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Abstract

Dempster–Shafer theory is an interesting and useful mathematical tool for uncertainty quantification and processing. From one point of view it can be seen as an alternative apparatus to probability theory and mathematical statistics based on this probability calculus, as D.–S. theory can be developed in a way quite independent of probability theory, beginning with a collection of more or less intuitive demands which an uncertainty degree calculus should meet. On the other side, however, D.–S. theory can be developed also as a particular sophisticated application of probability theory, using the notion of non–numerical, in particular, set–valued random variables (random sets) and their numerical characteristics. This later aspect enables to generalize D.–S. theory beyond its classical scopes using appropriately the apparatus of probability theory and measure theory.

This report is the first part of a surveyal work cumulating, and presenting in a systematic way, some former author's ideas and achievements dealing with applications of probability theory and mathematical statistics when defining, developing, and generalizing various parts of D.–S. theory. The more detailed contents of this report can be understood from the list of the titles of the particular chapters presented just below.

Keywords

Dempster–Shafer theory, probability theory, belief function, random variable, random set

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Contents

8 Belief Functions Induced by Partial Generalized Compatibility Relations

A common feature of the following three chapters consists in their aim to go beyond the framework of the already classical mathematical model for Dempster–Shafer theory, as explained and analyzed till now, in at least the three following directions:

(i) to weaken the demands imposed to the notion of compatibility relation as the basic relation binding the empirical data being at the user's (observer's) disposal with the hypothetical internal states of the system under investigation (this chapter);

(ii) to abandon the assumption that the state space S is finite and to extend the definition of degrees of beliefs to at least some subsets of an infinite space S (the next chapter);

(iii) to replace the probabilistic measures used in our definitions of basic probability assignments and belief functions by more general set functions, e. g., by measures or signed measures, in order to generalize the notion of basic probability assignment and belief function so that an operation inverse to the Dempster combination rule were definable if not totally, so at least for a large class of generalized basic probability assignments (Chapter 10).

Let S and E be nonempty, but not necessarily finite sets, let $\rho : S \times E \to \{0,1\}$ be a compatibility relation. This relation can be easily extended to a total relation $\rho^* : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$, setting for each $T \subset S$ and $F \subset E$ such that $T \neq \emptyset,\ F \neq \emptyset$,

$$\rho^*(T,F) = \max\{\rho(s,x) : s \in T,\ x \in F\}, \tag{8.1}$$

and setting $\rho^*(T,\emptyset) = \rho^*(\emptyset,F) = 0$ for each $T \subset S$ and each $F \subset E$. Obviously, $\rho^*(\{s\},\{x\}) = \rho(s,x)$ for each $s \in S$ and $x \in E$. Hence, $\rho^*(T,F) = 1$ iff there are $s \in T$ and $x \in F$ such that $\rho(s,x) = 1$. If the actual state s_0 of the system is defined by the value of a random variable σ, taking a fixed probability space $\langle \Omega, \mathcal{A}, P\rangle$ into a measurable space $\langle S, \mathcal{S}\rangle$ generated by a nonempty σ-field of subsets of S, if the observed empirical value x is defined by the value of a random variable X taking $\langle \Omega, \mathcal{A}, P \rangle$ into $\langle E, \mathcal{E} \rangle$, where $\mathcal{E}$ is a nonempty σ-field of subsets of E, and if the compatibility relation $\rho : S \times E \to \{0, 1\}$ is defined by

$$\begin{aligned} &\rho(s, x) = 1 \quad \text{iff} \\ &\{\omega \in \Omega : \sigma(\omega) = s\} \cap \{\omega \in \Omega : X(\omega) = x\} \neq \emptyset, \end{aligned} \tag{8.2}$$

then

$$\begin{aligned} &\rho^*(T, F) = 1 \quad \text{iff} \\ &(\exists\, s \in T)\,(\exists\, x \in F)\,(\{\omega \in \Omega : \sigma(\omega) = s\} \cap \{\omega \in \Omega : X(\omega) = x\} \neq \emptyset) \\ &\text{iff} \\ &\left(\bigcup\nolimits_{s \in T} \{\omega \in \Omega : \sigma(\omega) = s\}\right) \cap \left(\bigcup\nolimits_{x \in F} \{\omega \in \Omega : X(\omega) = x\}\right) \neq \emptyset \\ &\text{iff} \\ &\{\omega \in \Omega : \sigma(\omega) \in T\} \cap \{\omega \in \Omega : X(\omega) \in F\} \neq \emptyset. \end{aligned} \tag{8.3}$$

The extension of ρ to ρ^* defined by (8.1) and (8.3) agrees with our intuition imposed above on the notion of compatibility between states and empirical values. Or, $\rho^*(T, F) = 0$ should mean that if the observed value is in F, then the laws and rules governing the system and its environment as a whole are such that the membership of the actual state s_0 in T is impossible. In a more subjective way taken, knowing that the observed empirical value is in F, but not knowing anything more about it, we are able to prove that s_0 cannot be in T. From both these interpretations it follows immediately, that in such a case each state $s \in T$ must be incompatible with each $x \in F$, so that $\rho(s, x) = 0$ for each $s \in T,\ x \in F$, and (8.1) follows. The reasoning verifying the inverse implication, i.e., that $\rho(s, x) = 0$ for all $s \in T$ and $x \in F$ should imply $\rho^*(T, F) = 0$, is not so persuasive and immediate, and is charged with a great portion of Platonistic idealization, but we shall accept it as a useful simplification for our further considerations and computations. In more detail, the case that $\rho^*(T, F) = 0$ but $\rho(s, x) = 1$ for some $s \in T$ and some $x \in F$ evidently contradicts the intuition behind and the relation (8.3), but the case when $\rho^*(T, F) = 1$ and $\rho(s, x) = 0$ for all $s \in T$ and all $x \in F$, even if also contradicts (8.3), admits an interesting interpretation. Or, consider the case when, in order to arrive at the conclusion that $\emptyset \neq T \subset S$ and $F = \{x\} \subset E$ are incompatible, we have to prove, within an appropriate deductive formalism, that $\rho(s, x) = 0$ holds for each $s \in T$ in particular. If T is infinite, this cannot be sequentially done by a finite proof, so that we cannot arrive at the conclusion that $\rho^*(T, \{x\}) = 0$ and we must accept that $\rho^*(T, \{x\}) = 1$. The same situation occurs also for finite sets T supposing that only proofs not longer than a given threshold value are accepted as proofs, because of perhaps various reasons of mathematical as well as extra-mathematical nature. So, it may be also worth considering more general extensions of ρ to $\mathcal{P}(S) \times \mathcal{P}(E)$, namely, the mappings $\rho^{**} : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0, 1\}$ such that

$$\rho^{**}(T, F) \geq \max\{\rho(s, x) : s \in T,\ x \in E\} \tag{8.4}$$

holds for each $\emptyset \neq T \subset S,\ \emptyset \neq F \subset E$, with $\rho^{**}(T, \emptyset) = \rho^{**}(\emptyset, F) = 0$ as above.

