introduction - to convergence of *sequences* of functions. The principles of the combinatorial approach will become evident even in this traditional particular case; the combinatorial treatment of possibly uncountable families of functions is not essentially different.

We consider a set T and a sequence of (complex-valued) functions f_n defined on T such that $f_n(t) \to 0$ for every $t \in T$.

In general the convergence $f_s(t) \to 0$ depends heavily on t and may be considerably slower in some t than in others. In a number of cases it is possible to show that there are large subsets of T on which the convergence behaviour is to a certain extent comparable, that there is certain uniformity.

We shall use the symbol S for the set of all nonnegative integers since we are dealing, in this introductory section, with convergence of sequences. We shall keep the symbol S for the set of indices for families of functions in the general case to be treated later.

We start by reviewing the standard notions of different types of convergence and by showing how to reformulate their definitions in combinatorial terms.

We say that the convergence $f_n(t) \to 0$ is uniform, if, for each $\varepsilon > 0$, there exists an integer $n(\varepsilon)$ such that

$$|f_s(t)| < \varepsilon$$

for all t provided $s > n(\varepsilon)$.

For our purposes it will be convenient to reformulate the definition as follows:

For each $\varepsilon > 0$ and each infinite set $R \subset S$ there exists $r \in R$ such that, for each $t \in T$,

$$|f_r(t)| < \varepsilon$$

A weaker form of uniformity is the following notion of almost uniform convergence.

We say that the convergence $f_n(t) \to 0$ is almost uniform if, for each $\varepsilon > 0$ and each infinite $R \subset S$ there exists a finite $K \subset R$ such that, for each $t \in T$, the inequality $|f_k(t)| < \varepsilon$ is satisfied for at least one of the indices $k \in K$.

In the case of uniform convergence we may say that, for each $\varepsilon > 0$ and each infinite R the zero function may be approximated within ε uniformly on T by one of the f_r (the index r depending on ε and R).

In the case of almost uniform convergence the analogous formulation has to be modified as follows: The zero function may be thus approximated on the whole of T not necessarily by one function but by a finite family $f_k, k \in K$ in the sense that, for each $t \in T$, at least one function of this finite family will approximate zero.

In other words the zero function is uniformly approximated by the function

$$f_{\min}(t) = \min\{|f_k(t)|;\ k \in K\},$$

the function $f_{\min}$ depending on ε and R.

Now let us introduce yet another notion of convergence; we shall call it convergence in the mean:

For each positive ε and each infinite $R \subset S$ there exists a finite set $K \subset R$ and nonnegative numbers $\lambda(k)$, $k \in K$ with $\sum \lambda(k) = 1$ such that

$$|\sum \lambda(k) f_k(t)| < \varepsilon \quad \text{for all} \quad t \in T;$$

the zero function is uniformly approximated, this time, by the function

$$f_{\text{conv}}(t) = \sum \lambda(k) f_k(t).$$

The fact that the sequence f_n tends to zero pointwise on T means, in particular, that, for each $t \in T$ and each $\varepsilon > 0$, the set of those indices s for which $|f_s(t)| \geq \varepsilon$, is finite. Keeping ε fixed and varying t, we obtain a family of finite sets the structure of which clearly is closely related to the degree of uniformity of the convergence: let us examine this connection more closely.

Fix $\varepsilon > 0$; let W_ε be the relation on $S \times T$ defined by

$$[s,t] \in W_\varepsilon \quad \text{iff} \quad |f_s(t)| \geq \varepsilon.$$

This relation may also be considered as a multivalued mapping from S into T; in accordance with this we may use the notation $W(s)$ for the set of those t for which $[s,t] \in W_\varepsilon$ or the set of those t for which $|f_s(t)| \geq \varepsilon$. Using this notation, the conditions for uniform convergence and almost uniform convergence appear in the following form:

For each infinite $R \subset S$ and for each $\varepsilon > 0$ there exists $r \in R$ such that

$$W_\varepsilon(r) = \emptyset.$$

For each infinite $R \subset S$ and for each $\varepsilon > 0$ there exists a finite $K \subset R$ such that

$$\bigcap_{r \in K} W_\varepsilon(r) = \emptyset.$$

In a manner of speaking, the notion of almost uniform convergence is more natural than that of uniform convergence. The following observation shows how it appears in a natural situation.

Let T be a compact topological space and consider a sequence f_n of continuous functions on T such that $f_n(t) \to 0$ for each $t \in T$. Then the convergence is almost uniform.

Proof: Given $\varepsilon > 0$ and an infinite set $R \subset S$, consider for each $r \in R$, the set

$$W_\varepsilon(r) = \{t \in T;\ |f_r(t)| \geq \varepsilon\}.$$

Since the sets $W_\varepsilon(r)$ are closed and T compact, if follows that the family $W_\varepsilon(r)$, $r \in R$ cannot have the finite intersection property.

We have seen that, in the particular case of continuous functions on a compact space, almost uniform convergence appears in a natural manner: so does convergence in the mean, for that matter, if we restrict our attention to equibounded sequences. This fact, however, is less immediate: it is essentially nothing more than the equivalence of weak and pointwise convergence for bounded sequences of continuous functions on a compact space. Let us recall, in a few words, the statement and the proof of this result.

Let T be a compact topological space and consider a sequence f_n of continuous functions such that $f_n(t) \to 0$ for each $t \in T$. If the sequence is equibounded then f_n converges to zero in the mean.

To prove this fact it suffices to know that, for equibounded sequences, pointwise and weak convergence are equivalent. Knowing this, we may invoke the separation theorem from which it follows that, for convex sets, the weak and uniform closures coincide. The equivalence of pointwise and weak convergence for bounded sequences, in its turn, is based on two results which are far from superficial.

By the Riesz theorem every bounded linear functional m on $C(T)$ may be represented by a measure m on T; by the Lebesgue dominated convergence theorem, boundedness of a sequence $f_n \in C(T)$ together with pointwise convergence $f_n(t) \to 0$ implies $\langle f_n, m\rangle \to 0$.

This shows that, in the particular case of bounded sequences of continuous functions on a compact T, almost uniform convergence and convergence in the mean are equivalent; the proof, however, as sketched above, uses important theorems that are far from superficial.

It is not immediately obvious that the implication is essentially combinatorial in character.

It is natural to examine the relation between the two types of convergence in the general case; it turns out that the above equivalence remains in force for arbitrary bounded sequences - to prove this our main tool will be the combinatorial lemma that we now proceed to explain. We shall state the lemma in its full generality; we do admit that, at first glance, its connection with convergence will not be immediately obvious; we hope however, that the preceding discussion has given the reader a flavour of the questions that we are going to ask now.

Recall that we considered families of sections of a relation $W_\varepsilon \subset S \times T$. Here S was the set of integers used as indices of the sequence f_n. We shall drop this restriction on S now, it was imposed in the introductory section for didactic reasons only.

Suppose a family $W \subset \exp S$ is given; for each $s \in S$ we shall denote by $W(s)$ the subfamily of those $w \in W$ for which $s \in w$. The mapping $s \mapsto W(s)$ may be viewed as a multivalued mapping from s into $\exp S$. It is thus natural to write, for $K \subset S$,

$$\begin{aligned} W(K) &= \bigcup_{k \in K} \{w \in W;\ k \in w\} \\ &= \bigcup_{k \in K} W(k) \\ &= \{w \in W;\ w \cap K \neq \emptyset\}. \end{aligned}$$

A *convex mean* on a set S is a nonnegative function λ defined on S with the following properties

1° the carrier of λ, the set of those $s \in S$ for which $\lambda(s) > 0$, is finite; we shall denote it by $\mathrm{N}(\lambda)$,

2° $\sum_{s \in S} \lambda(s) = 1$.

Given a convex mean λ on S it is natural to define, for an arbitrary set $a \subset S$, the value $\lambda(a)$ as the sum $\sum_{s \in a} \lambda(s)$; in this manner clearly a convex mean on S may also be considered as a probability measure on S with finite support. The set of all convex means on S will be denoted by $P(S)$.

Let W be a family of subsets of a set S. Given a convex mean $\lambda \in P(S)$ consider the maximum of the (finite) set $\{\lambda(w); w \in W\}$. For reasons that will become clear in the sequel, we are interested in convex means λ for which this maximum will be as small as possible – the lower bound will be

$$\inf_{\lambda} \sup_{w \in W} \lambda(w).$$

In our considerations we shall have to take into account also convex means the carrier of which is contained in a given subset $R \subset S$. The following quantity

$$\inf_{N(\lambda) \subset R} \sup_{w \in W} \lambda(w)$$

– to be denoted by $e(W, R)$ – reflects certain aspects of the combinatorial structure of the family W. Let us consider two extreme cases of this optimization problem:

1° The union of the family W does not cover the whole if S, in other words, there is an s_0 for which $W(s_0)$ is void; clearly, in this case $e(W, S) = 0$. Indeed, the convex mean concentrated in s_0 will realize the minimum. In fact, $e(W, R) = 0$ for every R which contains s_0.

2° The other extreme case is the one when the family $W(s)$, $s \in S$ possesses the finite intersection property. In this case, for each λ, there exists a $w \in W$ such that $\lambda(w) = 1$: indeed, if $s_1, \ldots, s_n$ is the carrier of λ, consider an arbitrary $w \in W(s_1) \cap \ldots \cap W(s_n)$. The carrier of λ will thus be contained in w whence $\lambda(w) = 1$.

Now let us consider a situation where a weaker form of the finite intersection property is satisfied.

Suppose $r_1, r_2, \ldots$ is a sequence of distinct elements of S such that the intersection

$$W(r_1) \cap \ldots \cap W(r_n)$$

is nonvoid for every n; if R stands for the set consisting of the r_j then $e(W, R) = 1$. Indeed, if λ is a convex mean with carrier $N \subset R$ then $N \subset \{r_1, \ldots, r_m\}$ for m sufficiently large and $\lambda(w) = 1$ for any $w \in W(r_1) \cap \ldots \cap W(r_m)$ since $N \subset \{r_1, \ldots, r_m\} \subset w$.

Here we are confronted with a situation where it is impossible to find a convex mean λ with $\lambda(w)$ small for all $w \in W$, at least as long as we want the carrier to be contained in R.

The basic lemma to be proved now shows that this situation is essentially the only case when means λ with uniformly small $\lambda(w)$ may fail to exist.

We have seen that the existence of a sequence r_n of distinct elements with $W(r_1) \cap \ldots \cap W(r_n)$ nonvoid for every n implies $e(W, R) = 1$. In a manner of speaking, the main result is the converse of this statement.

The full statement of the main result is the following:

(1.1) The Combinatorial Lemma. *Let S be a set and let $W \subset \exp S$. Then these are equivalent:*

1° $e(W, R) > 0$ *for some infinite* $R \subset S$,

2° $e(W, R) = 1$ *for some infinite* $R \subset S$,

3° *there exists a sequence $r_1, r_2 \ldots$ of distinct elements of S such that $W(r_1) \cap \ldots \cap W(r_n)$ is nonvoid for every n.*

We have just discussed the implication 3° → 2°. The implication 2° → 1° being immediate, the substance of the combinatorial lemma lies in the implication 1° → 3°. Reversing the implications, the combinatorial lemma assumes the following form.

(1.1') Combinatorial lemma restated. *These are equivalent:*

1° $e(W, R) = 0$ *for every infinite* $R \subset S$,

2° *given any sequence $s_1, s_2, \ldots$ of distinct elements of S, the intersection $W(s_1) \cap \ldots \cap W(s_n)$ will be eventually void.*

In other words, either there exists, for every infinite $R \subset S$ and every $\varepsilon > 0$, a convex mean λ with carrier in R such that $\lambda(w) < \varepsilon$ for all $w \in W$, or there exists a sequence r_n of distinct elements such that $W(r_1) \cap \ldots \cap W(r_n)$ is nonvoid for every n.

The equivalence of 1° and 2° in the combinatorial lemma leads to a restatement of the result in the form of a

(1,1") Dichotomy. *The maximum of $e(W, R)$ as R ranges over all infinite sets $R \subset S$ exists and can only assume one of the values 0 and 1.*

We postpone the proof of the lemma to the next section: we first demonstrate its usefulness by showing how it will be applied. Its meaning will become evident as soon as we see how it is used. Again we restrict ourselves to convergence of sequences of functions only - even this very particular case is amply sufficient to illustrate the main idea used in the proofs of the stronger results to follow.

Assuming the combinatorial lemma proved we are ready to prove the equivalence of convergence in the mean and almost uniform convergence for bounded sequences.

(1.2) *Let T be a set and f_n a sequence of functions such that $f_n(t) \to 0$ for each t and $|f_n(t)| \leq 1$ for all n and t. If the sequence converges almost uniformly then it converges in the mean.*

Proof: Let $\varepsilon > 0$ be given. Define the relation $W_\varepsilon \subset S \times T$ by setting $[s,t] \in W_\varepsilon$ iff $|f_s(t)| \geq \frac{1}{2}\varepsilon$. Let W be the family of subsets of S defined as follows:

$$w \in W \quad \text{iff} \quad w = W_\varepsilon^{-1}(t) \ \text{ for some } \ t \in T.$$

In particular, $w \in W(s_1) \cap \ldots \cap W(s_n)$ if and only if $w = W_\varepsilon^{-1}(t)$ for some $t \in W_\varepsilon(s_1) \cap \ldots \cap W_\varepsilon(s_n)$. In this manner $W(s_1) \cap \ldots \cap W(s_n)$ is nonvoid iff $W_\varepsilon(s_1) \cap \ldots \cap W_\varepsilon(s_n)$ is nonvoid. It follows that the assumption of almost uniform convergence implies that $e(W,R) = 0$ for every infinite $R \subset S$.

Now let an infinite $R \subset S$ and a positive ε be given. The combinatorial lemma yields the existence of a convex mean λ on S with carrier in R such that $\lambda(w) < \frac{1}{2}\varepsilon$ for every $w \in W$. We intend to show that, for this λ, $|\sum \lambda(s) f_s(t)| < \varepsilon$ for every $t \in T$.

To see that, consider a fixed $t \in T$; the sum $\sum_s \lambda(s) f_s(t)$ will be split into two parts according to whether $s \in W_\varepsilon^{-1}(t)$ or $s \in S \backslash W_\varepsilon^{-1}(t)$. For s outside $W_\varepsilon^{-1}(t)$ we have $|f_s(t)| < \frac{1}{2}\varepsilon$ whence

$$\left|\sum \lambda(s) f_s(t)\right| < \frac{1}{2}\varepsilon$$

as s ranges over $S \backslash W_\varepsilon^{-1}(t)$.

To estimate the remaining sum $\sum_{s \in W(t)} \lambda(s) f_s(t)$ it suffices to observe that $\sum_{s \in W(t)} \lambda(s) < \frac{1}{2}\varepsilon$ and that $|f_s(t)| \leq 1$ for any s,t. In this manner we have split the sum into two parts: one, where the values of the functions are small and the other, where we have no bound for $f_s(t)$ except one but where the sum of the weights $\sum \lambda(s)$ is small.

It follows that $|\sum \lambda(s) f_s(t)| < \varepsilon$ for every $t \in T$. □

The proof of the converse implication being straightforward, this shows that, for equibounded sequences, the concepts of almost uniform convergence and convergence in the mean coincide.

2 Proof of the Combinatorial Lemma

Now it is time to present the proof of the combinatorial lemma. For brevity, we introduce the following notation. If $W \subset \exp S$, $R \subset S$, $\varepsilon > 0$, let

$$M(W,R,\varepsilon) = \{\lambda \in P(S);\ N(\lambda) \subset R,\ \lambda(w) < \varepsilon \quad \text{for all} \quad w \in W\}.$$

In particular, this set is nonvoid if and only if $e(W,R) < \varepsilon$.