Definition 8.1. Given a (total) compatibility relation ρ on $S \times E$, the relation ρ^* on $\mathcal{P}(S) \times \mathcal{P}(E)$, uniquely defined by (8.1), is called *the (total) generalized compatibility relation induced (on $\mathcal{P}(S) \times \mathcal{P}(E)$) by* ρ, and each relation ρ^{**} on $\mathcal{P}(S) \times \mathcal{P}(E)$ satisfying (8.4) is called a *quasi-compatibility relation induced (on $\mathcal{P}(S) \times \mathcal{P}(E)$ by ρ)*. A *partial generalized compatibility relation* (*partial quasi-compatibility relation*, resp.) on $\mathcal{P}(S) \times \mathcal{P}(E)$ is a mapping ρ^0 defined on a subset $Dom(\rho^0) \subset \mathcal{P}(S) \times \mathcal{P}(E)$, taking its values in $\{0,1\}$ and such that there exists a total generalized compatibility relation ρ^* (quasi-compatibility relation ρ^{**}, resp.) on $\mathcal{P}(S) \times \mathcal{P}(E)$ such that ρ^0 is the restriction of ρ^* (of ρ^{**}, resp.) to $Dom(\rho^0)$, in symbols, $\rho^0 = \rho^* \restriction Dom(\rho^0)$ ($\rho^0 = \rho^{**} \restriction Dom(\rho^0)$, resp.). □

Evidently, not every partial or total mapping $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ is a partial generalized compatibility relation or a partial quasi-compatibility relation on $\mathcal{P}(S) \times \mathcal{P}(E)$. As a counter-example let us consider any mapping ρ^0 such that, for some $T_1 \subset T_2 \subset S$ and for some $F_1 \subset F_2 \subset E$, $\{\langle T_1, F_1\rangle, \langle T_2, F_2\rangle\} \subset Dom(\rho^0)$ and $\rho^0(T_1, F_1) > \rho^0(T_2, F_2)$ holds. Let us investigate, first of all, under which conditions a (partial) mapping $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ is a partial generalized compatibility relation and when the corresponding total generalized compatibility relation is defined unambiguously. Consequently, we shall focus our attention to the cases when a partial generalized compatibility relation is the only knowledge about the investigated system and its environment being at hand. Then, we shall try to deduce, or at least to approximate, the original compatibility relation on $S \times E$ and to use this approximation in order to obtain reasonable approximations of the belief and plausibility functions defined by the original compatibility relation.

Given a partial mapping $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ with the domain $Dom(\rho^*) \subset \mathcal{P}(S) \times \mathcal{P}(E)$, we set for each $s \in S,\ x \in E$,

$$\overline{\rho}(s,x) = \min\left\{\rho^0(T,F) : \langle T,F\rangle \in Dom(\rho^0),\ s \in T,\ x \in F\right\}, \tag{8.5}$$

if there exists $\langle T,F\rangle \in Dom(\rho^0)$ such that $s \in T$ and $x \in F$, $\overline{\rho}(s,x) = 1$ otherwise. We also set, for each $T \subset S,\ F \subset E$,

$$\overline{\rho}^*(T,F) = \max\{\overline{\rho}(s,x) : s \in T,\ x \in F\} \tag{8.6}$$

with the conventions for $T = \emptyset$ or $F = \emptyset$ as in (8.1). In other words, $\overline{\rho}^*$ is a total mapping which takes $\mathcal{P}(S) \times \mathcal{P}(E)$ into $\{0,1\}$, defined by $\overline{\rho}^* = (\overline{\rho})^*$.

Theorem 8.1. Let $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ be a partial mapping with the domain $Dom(\rho^0)$, let $\overline{\rho}$ and $\overline{\rho}^*$ be defined by (8.5) and (8.6).

(i) For each $\langle T,F\rangle \in Dom(\rho^0)$ the inequality $\overline{\rho}^*(T,F) \leq \rho^0(T,F)$ holds.

(ii) Let ρ^0 be such that

(a) for each $\langle T,F\rangle \in Dom(\rho^0)$ such that $\rho^0(T,F) = 0$ and each $\langle T_1,F_1\rangle \in \mathcal{P}(S) \times \mathcal{P}(E)$ such that $T_1 \subset T$ and $F_1 \subset F$ hold, $\langle T_1,F_1\rangle \in Dom(\rho^0)$ and $\rho^0(T_1,F_1) = 0$ hold as well,

(b) for each nonempty parametric set Λ and for each $\{\langle T_\lambda, F_\lambda\rangle : \lambda \in \Lambda\} \subset Dom(\rho^0)$, if $\langle\bigcup_{\lambda\in\Lambda} T_\lambda, \bigcup_{\lambda\in\Lambda} F_\lambda\rangle \in Dom(\rho^0)$ holds, then $\rho^0\left(\bigcup_{\lambda\in\Lambda} T_\lambda, \bigcup_{\lambda\in\Lambda} F_\lambda\right) = \max\{\rho^0(T_\lambda, F_\lambda) : \lambda\in\Lambda\}$. Then $\overline{\rho}^* = \rho^0(T,F)$ for each $\langle T,F\rangle \in Dom(\rho^0)$.