The proof of the combinatorial lemma proceeds in three steps. The first essential step is a sort of "principe de condensation des singularités". The second is a standard combinatorial construction and the third a simple technicality.

First step. *For every infinite* $R \subset S$

$$e(W,R) = \sup e(W(K), R)$$

as K *ranges over all finite subsets of* R.

Proof: Observe that $e(W,R) \geq e(W',R)$ if $W \supset W'$. This inequality proves the assertion in the case $e(W,R) = 0$; furthermore, it shows that it suffices to prove the following:

If $e = e(W,R) > 0$ *and* $0 < e' < e$ *then there exists a finite* $K \subset R$ *such that* $e(W(K),R) \geq e'$.

According to our assumption $M(W,R,e') = \emptyset$.

Suppose that $M(W(K),R,e')$ is nonvoid for every finite $K \subset R$. Take an arbitrary nonvoid finite $A_1 \subset R$ and a $\lambda_1 \in M(W(A_1),R,e')$. Set $A_2 = A_1 \cup N(\lambda_1)$ so that A_2 is a nonvoid finite subset of R. Accordingly, there exists a $\lambda_2 \in M(W(A_2),R,e')$. Set $A_3 = A_2 \cup N(\lambda_2)$, choose a $\lambda_3 \in M(W(A_3),R,e')$ and continue this process inductively.

Let us show that, for each $w \in W$, the sequence $\lambda_1(w)$, $\lambda_2(w), \ldots$ contains at most one term $\geq e'$. Indeed, if $\lambda_p(w) \geq e'$ for some p then w intersects the carrier of λ_p. Since $N(\lambda_p) \subset A_m$ for every $m > p$ it follows that $w \in W(K_m)$ whence $\lambda_m(w) < e'$ for every $m > p$. Take n large enough so as to have

$$\frac{1}{n}(1 + (n-1)e') < e;$$

it suffices to take $n > \frac{1-e'}{e-e'}$. Then

$$\frac{1}{n}(\lambda_1 + \ldots + \lambda_n) \in M(W,R,e),$$

a contradiction. □

Second step. *Suppose that* $e(W,R) > 0$ *for some infinite* R. *Then there exists a sequence of mutually disjoint finite sets* $K_j \subset R$ *such that*

$$W(K_1) \cap \ldots \cap W(K_n)$$

is nonvoid for every n.

Proof: Write e for $e(W,R)$. By the preceding proposition, there exists K_1 such that $e(W(K_1),R) > \frac{1}{2}e$ whence $e(W(K_1),\ R \backslash K_1) \geq e(W(K_1),R) > \frac{1}{2}e$. Applying the proposition to the pair $W(K_1)$, $R \setminus K_1$, we obtain the existence of a finite $K_2 \subset R \backslash K_1$ such that

$$e(W(K_1) \cap W(K_2),\ R \backslash K_1) > \frac{1}{4}\varepsilon.$$

Now consider the pair $W(K_1) \cap W(K_2)$ and $R \backslash (K_1 \cup K_2)$ and choose a finite set $K_3 \subset R \backslash (K_1 \cup K_2)$ with

$$e(W(K_1) \cap W(K_2) \cap W(K_3),\ R \backslash (K_1 \cup K_2)) > \frac{1}{8}\varepsilon.$$

Continuing is this manner, we obtain a sequence of disjoint finite sets $K_1, K_2, \ldots$ such that $W(K_1) \cap \ldots \cap W(K_n)$ is nonvoid for every n. □

Third step. *Suppose W is a family of subsets of a set S and let $K_1, K_2, \ldots$ be a sequence of finite subsets of S such, that, for each n the set $W(K_1) \cap \ldots \cap W(K_n)$ is nonvoid; then there exists a sequence $k_j \in K_j$ such that, for each n the set $W(k_1) \cap \ldots \cap W(k_n)$ is nonvoid.*

It is easy to give a straightforward elementary proof of this fact. For brevity, we present a more sophisticated one. The space $X = K_1 \times K_2 \times \ldots$ is compact. For each n define a set $R_n \subset X$ as follows: a sequence $x \in X$, $x = (x_1, x_2, \ldots)$ belongs to R_n if and only if $W(x_1) \cap \ldots \cap W(x_n)$ is nonvoid. The sets R_n are nonvoid, closed in X and $R_n \supset R_{n+1}$. Use compactness of X. □

3 An Example

The following example is included to show the advantages of the combinatorial approach; it also indicates connections with other branches of mathematics.

Let S be the set of all positive integers and consider the family W of all finite subsets w of S such that card $w \leq \min w$. We intend to show that $e(W) = 0$. To see that, it suffices to show that, given an arbitrary sequence $s_1 < s_2 < \ldots$, the intersection

$$W(s_1) \cap \ldots \cap W(s_k)$$

will become void if k is large enough. Indeed, we prove the following fact:

If $s_1 < s_2 < \ldots$ is an arbitrary sequence then

$$W(s_1) \cap \ldots \cap W(s_k)$$

is void as soon as $k > s_1$.

Proof: If $w \in W(s_1) \cap \ldots \cap W(s_k)$ then $s_j \in w$ for $j = 1, \ldots, k$ so that

$$k \leq \text{card } w \leq \min w \leq \min\{s_1, \ldots, s_k\} = s_1.$$

□

This example shows the power of the combinatorial lemma. It is, of course, possible to give [2], for each $\varepsilon > 0$, an explicit construction of a convex mean λ with $\lambda(w) < \varepsilon$ for all $w \in W$. The construction is based on the divergence of the harmonic series and is not simple: indeed, it may be shown that arithmetic means are not sufficient to solve this optimization problem. To be more precise: a convex mean λ is said to be an arithmetic mean if λ is carried by a finite set $M \subset S$ of cardinality m and $\lambda(s) = \frac{1}{m}$ for all $s \in M$. It is not difficult to prove that, for this particular family W, $\inf_\lambda \sup_w \lambda(w) > 0$ if λ is only allowed to range over the set of arithmetic means.

This fact is an immediate consequence of the following observation.

Given any arithmetic mean λ, there exists a $w \in W$ such that $\lambda(w) \geq \frac{1}{2}$.

Given an arithmetic mean λ, there exists a finite set M of cardinality k such that $\lambda(s) = \frac{1}{k}$ for $s \in M$ and $\lambda(s) = 0$ otherwise. Suppose $s_1 < s_2 < \ldots < s_k$ are the elements of M. Let $2m$ be the smallest even number $\geq k+1$ so that $2m \geq k+1 \geq 2m-1$. The set

$$a = \{s_m, \ldots, s_k\}$$

has $k - m + 1$ elements. Since

$$\min a = s_m \geq m \geq k - m + 1 = \text{card } a,$$

it follows that $a \in W$. At the same time $\lambda(a) = \frac{1}{k}(k - m + 1) \geq \frac{1}{2}$.

This example is related to a problem treated in another context in [22].

4 Weak Compactness

The combinatorial lemma may be interpreted as an existence theorem for a system of inequations

$$\lambda(w) < \varepsilon, \quad w \in W.$$

Observe that no assumption on the cardinality of W is made. On the other hand the condition for the existence of a solution is of *countable* character.

This remarkable fact has important applications in analysis; we now proceed to explain how it may be used to extend classical results and present them in a new light; this will be done in two directions

(1) the sequential character of the condition may be used to explain why, in the weak topology of a Banach space, sequential compactness implies compactness

(2) the sequential character of the condition will permit us to prove results which say that the assumption of invertibility of the order of two simple sequential limit processes implies that the same holds for more complicated ones.

In the introductory section our main purpose was to explain the principles of the use of combinatorial methods; to that end its was convenient to study the convergence of sequences. We now pass to applications in their full generality, to families of functions (as opposed to only sequences). We adopt a duality approach; given a family F of functions on a set T we establish complete duality by considering the pair F, T as a function B defined on $F \times T$ by the formula

$$B(f, t) = f(t)$$

and by treating both variables in the same manner; in particular this point of view makes it natural to consider convex combinations of elements of T.