(iii) If $\rho^0 = \rho^*$ for a compatibility relation $\rho : S\times E \to \{0,1\}$, then $\rho \equiv \overline{\rho}$, i.e., $\rho(s,x) = \overline{\rho}(s,x)$ for each $s\in S$, $x\in E$. □

Proof. Let $Dom(\rho^0)=\emptyset$. Then the equality $\rho^0 = \overline{\rho}^*$ on $Dom(\rho^0)$ holds trivially. Let $\rho^0(T,F)=1$ for each $\langle T,F\rangle\in Dom(\rho^0)\neq\emptyset$. Then $\overline{\rho}(s,x)=1$ for each $s\in S$ and $x\in E$, hence $\overline{\rho}^*(T,F)=1$ for each $T\subset S$ and each $F\subset E$, and the equality between ρ^0 and $\overline{\rho}^*$ on $Dom(\rho^0)$ again immediately follows. So, let there exist $\langle T,F\rangle\in Dom(\rho^0)$ such that $\rho^0(T,F)=0$. Relation (8.5) yields then, for each $s\in T$ and $x\in F$, that

$$\overline{\rho}(s,x) = \min\left\{\rho^0(T_1,F_1) : \langle T_1,F_1\rangle\in Dom(\rho^0),\ s\in T_1,\ x\in F_1\right\} \leq \rho^0(T,F)=0. \quad (8.7)$$

Consequently, by (8.6),

$$\overline{\rho}^*(T,F) = \max\{\overline{\rho}(s,x) : s\in S,\ x\in F\} = 0, \quad (8.8)$$

so that the inequality $\overline{\rho}^*(T,F)\leq\rho^0(T,F)$ for each $\langle T,F\rangle\in Dom(\rho^0)$ immediately follows and (i) is proved.

Let the conditions of (ii) holds, let $\langle T,F\rangle\in Dom(\rho^0)$ be such that $\overline{\rho}^*(T,F)=0$. So, by (8.6), $\max\{\overline{\rho}(s,x) : s\in T,\ x\in F\}=0$, consequently, (8.5) yields that

$$\min\left\{\rho^0(T_1,F_1) : \langle T_1,F_1\rangle\in Dom(\rho^0),\ s\in T_1,\ x\in F_1\right\} = 0 \quad (8.9)$$

holds for each $s\in T$, $x\in F$. Hence, for each pair $\langle s,x\rangle\in T\times F$ there exists $\langle T_{\langle s,x\rangle}, F_{\langle s,x\rangle}\rangle\in Dom(\rho^0)$ such that $s\in T_{\langle s,x\rangle}$, $x\in F_{\langle s,x\rangle}$, and $\rho\left(T_{\langle s,x\rangle}, F_{\langle s,x\rangle}\right)=0$. Using the axiom of choice, let us choose just one such $\left\langle T_{\langle s,x\rangle}, F_{\langle s,x\rangle}\right\rangle$ for each $\langle s,x\rangle\in T\times F$. Set, for each $\langle s,x\rangle\in T\times F$, $T^0_{\langle s,x\rangle} = T\cap T_{\langle s,x\rangle}$, $F^0_{\langle s,x\rangle} = F\cap F_{\langle s,x\rangle}$, then $s\in T^0_{\langle s,x\rangle}$ and $x\in F^0_{\langle s,x\rangle}$ hold for each $s\in T$, $x\in F$, moreover $\bigcup_{\langle s,x\rangle\in T\times F} T^0_{\langle s,x\rangle} = T$ and $\bigcup_{\langle s,x\rangle\in T\times F} F^0_{\langle s,x\rangle} = F$. By (ii) (a) $\left\langle T^0_{\langle s,x\rangle}, F^0_{\langle s,x\rangle}\right\rangle\in Dom(\rho^0)$ and $\rho^0\left(T^0_{\langle s,x\rangle}, F^0_{\langle s,x\rangle}\right)=0$ hold for each $s\in T$, $x\in F$, so that, by (ii) (b)

$$\left\langle \bigcup_{\langle s,x\rangle\in T\times F} T^0_{\langle s,x\rangle},\ \bigcup_{\langle s,x\rangle\in T\times F} F^0_{\langle s,x\rangle}\right\rangle \in Dom(\rho^0) \quad (8.10)$$

and

$$\rho^0\left(\bigcup_{\langle s,x\rangle\in T\times F} T^0_{\langle s,x\rangle},\ \bigcup_{\langle s,x\rangle\in T\times F} F^0_{\langle s,x\rangle}\right) = \rho^0(T,F) = \quad (8.11)$$
$$= \max\left\{\rho^0\left(T^0_{\langle s,x\rangle}, F^0_{\langle s,x\rangle}\right) : \langle s,x\rangle\in T\times F\right\} = 0.$$

Consequently, $\overline{\rho}^*(T,F)=0$ implies $\rho^0(T,F)=0$ what, combined with (i), yields that $\overline{\rho}^*(T,F)=\rho^0(T,F)$ for each $\langle T,F\rangle\in Dom(\rho^0)$. So, (ii) is proved.

Let $\rho^0=\rho^*$ for a compatibility relation $\rho : S\times E\to\{0,1\}$, so that $\rho^0(T,F) = \max\{\rho(s,x) : s\in T,\ x\in F\}$ for each $T\subset S$ and each $F\subset E$. If $T_1\subset T$ and $F_1\subset F$, then, obviously, $\rho^0(T_1, F_1) \leq \rho^0(T, F)$ so that $\rho^0(T, F) \geq \rho^0(\{s\}, \{x\})$ holds for each $s \in T$ and each $x \in F$. Consequently,

$$\begin{aligned}\overline{\rho}(s, x) &= \min\{\rho^0(T, F) : s \in T,\, x \in F\} = \rho^0(\{s\}, \{x\}) = \rho^*(\{s\}, \{x\}) = \\ &= \max\{\rho(s_1, x_1) : s_1 \in \{s\},\, x_1 \in \{x\}\} = \rho(s, x)\end{aligned} \tag{8.12}$$

holds for each $s \in T$ and each $x \in F$, so that (iii) is proved. The proof of Theorem 8.1 is completed. □

Theorem 8.2. Let $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0, 1\}$ be a partial mapping with the domain $Dom(\rho^0)$ such that $\rho^0(T_1, F_1) \geq \rho^0(T_2, F_2)$ holds for each $\{\langle T_1, F_1\rangle, \langle T_2, F_2\rangle\} \subset Dom(\rho^0)$ such that $T_1 \supset T_2$ and $F_1 \supset F_2$, let $s \in S$ and $x \in E$ be such that $\langle\{s\}, \{x\}\rangle \in Dom(\rho^0)$, let $\overline{\rho}$ be defined by (8.5). Then $\overline{\rho}(s, x) = \rho^0(\{s\}, \{x\})$. □

Proof. By (8.5)

$$\overline{\rho}(s, x) = \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\, s \in T,\, x \in F\right\} \leq \rho^0(\{s\}, \{x\}), \tag{8.13}$$

as $\langle\{s\}, \{x\}\rangle \in Dom(\rho^0)$, $s \in \{s\}$, and $x \in \{x\}$. However, $\rho^0(T, F) \geq \rho^0(\{s\}, \{x\})$ holds for each $\langle T, F\rangle \in Dom(\rho^0)$ such that $s \in T$ and $x \in F$ due to the conditions of Theorem 8.2. Hence, $\overline{\rho}(s, x) \geq \rho^0(\{s\}, \{x\})$ immediately follows and the proof is completed. □

Theorem 8.3. Let $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E) \to \{0, 1\}$ be a partial generalized compatibility relation such that $\rho^0 = \rho^* \upharpoonright Dom(\rho^0)$ for a compatibility relation ρ on $S \times E$, let $\overline{\rho}$ be defined by ρ^0 using (8.5). Then $\overline{\rho}(s, x) \geq \rho(s, x)$ holds for each $s \in S$ and each $x \in E$. □

Proof. An easy calculation yields that

$$\begin{aligned}\overline{\rho}(s, x) &= \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\, s \in T,\, x \in F\right\} \geq \\ &\geq \min\{\rho^*(T, F) : s \in T \subset S,\, x \in F \subset E\} = \\ &= \rho^*(\{s\}, \{x\}) = \rho(s, x),\end{aligned} \tag{8.14}$$

as the inequality $\rho^*(T, F) \geq \rho^*(\{s\}, \{x\})$ obviously holds for each $T \subset S$ and $F \subset E$ such that $s \in T$ and $x \in F$. □

As can be easily proved, the inequality in the assertion of Theorem 8.3 cannot be, in general, replaced by equality. Or, let $f : E \to S$ be a total function such that $\rho(s, x) = 1$ iff $s = f(x)$, $\rho(s, x) = 0$ otherwise. So,

$$\rho^*(S, E) = \max\{\rho(s, x) : s \in S,\, x \in E\} = \max\{\rho(f(x), x) : x \in E\} = 1. \tag{8.15}$$

Consequently, for each $s \in S$ and each $x \in E$, if $Dom(\rho^0) = \{\langle S, E\rangle\}$, then

$$\begin{aligned}\overline{\rho}(s, x) &= \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\, s \in T,\, x \in F\right\} = \\ &= \rho^0(S, E) = \rho^*(S, E) = 1,\end{aligned} \tag{8.16}$$

so that $\overline{\rho}(s,x) > \rho(s,x)$ holds for each $s \in S$, $x \in E$ such that $s \neq f(x)$.

Before focusing our attention on a more detailed investigation of partial generalized compatibility relations we take as worth saying explicitly, that compatibility relations on $\mathcal{P}(S) \times \mathcal{P}(E)$ can be defined not only by extending compatibility relations defined on $S \times E$ to $\mathcal{P}(S) \times \mathcal{P}(E)$ by (8.1), but also directly, taking $S_0 = \mathcal{P}(S)$ instead of S and $E_0 = \mathcal{P}(E)$ instead of E in the general definition of compatibility relation. Such a compatibility relation $\rho^0 : S_0 \times E_0 \rightarrow \{0,1\}$ cannot be, in general, defined by an extension of a compatibility relation defined on $S \times E$, or as a fragment of such an extension, if ρ^0 is partial, as it is possible that $\rho^0(T,F) = 0$, but $\rho^0(T_1,F_1) = 1$ for some $T_1 \subset T \subset S$, $F_1 \subset F \subset E$. So, such a compatibility relation on $\mathcal{P}(S) \times \mathcal{P}(E)$ can be taken as a relation between a metasystem the states of which are sets of states of the original system, and an enriched observation space the elements of which are sets of original empirical values. A more detailed mathematical investigation and possible interpretations of such meta-systems, meta-observations and corresponding compatibility relations would be interesting and perhaps useful, but it would exceed the intended scope and extent of this chapter and will be postponed till another occasion.

In order to simplify our further reasonings by avoiding technical difficulties we shall suppose, till the end of this chapter and unless stated otherwise, that both the spaces S and E are finite. As above, we shall suppose that the empirical values (values from E) being at the subject's disposal are of random character and can be described, quantified and processed by the tools of the classical (Kolmogorov axiomatic) probability theory. In the way described above we arrive at the notion of belief function and plausibility function induced by a compatibility relation $\rho : S \times E \rightarrow \{0,1\}$.

Let $\rho^0 : \mathcal{P}(S) \times \mathcal{P}(E)$ be a partial mapping, let $\overline{\rho}$ be defined by (8.5). Then we set $bel_{\rho^0}(T) = bel_{\overline{\rho}}(T)$ for each $T \subset S$. A compatibility relation ρ defined on $S \times E$ is called *consistent*, if for each $x \in E$ there exists $s \in S$ such that $\rho(s,x) = 1$.

Theorem 8.4. Let the notations and conditions of Theorem 8.3 hold, let ρ be consistent. Then $bel_{\overline{\rho}}(T) \leq bel_{\rho}(T)$ and $pl_{\overline{\rho}}(T) \geq pl_{\rho}(T)$ hold for each $T \subset S$. □

Proof. By Theorem 8.3, $\overline{\rho}(s,x) \geq \rho(s,x)$ holds for each $s \in S$, $x \in E$, so that

$$U_{\overline{\rho}}(x) = \{s \in S : \overline{\rho}(s,x) = 1\} \supset \{s \in S : \rho(s,x) = 1\} = U_{\rho}(x). \tag{8.17}$$

Hence, $U_{\overline{\rho}}(x) \neq \emptyset$ holds for each $x \in E$, $bel_{\overline{\rho}}(T) = P(\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T\})$, moreover, $U_{\overline{\rho}}(X(\omega)) \supset U_{\rho}(X(\omega))$ is valid for each $\omega \in \Omega$. Consequently, for each $T \subset S$, if $U_{\overline{\rho}}(X(\omega)) \subset T$, then $U_{\rho}(X(\omega)) \subset T$. In other terms,

$$\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T\} \subset \{\omega \in \Omega : U_{\rho}(X(\omega)) \subset T\}, \tag{8.18}$$

and this inclusion immediately yields that

$$\begin{aligned} bel_{\overline{\rho}}(T) &= P(\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T\}) \leq \\ &\leq P(\{\omega \in \Omega : U_{\rho}(X(\omega)) \subset T\}) = bel_{\rho}(T). \end{aligned} \tag{8.19}$$

The dual inequality for plausibility functions is obvious so that the assertion is proved. □

It is perhaps worth stating explicitly that if the basic compatibility relation ρ is not consistent, then the inequality (8.19) need not hold, as the following example illustrates.