We have seen that an ingenious application of the lemma yields (for equibounded sequences) the equivalence of almost uniform convergence and convergence in the mean. For families of functions the condition of almost uniform convergence will be replaced by a closely related condition known as the iterated limit condition.

Definition. *A (complex valued) function f defined on the cartesian product of two sets X and Y is said to satisfy the iterated limit condition if the following implication holds: given two sequences $x_n \in X$, $y_n \in Y$ such that the limits*

$$\begin{aligned} \lim_n f(x_p, y_n) &= f_{p,\infty} \\ \lim_n f(x_n, y_q) &= f_{\infty,q} \\ \lim_p f_{p,\infty} &= f_{*,\infty} \\ \lim_q f_{\infty,q} &= f_{\infty,*} \end{aligned}$$

exist, then $f_{*,\infty} = f_{\infty,*}$.

This important condition appears first, in a particular form, in the work to S. Banach – the present symmetric form is due to A. Grothendieck.

The following proposition exhibits a typical situation where the iterated limit condition is satisfied.

A completely regular topological space T is said to be *countably compact* if every sequence $F_n \supset F_{n+1}$ of closed subsets of T has a nonvoid intersection.

In particular, given a sequence $t_n \in T$, define, for each n, F_n are the closure of the set consisting of all points t_j for $j \geq n$. Then the intersection $\cap F_n$ is nonvoid. A point $t \in \cap F_n$ is called a *cluster point* of the sequence t_n. If f is a continuous function on T and $t_n \in T$ a sequence for which the limit $\lim_n f(t_n) = \alpha$ exists then $\alpha = f(t)$ for each cluster point t of the sequence t_n.

(4.1) *Let T be a countably compact completely regular topological space. Let $A \subset C(T)$ be bounded and countably compact in the topology of pointwise convergence on $C(T)$. Then A satisfies the iterated limit condition: more precisely, the function $a, t \to a(t)$ defined on $A \times T$ satisfies the iterated limit condition.*

Proof: Suppose $a_k \in A$, $t_j \in T$ are two sequences such that

$$\begin{aligned} \lim_j a_k(t_j) &= f_{k\infty} \qquad \text{for all } k, \\ \lim_k a_k(t_j) &= f_{\infty j} \qquad \text{for all } j; \end{aligned}$$

furthermore, suppose $\lim_k f_{k\infty} = f_{*\infty}$ and $\lim_j f_{\infty j} = f_{\infty *}$.

Let t be a cluster point of the sequence t_j and a a cluster point of sequence a_k. It follows that $f_{k\infty} = a_k(t)$ for every t. Furthermore $a(t_j) = \lim_k a_k(t_j) = f_{\infty j}$ for every j and $a(t) = \lim a_k(t) = \lim f_{k\infty} = f_{*\infty}$. Also, $f_{\infty,*} = \lim_j f_{\infty j} = \lim_j a(t_j) = a(t)$. □

The preceding proposition exhibits a typical concrete situation in which the iterated limit condition may easily be verified; we do not aim at the greatest generality, countable compactness could easily have been replaced by pseudocompactness or by other weaker conditions [10].

The iterated limit condition closely related to the notion of almost uniform convergence. It may be restated in several other forms in which the connection becomes more evident. It will be convenient to introduce a

Definition. *A double sequence a_{pq} of complex numbers is said to be convergent if $\lim\limits_q a_{pq} = a_{p\infty}$ exists for each p and $\lim\limits_p a_{pq} = a_{\infty q}$ exists for each q.*

The following equivalence throws more light on the meaning of the iterated limit condition.

(4.2) *Let a_{pq} be a bounded convergent double sequence. Then the following statements are equivalent:*

1° *the convergence $\lim\limits_q a_{pq} = a_{p\infty}$ is almost uniform with respect to p,*

2° *the convergence $\lim\limits_p a_{pq} = a_{\infty}$ is almost uniform with respect to q,*

3° *both limits $\lim\limits_p a_{p\infty}$ and $\lim_q a_{\infty q}$ exist and are equal to each other,*

4° *the covergence $\lim\limits_q a_{pq} = a_{p\infty}$ is uniform in the mean with respect to p,*

5° *the convergence $\lim\limits_q a_{pq} = a_{\infty q}$ is uniform in the mean with respect to q.*

We intend to concentrate on the purely combinatorial essence of the results. Observe that in the following two propositions no assumptions of continuity are made on the function $B(\cdot,\cdot)$.

The iterated limit condition is closely related to the notion of almost uniform convergence; the connection will be evident from the following lemma.

(4.3) *Let f_n be a sequence of functions defined on a set T such that $f_n(t) \to f_\infty(t)$ for each $t \in T$. Suppose that $|f_n(t)| \leq 1$ for all n and t and that the function*

$$[s,t] \to f_s(t)$$

satisfies the iterated limit condition. Then the convergence is almost uniform; thus f_n converges to f_∞ in the mean.

Proof: Suppose that, for some $\varepsilon > 0$, there exists, for each n, a point t_n such that

$$|f_i(t_n) - f_\infty(t_n)| \geq \varepsilon$$

for $i = 1, 2, \ldots, n$. Thus, for each i, the difference $|f_i(t_n) - f_\infty(t_n)| \geq \varepsilon$ for all $n \geq i$. Use the assumption of boundedness to select a subsequence t'_n of the sequence t_n in such a manner that the limits $\lim\limits_n f_i(t'_n)$ exist for each i including ∞. The double sequence $f_i(t'_j)$ contradicts the iterated limit condition. □

Now we are ready to reproduce the historically first application of the combinatorial lemma [6], [7]. The result presents an opportunity to exhibit the use of combinatorial methods in their purest form, without any additional topological assumptions. In conformity with our general duality principle we shall consider a bounded function f defined on the cartesian product of two sets $U \times A$. It will be convenient to regard f as the restriction of a bilinear form (denoted again by f). Indeed, it is natural to extend f to a bilinear form $f^\wedge$ in the following manner

$$f^\wedge \left(\sum_{j=1}^{p} \alpha_j u_j, \ \sum_{k=1}^{q} \beta_k a_k \right) = \sum_{j,k} f(u_j, a_k) \alpha_j \beta_k$$

(the sum $\sum_{j=1}^{p} \alpha_j u_j$ has the obvious meaning if U is contained in a vector space, otherwise it is to be taken as a formal sum; the same applies to the other sum). This convention makes it possible to formulate the following

(4.4) Theorem. *Let f be a bounded real valued function on $U \times A$. If f satisfies the iterated limit condition on $U \times A$ then f satisfies the iterated limit condition also on $U \times conv\ A$.*

Proof: Consider two sequences $u_i \in U$, $c_j \in$ conv A such that

$$\begin{array}{ccc} f(u_i, c_j) & \vec{j} & f(u_i, \infty) \\ i \downarrow & & \\ f(\infty, j) & \vec{j} & f(\infty, \infty) \end{array}$$

and such that $f(u_i, \infty) - f(\infty, \infty) \geq \alpha > 0$ for all i. There exists a countable $T \subset A$ such that all the c_j are convex combinations of elements of T. It is possible to extract a subsequence of the sequence u_i (we shall call it u_i again) such that, for each $t \in T$, the limit $\lim_i f(u_i, t) = f(\infty, t)$ exists. Since f is bounded and satisfies the iterated limit condition, it follows from lemma (4.3) that there exists a convex mean $\lambda(k)$, $k \in F$ such that

$$\left| \sum \lambda(k)(f(u_k, t) - f(\infty, t)) \right| < \varepsilon$$

for every $t \in T$. If we agree to write $f(\infty, \sum \alpha_r t_r)$ for $\sum \alpha_r f(\infty, t_r)$ the previous estimate extends to conv A, so that $| \sum \lambda(k)(f(u_k, c) - f(\infty, c))| < \varepsilon$ for all $c \in$ conv T. Now choose c_j such that

$$|f(u_k, c_j) - f(u_k, \infty)| < \varepsilon$$

for all $k \in F$ and $|f(\infty, c_j) - f(\infty, \infty)| < \varepsilon$. We have then

$$\begin{aligned} \alpha \ \leq \ & \sum \lambda(k)(f(u_k, \infty) - f(\infty, \infty)) = \sum \lambda(k)(f(u_k, \infty) - f(u_k, c_j)) \\ & + \sum \lambda(k)(f(u_k, c_j) - f(\infty, c_j)) + \sum \lambda(k)(f(\infty, c_j) - f(\infty, \infty)). \end{aligned}$$

Each of these summands being less than ε in absolute value, the preceding inequality yields a contradiction if $\varepsilon < \frac{1}{3}\alpha$. □

In conformity with our general principle we deal with bounded functions of two variables. In applications these functions will mostly be pairings of elements of a Banach space and of functionals from its dual so that in most cases the arguments in the function

$$B(s,t) \quad \text{on} \quad S \times T$$

will come from sets with a linear structure of their own. Even if this is not the case, we shall consider B as the restriction of a bilinear form. Indeed, it is possible to extend B to an bilinear form on the linear span of S and the linear span of T in the obvious manner. If T is a topological space and the functions $f \in F$ are continuous on T then the following statements are equivalent:

(1) $B(f,t)$ is separately continuous,

(2) F is taken in the topology of pointwise convergence.

The algebraic extension to a bilinear form may go a little further in this case: in fact given s, the function $B(s,\cdot)$ is a continuous function on T, so that for any continuous linear functional q on $C(T)$, the scalar product $\langle B(s,\cdot), q\rangle$ is a natural candidate for $B(s,q)$. Given a continuous linear functional p on $C(S)$, it would seem natural to take $B(p,q) = \langle B(\cdot,q), p\rangle$.

This, of course, is not possible in general since it presupposes that $B(\cdot,q)$ is continuous on S. This raises the question under what conditions $B(\cdot,q)$ will be continuous, a problem interesting on its own. The problem will turn out to be important for applications. We now proceed to treat it in detail.

If T is a completely regular topological space, denote by $C(T)$ the Banach space of all bounded continuous functions on T equipped with the norm

$$|x| = \sup\{|x(t)|;\ t \in T\}.$$

Let $C(T)'$ be the dual of $C(T)$ taken in the w^* topology, $\sigma(C(T)', C(T))$. For each $t \in T$ the mapping

$$x \to x(t)$$

clearly defines a linear functional on $C(T)$, obviously of norm one. We call it the evaluation functional corresponding to t and denote it by $e(t)$. In this manner

$$\langle x, e(t)\rangle = x(t)$$

for $x \in C(T)$ and $t \in T$. In view of the complete regularity of T the mapping

$$e : T \to C(T)'$$

is one-to-one; at the same time the topology in $C(T)'$ is chosen in such a manner that e is homeomorphic. Identifying t with $e(t)$ we obtain an embedding of T in $C(T)'$ as a topological space, not only as a set. Since $e(T)$ is contained in the unit ball of $C(T)'$ its closure is easily seen to be homeomorphic with the Stone-Čech compactification of T. The mapping e makes of possible to consider each completely regular topological space T as a subset of a topological *vector* space $(C(T)',\ \sigma(C(T)', C(T))$.

In this manner we obtain an embedding of T in a topological space with an additional algebraic structure; a number of important problems may be formulated in the following form: given a continuous mapping of T into a topological vector space, under what conditions does it possess a continuous extension to $C(T)'$ which also respects the linear structure of $C(T)'$? We intend to show how this approach may be used with advantage to treat weak compactness.

As a first application of the combinatorial method we state the following general theorem which essentially says that invertibility of the simplest sequential limit operations implies the invertibility of substantially more complicated ones.

Now we are ready to present the main result. It will be stated in two theorems; though formally different, essentially their information content is the same.

(4.5) A theorem of Fubini type. *Let S and T be two completely regular topological spaces.*

Suppose f is a bounded separately continuous function or $S \times T$ which satisfies the iterated limit condition on $S \times T$. Given $p \in C(S)'$, $q \in C(T)'$, consider the functions

$$\begin{aligned} \langle f(s,\cdot), q\rangle &= \varphi(s) \qquad \text{for} \quad s \in S, \\ \langle f(\cdot,t), p\rangle &= \psi(t) \qquad \text{for} \quad t \in T. \end{aligned}$$

Then

1° *the functions φ and ψ are continuous,*

2° $\langle \varphi, p\rangle = \langle \psi, q\rangle$.

(4.6) The Extension Theorem for separately continuous functions.
Let S and T be two completely regular topological spaces and f a bounded separately continuous function on $S \times T$.

There exists a separately continuous bilinear form on $C(S)' \times C(T)'$ which extends f if and only if f satisfies the iterated limit condition on $S \times T$.

Sketch of proof. The only if part is immediate: indeed, if a separately continuous extension to $C(S)' \times C(T)'$ exists it suffices to consider its restriction to $\beta S \times \beta T$, βS and βT being the closures of S and T in $C(S)'$ and $C(T)'$; these closures being compact, proposition (4.1) applies.

To explain the notation βS and βT: The reader will have observed that the closure of a completely regular space M in $C(M)'$ may be identified with the Stone-Čech compactification of M.

On the other hand, let us turn to the construction of the extension. A moment's reflection shows that such an extension B, if it exists, must satisfy

$$B(s,q) = \langle f(x,\cdot), q\rangle \qquad \text{for} \quad s \in S$$

and that the value $B(p,q)$ is obtained by applying the functional p to the function $s \to B(s,q)$. The same applies if this process is applied starting with the function

$$t \to \langle f(\cdot,t), p\rangle$$

and applying the functional q to it.

In this manner we see that B is uniquely defined if it exists. To prove the existence, consider a fixed $q \in C(T)'$ and the corresponding function $s \to \langle f(s,\cdot), q\rangle = B(s,q)$. Using the iterated limit condition, prove that the function $B(\cdot, q)$ is continuous on S so that $\langle B(\cdot,q), p\rangle$ is meaningful. Repeating this construction with the order of p and q inverted, we obtain

$$\langle B(p,\cdot), q\rangle \,.$$

The preceding theorem shows that both these processes are meaningful and lead to the same result.

5 Application to Weak Compactness

For metrizable topological spaces the notions of countable compactness and compactness coincide; in the general case, without metrizability, countable compactness can be weaker than compactness. Although the weak topology of a Banach space is far from metrizable, W.F. Eberlein was able to prove, in 1947, the following surprising theorem.

Let E be a Banach space taken in its weak topology. If $A \subset E$ is countably compact then the closure of A is compact.

We shall see how this may be deduced from the extension theorem. In fact, we get at the same time considerably more, the compactness of the closure of the convex hull of A. In view of (4.6) if suffices to prove the following.