Let $S = \{s_1, s_2, s_3\}$, let $E = \{x_1, x_2, x_3\}$, let $p(x_i) = P(\{\omega \in \Omega : X(\omega) = x_1\}) = 1/3$ for each $i = 1, 2, 3$. Let the compatibility relation ρ on $S \times E$ be defined as follows: $\rho(s_1, x_1) = 1,\ \rho(s_i, x_3) = 1$ for each $i = 1, 2, 3,\ \rho(s_i, x_j) = 0$ otherwise. Hence, ρ is not consistent, as there is no state s_i compatible with the empirical value x_2. Recalling that $U_\rho(x_i) = \{s \in S : \rho(s, x_i) = 1\}$ we obtain easily that $U_\rho(x_1) = \{s_1\},\ U_\rho(x_2) = \emptyset,\ U_\rho(x_3) = \{s_1, s_2, s_3\} = S$. Setting $T_0 = \{s_1, s_2\} \subset S,\ F_0 = \{x_1, x_2\} \subset E$, an easy calculation yields that

$$\begin{aligned} bel_\rho(T_0) &= P\left(\{\omega \in \Omega : U_\rho(X(\omega)) \subset T_0\} \,/\, \{\omega \in \Omega : U_\rho(X(\omega)) \neq \emptyset\}\right) = \qquad (8.20) \\ &= \frac{P(\{\omega \in \Omega : \emptyset \neq U_\rho(X(\omega)) \subset T_0\})}{P(\{\omega \in \Omega : \emptyset \neq U_\rho(X(\omega))\})} = \frac{\sum_{x \in E, \emptyset \neq U_\rho(x) \subset T_0} p(x)}{\sum_{x \in E, \emptyset \neq U_\rho(x)} p(x)} = \\ &= \frac{p(x_1)}{p(x_1) + p(x_3)} = \frac{1/3}{1/3 + 1/3} = \frac{1}{2}. \end{aligned}$$

For the generalized compatibility relation ρ^* induced by ρ we obtain that

$$\rho^*(T_0, F_0) = \max\{\rho(s, x) : s \in T_0,\ x \in F_0\} \geq \rho(s_1, x_1) = 1, \qquad (8.21)$$

as $s_1 \in T_0$ and $x_1 \in F_0$. Moreover,

$$\rho^*(T, \{x_3\}) = \max\{\rho(s, x) : s \in T,\ x \in \{x_3\}\} = 1 \qquad (8.22)$$

for each T, $\emptyset \neq T \subset S$, and

$$\rho^*(\{s_3\}, \{x_1\}) = \rho(s_3, x_1) = 0 = \rho(s_3, x_2) = \rho^*(\{s_3\}, \{x_2\}). \qquad (8.23)$$

Let $\rho^0 = \rho^* \restriction Dom(\rho^0)$, where

$$Dom(\rho^0) = \{\langle T_0, F_0\rangle,\ \langle\{s_3\}, \{x_1\}\rangle,\ \langle\{s_3\}, \{x_2\}\rangle\} \cup \{\langle T, \{x_3\}\rangle : \emptyset \neq T \subset S\}. \qquad (8.24)$$

We obtain easily that for both $i = 1, 2,\ j = 1, 2$

$$\begin{aligned} \overline{\rho}(s_i, x_j) &= \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\ s_i \in T,\ x_j \in F\right\} = \qquad (8.25) \\ &= \rho^0(T_0, F_0) = \rho^*(T_0, F_0) = 1, \end{aligned}$$

as $\langle T_0, F_0\rangle$ is the only pair $\langle T, F\rangle$ in $Dom(\rho^0)$ such that $s_i \in T$ and $x_j \in F$ hold simultaneously for $i = 1$ or 2 and $j = 1$ or 2. Moreover, for $i = 1, 2, 3$,

$$\begin{aligned} \overline{\rho}(s_i, x_3) &= \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\ s_i \in T,\ x_3 \in F\right\} = \qquad (8.26) \\ &= \min\left\{\rho^0(T, \{x_3\}) : s_i \in T\right\} = \min\{\rho^*(T, \{x_3\}) : s_i \in T\} = 1, \end{aligned}$$

and

$$\begin{aligned} \overline{\rho}(s_3, x_1) &= \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\ s_3 \in T,\ x_1 \in F\right\} = \qquad (8.27) \\ &= \rho^0(\{s_3\}, \{x_1\}) = \rho^*(\{s_3\}, \{x_1\}) = \rho(s_3, x_1) = 0, \end{aligned}$$

as well as

$$\begin{aligned} \overline{\rho}(s_3, x_2) &= \min\left\{\rho^0(T, F) : \langle T, F\rangle \in Dom(\rho^0),\ s_3 \in T,\ x_2 \in F\right\} = \\ &= \rho^0(\{s_3\}, \{x_2\}) = \rho^*(\{s_3\}, \{x_2\}) = \rho(s_3, x_2) = 0. \end{aligned} \tag{8.28}$$

So,

$$U_{\overline{\rho}}(x_1) = \{s \in S : \overline{\rho}(s, x) = 1\} = \{s_1, s_2\} \tag{8.29}$$

by (8.25) and (8.27), analogously, by (8.25) and (8.28), we obtain that

$$U_{\overline{\rho}}(x_2) = \{s_1, s_2\}. \tag{8.30}$$

Finally, (8.26) yields that

$$U_{\overline{\rho}}(x_3) = \{s_1, s_2, s_3\} = S. \tag{8.31}$$

So, $U_{\overline{\rho}}(x) \neq \emptyset$ for all $x \in E$, and an easy calculation yields that

$$\begin{aligned} bel_{\overline{\rho}}(T_0) &= P\left(\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T_0\} \ / \ \{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \neq \emptyset\}\right) = \\ &= P\left(\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T_0\}\right) = \textstyle\sum_{x \in E,\, U_{\overline{\rho}}(x) \subset T_0} p(x) = \\ &= p(x_1) + p(x_2) = 1/3 + 1/3 = 2/3 > 1/2 = bel_{\overline{\rho}}(T_0) \end{aligned} \tag{8.32}$$

by (8.20). Hence, the inequality (8.19) does not hold.