(5.1) *Let S be a bounded subset of a Banach space E. Denote by T the unit ball of the dual space E'. Suppose that the scalar product $\langle s,t\rangle$ on $S \times T$ satisfies the iterated limit condition. Then the bipolar of S is weakly compact.*

Proof: Take $T = (U^0, \sigma(E', E))$ and define f on $S \times T$ as the scalar product

$$f(s,t) = \langle s,t\rangle \,.$$

In this manner f is bounded, separately continuous and satisfies the iterated limit condition on $S \times T$. By the extension theorem, there exists a bounded separately continuous bilinear from B on $C(S)' \times C(T)'$ which extends f. Consider a fixed $\widetilde{s} \in C(S)'$ and the corresponding linear form

$$B(\widetilde{s}, \cdot)$$

on $C(T)'$. When restricted to the linear hull of T of U^0 (which is nothing more than E') it may be considered as an algebraic linear form on E'. It follows from the continuity of B in the second variable that $B(\widetilde{s}, \cdot)$, when restricted to U^0, is $\sigma(E', E)$ continuous; accordingly, it may be identified with an element of E. This element will be denoted by $P(\tilde{s})$. For $y \in U^0$ we have thus

$$\langle P(\tilde{s}), y\rangle = B(\tilde{s}, y).$$

Observe that the linear mapping $P : C(S)' \to E$ obtained in this manner acts as the identity on S: indeed, if $e \in S$, we have – B being an extension of f –

$$\langle P(e), y\rangle = B(e, y) = \langle e, y\rangle\,,$$

whence $P(e) = e$ for $e \in S$. Since B is continuous in the first variable, it is easy to see that P is a continuous mapping of $C(S)'$ (in its w^* topology) into $(E,\, \sigma(E, E'))$. If V is the unit ball of $C(S)'$ its image $P(V)$ will thus be $\sigma(E, E')$ compact. Thus

$$S = P(S) \subset P(V)$$

and $P(V)$ is an absolutely convex weakly compact subset of E. □

6 Duality

Among the many possible interpretations of the combinatorial problems discussed above the formulation as an optimization problem is of particular interest. The quantity

$$\inf_{\lambda \in P} \sup_{w \in W} \lambda(w)$$

represents a numerical characteristic of the family W which reflects some aspects of its combinatorial structure. When interpreted in geometric terms if assumes a form the intuitive meaning of which we now proceed to explain.

This characteristic may be given an intuitive geometric interpretation in terms of what we call the *thickness of a set* in a normed vector space. If M is a subset of a normed vector space E we define the thickness $e(M)$ of M by the formula

$$e(M) = \inf \sup f(m_1 - m_2)$$

as m_1, m_2 range over the set M and f over all linear functionals on E of norm not exceeding one. In this manner, $e(M)$ is the smallest distance of two parallel hyperplanes in E such that the set M lies between them.

In our case, E will be the Banach space $B(S)$ of all bounded complex functions on S with the norm

$$|x| = \sup\{|x(s)|;\ s \in S\}.$$

Given a complex function $\alpha(s)$ on S with finite support, the mapping

$$x \to \sum \alpha(s)x(s)$$

is clearly a bounded linear functional on $B(S)$; its norm equals $\sum |\alpha(s)|$. In particular, each probability measure $\lambda \in P(S)$ is a linear functional on $B(S)$ of norm 1.

In our case we shall use this geometric idea to describe a combinatorial characteristic of families of subsets of S. Given a family $W \subset \exp S$ and identifying each $w \in W$ with its characteristic function we may consider W as a subset of $B(S)$.

It will be convenient to introduce the notion of thickness of a family of subsets of S.

For each $\lambda \in P(S)$, considered as a functional on $B(S)$, and each $w \in W$ considered as an element of $B(S)$, we have

$$\langle w, \lambda \rangle = \lambda(w).$$

In this manner, the whole family W is contained in the set

$$\{x;\ 0 \leq \langle x, \lambda \rangle \leq \sup\{\lambda(w);\ w \in W\}.$$

The study of the combinatorial structure of families of sets discussed in the preceding chapter made it possible to interpret $e(W)$ as a characteristic of an optimization problem, giving it, in this manner, an intuitive geometric meaning. It is to be expected that the dual interpretation of the optimization problem will provide further intuitive insight into the matter; this is indeed so – a standard application of the separation lemma for convex sets will present $e(W)$ in a different light, giving further support to the intuitive interpretation as thickness.

In a manner of speaking, the two mutually dual interpretations of $e(W)$ correspond to two natural ways of visualizing a relation $R \subset P \times Q$: we may either view it as a multivalued mapping of P into Q setting

$$\begin{aligned} p \quad &\rightarrow \quad \{q \in Q;\ [p, q] \in R\}, \\ &p \rightarrow R(p), \end{aligned}$$

or as a family of subsets of P parametrized by Q

$$\begin{aligned} q \quad &\rightarrow \quad \{p \in P;\ [p, q] \in R\}, \\ &q \rightarrow R^{-1}(q). \end{aligned}$$

In order to obtain a dual description of $e(W)$ it will be convenient to use the second approach to $S \times W$ and assign to every $w \in W$ the characteristic function of $R^{-1}(w)$, in other words, to write

$$w(s) = 1 \quad \text{iff} \quad s \in w, \quad \text{otherwise} \quad w(s) = 0.$$

Our first observation will be the following estimate.

(6.1) *Let W be an arbitrary family of subsets of S. Then*

$$e(W) \geq \inf_F \sup_{\mu \in P(W)} \inf_{s \in F} \sum_{w \in W} \mu(w) w(s)$$

as F ranges over all finite subsets of S.

The intuitive meaning of this estimate is obvious: if β is such that, for each finite $F \subset S$, there exists a convex combination b of the functions w with $b(F) \geq \beta$, then $e(W) \geq \beta$.

Proof: Write β for

$$\beta = \inf_F \sup_{\mu \in P(W)} \inf_{s \in F} \sum \mu(w) w(s)$$

and suppose that $e(W) < \beta$. Choose β', β'' so as to have $e(W) < \beta' < \beta'' < \beta$. Since $e(W) < \beta'$ there exists a $\lambda \in P(S)$ such that $\lambda(w) < \beta'$ for every $w \in W$. Set $F = N(\lambda)$. Since $\sup_{\mu \in P(W)} \inf_{s \in F} \sum \lambda(w)w(s) \geq \beta$ there exists a $\mu \in P(W)$ such that $\inf_{s \in F} \sum \mu(w)w(s) \geq \beta''$. Thus $\beta'' \leq \langle \sum \mu(w)w(\cdot), \lambda \rangle \leq \sum \mu(w)\lambda(w) < \sum \mu(w)\beta' = \beta'$, a contradiction. □

If the $w \in W$ are interpreted as functions on S then, for each $\mu \in P(W)$, the sum $\sum \mu(w)w(s)$ is nothing more than the value at the point s of the convex combination $\sum \mu(w)w$ of the functions w: as λ ranges over $P(W)$ these functions range over the convex hull of W. The inequality above may thus be rewritten in the form

$$e(W) \geq \inf_F \sup_{b \in \text{conv } W} \inf_{s \in F} b(s).$$

The dual characterization of $e(W)$ may be formulated as follows.

(6.2) *Let W be an arbitrary family of subsets of a set S. Then*

$$\inf_{\lambda \in P(S)} \sup_{w \in W} \lambda(w) = \inf_F \sup_{b \in \text{conv } W} \inf b(F)$$

as F ranges over all finite subsets of S.

Proof: The preceding lemma may be interpreted as the inequality $\geq$. Thus it remains to prove the opposite inequality. Let us show that, for every finite $F \subset S$ there exists a $b \in \text{conv } W$ such that $b(s) \geq e(W)$ for all $s \in F$. If $s_1, \ldots, s_n$ are the elements of F, define a mapping G of $B(S)$ into R^n by the formula

$$G(x) = (x(s_1), \ldots, x(s_n)).$$

Denote by M the subset of R^n consisting of all $(y_1, \ldots, y_n) \in R^n$ for which all $y_j \geq e(W)$. Suppose the intersection $G(\text{conv } W) \cap M$ is void. Since $G(\text{conv } W)$ is compact there exists a linear form α on R^n

$$\alpha(x) = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

such that $\sup \alpha(G(\text{conv } W)) < \inf \alpha(M)$. Since $\inf \alpha(M)$ is finite, it follows that all α_j are nonnegative. Since α is nonzero we may assume that $\sum \alpha_j = 1$, in other words $\alpha \in P(S)$. Now $\inf \alpha M \leq e(W)$ so that

$$\sup \alpha(W) = \sup \alpha(\text{conv } W) = \sup \alpha(G \text{ conv } W) < \inf \alpha M \leq e(W),$$

a contradiction. The proof is complete. □

7 Number Theory

The motivation for the optimization problem $\inf_\lambda \sup_w \lambda(w)$ was a study of limit processes – accordingly, the set S was infinite. In this section we intend to show that the optimization problem is not without interest in the case when S is finite. Among the many relationships with other branches of mathematics we single out, in this section, the connection with number theory.

If the set S is finite then so is the family W. Identifying each set $w \in W$ with its characteristic function we may write

$$\lambda(w) = \sum \lambda(s)w(s).$$

Thus

$$\sum_w \lambda(w) = \sum_w \sum_s \lambda(s)w(s);$$

inverting the order of the summations we obtain the identity

$$\sum_w \lambda(w) = \sum_s \lambda(s)n(s)$$

where $n(s)$ is the number of w such that $s \in w$.

This leads to a lower estimate for $e(W)$.

(7.1) *If the family W is finite then*

$$e(W) \geq \min_s \frac{\text{card } W(s)}{\text{card } W}.$$

Proof: Writing n for card W and $n(s)$ for card $W(s)$ we have the following estimate.

$$ne(W) = n \inf_\lambda \sup_w \lambda(w) \geq \inf_\lambda \sum_w \lambda(w) = \inf_\lambda \sum_s \lambda(s)n(s) \geq \inf n(s).$$

$\square$

This estimate has a dual counterpart; to state it, we introduce an abbreviation: the cardinality of a set M will be denoted by $\|M\|$.

(7.2) *Consider a relation $R \subset S \times T$ and denote by W the family $R^{-1}(t)$, $t \in T$. Then*

$$\inf_s \frac{\|R(s)\|}{\|T\|} \leq e(W) \leq \sup_t \frac{\|R^{-1}(t)\|}{\|S\|}.$$

Proof: $e(W) = \inf_\lambda \sup_w \lambda(w) = \inf_\lambda \sup_t \lambda(R^{-1}(t)) \leq \sup_t \lambda_0(R^{-1}(t))$ where λ_0 is the arithmetic mean defined by $\lambda(s) = \frac{1}{\|S\|}$ for all s. Thus $\lambda_0(R^{-1}(t)) = \frac{\|R^{-1}(t)\|}{\|S\|}$.

The lower bound for $e(W)$ is a consequence of (7.1). $\square$

This section will be devoted to the study of the relationship between the combinatorial structure of a family W and its thickness – in particular to the question of constructing, for a given number α, $0 < \alpha < 1$, a family W for which $e(W) = \alpha$.

There are many ways to represent a given rational number $\frac{k}{n}$ in the form $e(W)$ for a suitable family W. The most economic one would be one for which

$$\begin{aligned} \|S\| &= \|T\| = n, \\ \|R(s)\| &= \|R^{-1}(t)\| = k \qquad \text{for all} \quad s, t. \end{aligned}$$

Such a relation may be described as follows: $S = T = \{0, 1, \ldots, n-1\}$. To define R, we distinguish two cases. If $i + k - 1 \leq n - 1$ we set

$$[i, j] \in R \qquad \text{iff} \qquad i \leq j \leq i + k - 1.$$

For $i + k - 1 \geq n$

$$[i, j] \in R \qquad \text{iff} \qquad i \leq j \leq n - 1 \qquad \text{or} \qquad 0 \leq j \leq i + k - 1 - n.$$

For $k = 3$, $n = 7$, we obtain the following pattern

$$\begin{array}{ccccccc} \ast & \ast & \ast & 0 & 0 & 0 & 0 \\ 0 & \ast & \ast & \ast & 0 & 0 & 0 \\ 0 & 0 & \ast & \ast & \ast & 0 & 0 \\ 0 & 0 & 0 & \ast & \ast & \ast & 0 \\ 0 & 0 & 0 & 0 & \ast & \ast & \ast \\ \ast & 0 & 0 & 0 & 0 & \ast & \ast \\ \ast & \ast & 0 & 0 & 0 & 0 & \ast \\ \ast & \ast & \ast & 0 & 0 & 0 & 0 \end{array}$$

Given n and k $(0 \leq k < n)$, S may be viewed as the set of residue classes of integers modulo n and the family W consists of all intervals of length k arranged on a circle; there are n such intervals.

Let us turn now to the problem of representing an irrational number. Intertwining the families corresponding to rational approximations of the number in a suitable manner [2] we obtain.

(7.3) *Suppose α is an irrational number, $0 < \alpha < 1$. Then there exists a set S and a family W of subsets of S such that $e(W) = \alpha$.*

Proof: For each $m = 1, 2, \ldots$ let k_m be the integer defined by the requirement that

$$k_m - 1 < 2^m \alpha \leq k_m.$$

It follows that $1 \leq k_m \leq 2^m$ and $k_m 2^{-m} \downarrow \alpha$.

Let S be the set of rational numbers defined as the union $S = S_1 \cup S_2 \cup \ldots$ where S_k is the set of all rational numbers of the form $m/2^k$ with $0 < m \leq 2^k$. For each $m = 1, 2, \ldots$ we define a family W_m of subsets of S_m

$$\begin{aligned} W_m &= \{w_{mj};\ j = 1, 2, \ldots, 2^m\}, \\ w_{mj} &= \left\{ \frac{j}{2^m}, \frac{j+1}{2^m}, \ldots, \frac{j + k_m - 1}{2^m} \right\}, \end{aligned}$$

the numerators being taken modulo 2^m. In this manner the family W_m consists of 2^m subsets of S_m, each of cardinality k_m. The family W will be defined as the union of the families $W_1, W_2, \ldots$.

Given $\lambda \in P(S)$, there exists an m such that the support of λ is contained in S_m. For every $s \in S_m$ there are exactly k_m sets w_{mj} such that $s \in w_{mj}$.

We begin by proving the inequality $e(W) \geq \alpha$; identifying the sets w with their characteristic functions on S our task is to compute the infimum

$$\inf_F \sup_b \inf b(F);$$

here F are arbitrary finite subsets of S while b ranges over all convex combinations of the functions w.

Observe that $\sup_b \inf b(F) \geq \sup_b \inf b(F')$ if $F \subset F'$; since every finite F is contained in one of the S_m if m is large enough the infimum above equals

$$\inf_m \sup_b \inf b(S_m)$$

and is bounded below by $\inf_m \inf b_m(S_m)$ for any sequence b_m of convex combinations of the functions w. Taking $b_m = \frac{1}{2^m} \sum_{j=1}^{2^m} w_{mj}$ we have $b_m(s) = \frac{k_m}{2^m}$ for all $s \in S_m$ and the estimate $e(W) \geq \alpha$ follows.

On the other hand, fix an n and define $\lambda_n \in P(S)$ by setting $\lambda_n(s) = \frac{1}{2^n}$ for $s \in S_n$ and $\lambda(s) = 0$ otherwise. Consider an arbitrary $w \in W$, $w = w_{mr}$ for some r, $1 \leq r \leq 2^m$. Clearly $\lambda_n(w_{mr})$ equals $\frac{1}{2^m}$ times the cardinality of the intersection $w_{mr} \cap S_n$.

To estimate the cardinality of $w_{mr} \cap S_n$ we consider a sequence of k_m consecutive numbers of the form $\frac{x}{2^m}$ and ask for the number of those among them that are of the form $\frac{y}{2^n}$. We distinguish two cases. If $m \leq n$ we have $w_{mr} \subset S_m \subset S_n$ so that the cardinality is k_m. Thus

$$\lambda_n(w_{mr}) = \frac{1}{2^n} k_m = \frac{1}{2^n}(k_m - 1) + \frac{1}{2^n} \leq \frac{1}{2^m}(k_m - 1) + \frac{1}{2^n} < \alpha + \frac{1}{2^n}.$$

In the case $m > n$ the cardinality does not exceed $\frac{k_m}{2^{m-n}} + 1$ so that

$$\lambda_n(w_{mr}) = \frac{1}{2^n} \text{ card } (w_{mr} \cap S_n) \leq \frac{1}{2^m} k_m + \frac{1}{2^n} < \frac{1}{2^m}(k_m - 1) + \frac{1}{2^{n-1}}.$$

It follows that $e(W) < \alpha + \frac{1}{2^{n-1}}$ for every n whence $e(W) \leq \alpha$.