As the example just presented shows, if the basic compatibility relation ρ on $S \times E$ is not consistent, then its behaviour and the properties of the corresponding belief functions are rather counter-intuitive. Namely, having at our disposal only a partial knowledge about the compatibility relation ρ, i. e. the knowledge encoded by a fragment of the induced generalized compatibility relation, we can arrive at *higher* values of the belief function for some subsets of S. This fact follows from a more general paradoxal property of belief functions according to which enriching the database by new items which are inconsistent with the former ones can augment the degree of belief for some sets of states. It is just this strange property which, together with the technical difficulties involved by the apparatus of conditional probabilities, makes a great portion of specialists dealing with the D.–S. theory to consider just the case of consistent compatibility relations. Another solution may be, to abandon the assumption of closed world, i. e., to admit that there are also some possible internal states of the system not contained in S, and to take the case when the data are inconsistent as the indication that the actual state of the system is beyond the set S. At the formalized mathematical level this approach leads to the case of non-normalized belief functions when the inequality $bel_\rho(S) < 1$ can hold.

The next assertion generalizes Theorem 8.4 in the sense that two partial generalized compatibility relations induced by the same compatibility relation on $S \times E$ and with domains ordered by set-theoretic inclusion as far as the corresponding belief functions are concerned.

Theorem 8.5. Let $\rho^1, \rho^2 : \mathcal{P}(S) \times \mathcal{P}(E) \rightarrow \{0,1\}$ be two partial generalized compatibility relations such that $\rho^i = \rho^* \upharpoonright Dom(\rho^i)$ for both $i = 1,2$, and for a consistent compatibility relation ρ on $S \times E$, let $Dom(\rho^1) \subset Dom(\rho^2) \subset \mathcal{P}(S) \times \mathcal{P}(E)$ hold. Let

$$\overline{\rho}^i(s,x) = \min\left\{\rho^i(T,F) : \langle T,F\rangle \in Dom(\rho^i),\ s \in T,\ x \in F\right\} \qquad (8.33)$$

for both $i = 1,2$, and for all $s \in S$, $x \in E$ for which this value is defined, let $\overline{\rho}^i(s,x) = 1$ otherwise. Then the inequalities $bel_{\overline{\rho}^1}(T) \leq bel_{\overline{\rho}^2}(T)$ and $pl_{\overline{\rho}^1}(T) \leq pl_{\overline{\rho}^2}(T)$ hold for each $T \subset S$. □

Proof. For each $\langle T,F\rangle$, $T \subset S$, $F \subset E$, if $\langle T,F\rangle \in Dom(\rho^1)$, then $\langle T,F\rangle \in Dom(\rho^2)$ and, moreover, $\rho^1(T,F) = \rho^*(T,F) = \rho^2(T,F)$, as both ρ^1 and ρ^2 result from restrictions of the same generalized compatibility relation ρ^* to various domains. Hence, for each $s \in S$ and $x \in E$ such that $\overline{\rho}^1(s,x)$ is defined by (8.33) we obtain that

$$\begin{aligned}\overline{\rho}^1(s,x) &= \min\left\{\rho^1(T,F) : \langle T,F\rangle \in Dom(\rho^1),\ s \in T,\ x \in F\right\} \geq \\ &\geq \min\left\{\rho^2(T,F) : \langle T,F\rangle \in Dom(\rho^2),\ s \in T,\ x \in F\right\} = \\ &= \overline{\rho}^2(s,x).\end{aligned} \qquad (8.34)$$

If $\overline{\rho}^1(s,x)$ is not defined by (8.33), then $\overline{\rho}^1(s,x) = 1$ and the inequality $\overline{\rho}^1(s,x) \geq \overline{\rho}^2(s,x)$ holds trivially. Setting, for both $i = 1,2$ and for each $x \in E$,

$$U_{\overline{\rho}^i}(x) = \{s \in S : \overline{\rho}^i(s,x) = 1\}, \qquad (8.35)$$

we obtain easily that for each $x \in E$, $U_{\overline{\rho}^1}(x) \supset U^2_{\overline{\rho}}(x)$, and both these sets are nonempty (both of them contain $U_\rho(s,x) \neq \emptyset$, as the relation ρ is supposed to be consistent). As in the proof of Theorem 8.4 we obtain that the inclusion $U_{\overline{\rho}^1}(X(\omega)) \supset U_{\overline{\rho}^2}(X(\omega))$ holds for each $\omega \in \Omega$, consequently, for each $T \subset S$,

$$\left\{\omega \in \Omega : U_{\overline{\rho}^1}(X(\omega)) \subset T\right\} \subset \left\{\omega \in \Omega : U_{\overline{\rho}^2}(X(\omega)) \subset T\right\}, \qquad (8.36)$$

what immediately yields that

$$\begin{aligned}bel_{\overline{\rho}^1}(T) &= P\left(\{\omega \in \Omega : U_{\overline{\rho}^1}(X(\omega)) \subset T\}\right) \\ &\leq P\left(\{\omega \in \Omega : U_{\overline{\rho}^2}(X(\omega)) \subset T\}\right) = bel_{\overline{\rho}^2}(T).\end{aligned} \qquad (8.37)$$

The dual inequality for the plausibility functions follows trivially, so that the assertion is proved.

As follows from Theorems 8.4 and 8.5, belief function $bel_{\overline{\rho}}$, defined by fragments of the generalized compatibility relation induced by an original compatibility relation ρ, is a lower approximation of the original belief function bel_ρ. This approximation can be improved, i. e., bel_ρ can be approximated more closely, if the fragments being at our disposal are enriched by a new part. Consequently, when using the original belief function bel_ρ in decision rules according to which the hypothesis that the actual state of the system is in $T \subset S$ is accepted, if $bel_\rho(T) \geq \alpha$ holds for some threshold value α close enough to one, this decision rule can be replaced, conserving the pessimistic worst – case principle typical for the D.–S. theory way of reasoning, by a more severe rule which accepts the same hypothesis when $bel_{\overline{\rho}}(T) \geq \alpha$ holds. On the other hand, knowing that the last inequality holds, we do not need to compute the value $bel_\rho(T)$, what may be much more time and space consuming, to be able to decide that $bel_\rho(T) \geq \alpha$ holds. At least the two following ways of further development are worth considering:

(i) to apply our reasonings to particular partial generalized compatibility relations, e. g. to those generated by appropriate equivalence relations on the spaces S and E, to arrive at more detailed results than those introduced above, and

(ii) to compute the time and/or space computational complexity savings achieved when replacing bel_ρ by $bel_{\overline{\rho}}$ in decision rules like that one mentioned above.