In this manner every number between zero and one may be represented as the thickness of a suitable family of finite sets. This representation not only reflects some number theoretical properties of the number; it has the additional advantage that arithmetical operations with the numbers correspond in a natural manner to combinatorial constructions with the corresponding families of sets.

We now define combinatorial operations on families of sets in such a manner that a combinatorial operation performed on two families of sets corresponds to a corresponding arithmetical operation performed on their thicknesses.

We begin by showing that the product of numbers corresponds to the cartesian product of the representing families. More precisely.

(7.4) *Let W_1 and W_2 be two families of subsets of S_1 and S_2 respectively. Set $S = S_1 \times S_2$ and let W the family of sets of the form $w_1 \times w_2$ with $w_i \in W_i$. Then $e(W) = e(W_1)e(W_2)$.*

Proof: Given $\varepsilon > 0$, let $\lambda_i \in P(S_i)$ be convex means such that

$$\lambda_i(w_i) < e(W_i) + \varepsilon$$

for all $w_i \in W_i$. Define $\lambda_0 \in P(S)$ as the product of λ_1 and λ_2,

$$\lambda_0([s_1, s_2]) = \lambda_1(s_1)\lambda_2(s_2).$$

Given $w_1 \in W_1$ and $w_2 \in W_2$ we have

$$\lambda_0(w_1 \times w_2) = \lambda_1(w_1)\lambda_2(w_2) \leq (e(W_1) + \varepsilon)(e(W_2) + \varepsilon).$$

Thus

$$e(W) \leq \sup_{w_1, w_2} \lambda_0(w_1 \times w_2) \leq (e_1 + \varepsilon)(e_2 + \varepsilon)$$

whence

$$e(W) \leq e(W_1)e(W_2).$$

The opposite inequality may be obtained in a similar manner using the dual characterization of e. □

In this manner the product of numbers corresponds to the cartesian product of their representations. There is a similar correspondence between the operation of addition and the union of the representations.

Consider two pairs (S_1, W_1) and (S_2, W_2). To avoid complications assume that S_1 and S_2 are disjoint. Set $S = S_1 \cup S_2$ and $W = W_1 \cup W_2$. For every convex mean λ on one of the S_i we write λ^0 for the convex mean on S obtained by setting $\lambda(s)$ zero on the other set.

(7.5) $\frac{1}{e(W)} = \frac{1}{e(W_1)} + \frac{1}{e(W_2)}$.

Proof: For $i = 1, 2$ let $\lambda_i \in P(S_i)$ satisfy

$$\lambda_i(w_i) < e(W_i) + \varepsilon$$

for all $w_i \in W_i$. Set

$$\lambda_0 = \frac{e_2}{e_1 + e_2}\lambda_1^0 + \frac{e_1}{e_1 + e_2}\lambda_2^0.$$

Consider a $w \in W$. Suppose $w \in W_1$. Then $\lambda_0(w) = \frac{e_2}{e_1+e_2}\lambda_1(w) \leq \frac{e_2}{e_1+e_2}(e_1 + \varepsilon) \leq \frac{e_2 e_1}{e_1+e_2} + \varepsilon$; we obtain the same estimate if $w \in W_2$. It follows that

$$e(W) \leq \sup_{w} \lambda_0(w) \leq \frac{e_1 e_2}{e_1 + e_2} + \varepsilon$$

so that $e(W) \leq \frac{e_1 e_2}{e_1+e_2}$.

The opposite inequality may be obtained in a similar manner using the dual characterization of e.

Given a finite set $F \subset S_1 \cup S_2$ and a positive ε, there exist $b_i \in$ conv W_i such that $v_i(F \cap S_i) > e(W_i) - \varepsilon$. Let b_0 be the function defined on $S_1 \cup S_2$ as follows:

$$\begin{aligned} b_0(s) &= \frac{e_2}{e_1+e_2} b_1(s) \quad \text{for } s \in S_1, \\ b_0(s) &= \frac{e_1}{e_1+e_2} b_2(s) \quad \text{for } s \in S_2; \end{aligned}$$

thus $b_0 \in$ conv W. Furthermore $b_0(s) \geq \frac{e_1 e_2}{e_1+e_2} - \varepsilon$ for every $s \in S$. In this manner

$$\inf_F \sup_{b \in \text{ conv } W} \inf b(F) \geq \frac{e_1 e_2}{e_1+e_2} - \varepsilon$$

for every $\varepsilon > 0$. It follows that $e(W) \geq \frac{e_1 e_2}{e_1+e_2}$. □

In view of these facts it seems that the relationship between numbers and their combinatorial representations could be worth investigating.

It would be interesting to relate the properties of a number and the combinatorial structure of its representation; in particular, is there a combinatorial characterization of algebraic numbers?

Bibliography

[1] M. Ajlani: Critères de compacité faible par le lemme combinatoire de Pták Séminaire Choquet 7 (1968), B4, 1-10.

[2] A. Lešanovský and V. Pták: A measure of thickness for families of sets, Discrete Math. 58 (1986) 35-44.

[3] J. Kindler: Some consequences of a double limit condition, Game Theory Math. Econ. (1981) 73-82.

[4] J. Kindler: Minimaxtheoreme für die discrete gemischte Erweiterung von Spielen und ein Approximationssatz, Math. Oper. Statist. Ser. Optim. 11 (1980) 473-485.

[5] J. Kindler: A minimax version of Pták's combinatorial lemma, Journal of Math. Analysis and Appl. 94 (1983), 454-459.

[6] G. Köthe: Topologische lineare Räume, Grundlehren der Math. Wissenschaften Bd 107 (Springer, Berlin, 1960).

[7] G. Köthe: Topological Vector Spaces, Grundlehren Math. Wiss. Vol. 159 (Springer, Berlin, 1983).

[8] I. Namioka: A substitute for Lebesgue's bounded convergence theorem, Proc. Amer. Math. Soc. 12 (1961) 713-716

[9] S. Negrepontis, S. Yiotopoulos, E. Yiannakoulias: Apirostikos logismos, tomos II, Ekdosis Symmetria, Athina 1993.

[10] V. Pták: On a theorem of W. F. Eberlein, Studia Math. 14 (1954), 276-284.

[11] V. Pták: A combinatorial theorem on systems of inequalities and its application to analysis, Czech. Math. J. 84 (1959) 629-630.

[12] V. Pták: An extension theorem for separately continuous functions and its application to functional analysis, Comment. Math. Univ. Carolin. 4 (1963) 109-116.

[13] V. Pták: A combinatorial lemma on the existence of convex means and its application to weak compactness, Amer. Math. Soc. Proc. Symp. Pure Math. 7 (1963), 437-450.

[14] V. Pták: An extension theorem for separately continuous and its application to functional analysis, Czechoslovak Math. J. 89 (1964) 562-581.

[15] V. Pták: An extension theorem for separately continuous functions and its applications to functional analysis and game theory, in: Proc. Coll. Convexity, Copenhagen, 1965 (1967) 242-243.

[16] V. Pták: Algebraic extension of topological spaces, in: Proc. Internat. Symp. on Extension Theory of Topological Structures held in Berlin in 1967 (1969) 179-188.

[17] V. Pták: Thickness of families of sets and a minimax lemma, Discrete Math. 108 (1992), 175-181.

[18] S. Simons: On Pták's combinatorial lemma, Pacific J. Math. 40 (1972) 719-721.

[19] S. Simons: A theorem on lattice ordered groups, results of Pták, Namioka and Banach, and a frontended proof of Lebesgue's theorem, Pacific J. Math. 20 (1967), 149-153.

[20] N. J. Young: On Pták's double limit theorems, Proc. Edinburgh Math. Soc. 17 (1971) 193-200.

[21] N. J. Young: Admixtures of two-person games, Proc. London Math. Soc. 25 (1972) 736-750.

[22] Z. Zalewasser: Sur une propriété du champ des fonctions continues, Studia Math. 2 (1930) 63-67.