It can happen, because of many reasons of practical nature, that the investigator is not able to distinguish two or more states of the system (two or more empirical values, resp.) from each other. E. g., when states of the system are numerical real-valued parameters, they can be processes only within a limited number of decimal digits and the same may hold true for the observed values of real-valued random variables. Here we shall limit ourselves to the most simple case when the indistinguishable states or empirical values are just the equivalence classes generated by certain equivalence relations on the corresponding sets. Hence, $\approx_S$ ($\approx_E$, resp.) is supposed to be an equivalence relation on S (on E, resp.), and for each $s \in S$ ($x \in E$, resp.) $[s]_{\approx_S}$ ($[x]_{\approx_E}$, resp.) denotes the class of equivalence in $S/\approx_S$ (in $E/\approx_E$, resp.) such that $s \in [s]_{\approx_S}$ ($x \in [x]_{\approx_E}$, resp.). The relations $\approx_S$ and $\approx_E$ are fixed in what follows, so that the indices $\approx_S$ an $\approx_E$ are omitted, if it is clear from the context to which set the element between [ ] belongs. For $T \subset S$ ($F \subset E$, resp.) we set $[T]_{\approx_S} = \bigcup_{s \in T}[x]$ ($[F]_{\approx_E} = \bigcup_{x \in F}[x]$, resp.) with the same convention adopted concerning the indices as in the case of single elements. In the rest of this chapter we shall discuss the case when the only information about relations between states and observations is given in the form of a compatibility relation between classes of states and classes of empirical values. I. e., ρ^0 takes $S|_{\approx_S} \times E|_{\approx_E}$ into $\{0,1\}$ and ρ^0 is supposed to be defined by an unknown compatibility relation $\rho : S \times E \to \{0,1\}$ by the relation

$$\begin{aligned} \rho^0([s],[x]) &= \max\{\rho(s_1,x_1) : s_1 \approx_S s, x_1 \approx_E x\} = \\ &= \max\{\rho(s_1,x_1) : s_1 \in [s],\, x_1 \in [x]\}. \end{aligned} \tag{8.38}$$

This relation is obviously the restriction of ρ^* induced by ρ and defined by (8.1) to the domain $Dom(\rho^0) = \{\langle [s],[x]\rangle : s \in S,\, x \in E\}$. In order to simplify our further reasonings we shall suppose that the underlying basic compatibility relation ρ is consistent in the sense that for each $x \in E$ there exists $s \in S$ such that $\rho(s,x) = 1$, so that $U(x) = U_\rho(x) \neq \emptyset$ for all $x \in E$. Let us also define, for each $T \subset S$ (for each $F \subset E$, resp.), $\langle T \rangle = \bigcup\{[s] : [s] \subset T\}$ ($\langle F \rangle = \bigcup\{[x] : [x] \subset F\}$ resp.), so that $\langle T \rangle \subset T \subset [T]$ and $\langle F \rangle \subset F \subset [F]$ hold for each $T \subset S$ and $F \subset E$.

Setting, for each $s \in S,\ x \in E$,

$$\overline{\rho}(s,x) = \min\left\{\rho^0([t],[y]) : t \in S,\, y \in E,\, s \in [t],\, x \in [y]\right\}, \tag{8.39}$$

we obtain immediately that $\overline{\rho}(s,x) = \rho([s],[x])$, as $[s]$ ($[x]$, resp.) is the only class in $S|_{\approx_S}$ (in $E_{\approx_E}$, resp.) containing s (x, resp.). Given $T \subset S$, $x \in E$, and $Y \subset S|_{\approx_S}$, setting $T^0 = \{[s] : [s] \subset T\} \subset S|_{\approx_S}$ for each $T \subset S$ (notice the difference between T^0 and $\langle T \rangle$), supposing that S is finite and considering the random variable $X : \langle \Omega, \mathcal{A}, P \rangle \to \langle E, \mathcal{E} \rangle$ as above, we can define

$$\begin{aligned} U_\rho(x) &= \{s \in S : \rho(s,x) = 1\}, \\ bel_\rho(T) &= P\left(\{\omega \in \Omega : U_\rho(X(\omega)) \subset T\}\right), \\ U_{\overline{\rho}}(x) &= \{s \in S : \overline{\rho}(s,x) = 1\}, \\ bel_{\overline{\rho}}(T) &= P\left(\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T\}\right), \\ U_{\rho^0}([x]) &= \left\{[s] \in S|_{\approx_S} : \rho^0([s],[x]) = 1\right\}, \\ bel_{\rho^0}(Y) &= P\left(\{\omega \in \Omega : U_{\rho^0}([X(\omega)]) \subset Y\}\right). \end{aligned} \tag{8.40}$$

Theorem 8.6. Let S be finite, let $\mathcal{S} = \mathcal{P}(\mathcal{P}(S))$, let $\rho : S \times E \to \{0,1\}$ be consistent. Then for each $T \subset S$ the following relations hold:

$$bel_{\overline{\rho}}(T) = bel_{\rho^0}(T) \leq bel_\rho(T). \tag{8.41}$$ □

Proof. As $s \approx_S s$ and $x \approx_S x$ hold for each $s \in S$, $x \in E$, (8.38) and (8.40) yield that $\overline{\rho}(s,x) \geq \rho(s,x)$ for all $s \in S$, $x \in E$. Hence, $U_{\overline{\rho}}(X(\omega)) \supset U_\rho(X(\omega))$ for all $\omega \in \Omega$, so that the inequality $bel_{\overline{\rho}}(T) \leq bel_\rho(T)$ immediately follows. If $s_1 \approx_S s_2$ and $s_1 \in U_{\overline{\rho}}(x)$, then $1 = \overline{\rho}(s_1,x) = \rho^0([s_1],[x]) = \rho^0([s_2],[x]) = \overline{\rho}(s_2,x)$, so that $s_2 \in U(x)$. Hence, $U_{\overline{\rho}}(x) = \bigcup_{s \in U_\rho(x)} [s] = [U_{\overline{\rho}}(x)]$. If $x_1 \approx x_2$ and $s \in U_\rho(x_1)$, then $1 = \overline{\rho}(s,x_1) = \rho^0([s],[x_1]) = \rho^0([s],[x_2]) = \overline{\rho}(s,x_2)$, so that $s \in U_{\overline{\rho}}(x_2)$ as well, hence, $U_{\overline{\rho}}(x_1) = U_{\overline{\rho}}(x_2)$. For any sets $A \subset T \subset S$ the inclusion $[A] \subset T$ holds iff $[A] \subset \langle T \rangle$ hold. Consequently, $U_{\overline{\rho}}(x) \subset T$ holds iff $[U_{\overline{\rho}}(x)] \subset \langle T \rangle$ holds. For each $s \in S$, $[s] \subset [U_\rho(x)]$ iff $\overline{\rho}(s,x) = 1$ iff $\rho^0([s],[x]) = 1$ iff $[s] \in U_{\rho^0}([x])$, so that $U_{\overline{\rho}}(x) \subset T$ holds iff $U_{\rho^0}([x]) \subset T^0 = \{[s] : [s] \subset \langle T \rangle\}$. Hence, $\{\omega \in \Omega : U_{\overline{\rho}}(X(\omega)) \subset T\} = \{\omega \in \Omega : [U_{\overline{\rho}}(X(\omega))] \subset \langle T \rangle\} = \{\omega \in \Omega : U_{\rho^0}([x]) \subset T^0\}$ and the equality $bel_{\overline{\rho}}(T) = bel_{\rho^0}(T^0)$ immediately follows. □

The inequality in (8.41) yields that $bel_{\overline{\rho}}$, which can be obtained from ρ^0, is a lower approximation of bel_ρ, hence, if we accept the hypothesis that the actual state s_0 of a system is in T supposing that $bel_{\overline{\rho}}(T) \geq 1 - \alpha$ for some fixed threshold value $\alpha \geq 0$ holds, we can accept the same hypothesis if $bel_\rho(T) \geq 1 - \alpha$ holds without computing $bel_\rho(T)$ as we know that the last condition of acceptance is at least as strict as the original one. On the other side, the equality in (8.41) yields that the value $bel_{\overline{\rho}}(T)$ can be computed much more easily than the value $bel_\rho(T)$. Let S be finite, let $\text{card}S = n$, let ρ be consistent, let $m(A) = P(\{\omega \in \Omega : U_\rho(X(\omega)) = A\})$ for each $A \subset S$, let $T \subset S$. Then $bel_\rho(T) = \sum_{A \subset T} m(A)$ can be computed using $2^{\text{card}T}$ applications of the operation of addition.If the equivalence relation $\approx_S$ is such that $\text{card}(S|_{\approx_S}) = n/K$ for some $K > 1$, if $\text{card}T^0 = (\text{card}T)/K$, and if $m^0(A^0) = P(\{\omega \in \Omega : U_{\rho^0}([X(\omega)]) = A^0\})$ for each $A^0 \subset S|_{\approx_S}$, then $bel_{\rho^0}(T^0) = \sum_{A^0 \subset T^0} m^0(A)$ can be computed by $2^{\text{card}T_0} = 2^{(\text{card}T)/K} = 2^{(1/K)\text{card}T}$ operations of addition. The strong law of large numbers (cf. [31], e. g.) yields that the relative frequence of such A's, $A \subset S$, for which $(\mathrm{card}A/\mathrm{card}S) - (1/2) < \varepsilon$ holds, tends to 1, with $\mathrm{card}S$ increasing, for each $\varepsilon > 0$, so that a subset $T \subset S$ such that $\mathrm{card}T = n/2$ (for n even) can be seen as a "typical" subset of S. For such $T \subset S$

$$\frac{2^{(1/K)\mathrm{card}T}}{2^{\mathrm{card}T}} = \left(\frac{2^{(1/K)}}{2}\right)^{n/2} = \frac{1}{2^{(n/2)\,(1-(1/K))}}, \tag{8.42}$$

so that the reduction of computational complexity is obvious.

9 Belief Functions over Infinite State Spaces

In order to make the following considerations more transparent, let us recall the basic idea of our definition of belief function (Def. 4.2) in the terms of set-valued (generalized) random variables and their probabilistic numerical characteristics (generalized quantiles). Let S be a nonempty set, let $\mathcal{S} \subset \mathcal{P}(\mathcal{P}(S))$ be a nonempty σ-field of systems of subsets of S, let $\langle \Omega, \mathcal{A}, P \rangle$ be a fixed abstract probability space. Let $\langle E, \mathcal{E} \rangle$ be a measurable space over the nonempty space E of possible empirical values, let $X : \langle \Omega, \mathcal{A}, P \rangle \rightarrow \langle E, \mathcal{E} \rangle$ be a random variable, let $\rho : S \times E \rightarrow \{0, 1\}$ be a compatibility relation, let $U_{\rho,X}(x) = \{s \in S : \rho(s, x) = 1\}$ for each $x \in E$. Then the value $bel^*_{\rho,X}(A)$ is defined by

$$bel^*_{\rho,X}(A) = P\left(\{\omega \in \Omega : \emptyset \neq U_{\rho,X}(X(\omega)) \subset A\}\right) \tag{9.1}$$

for each $A \subset S$ for which this probability is defined. In other terms we can say: let $U_{\rho,X}(X(\cdot))$ be a set-valued (generalized) random variable, i. e. measurable mapping, which takes the probability space $\langle \Omega, \mathcal{A}, P \rangle$ into a measurable space $\langle \mathcal{P}(S), \mathcal{S} \rangle$. Then the (non-normalized) degree of belief $bel^*_{\rho,X}(A)$ is defined by (9.1) for each $A \subset S$ such that $\mathcal{P}(A) \in \mathcal{S}$ holds. If, moreover, $\{\emptyset\} \in \mathcal{S}$ and $P(\{\omega \in \Omega : U_{\rho,X}(\omega) = \emptyset\}) < 1$ hold, the (normalized) degree of belief $bel_{\rho,X}(A)$ is defined by the conditional probability

$$bel_{\rho,X}(A) = P\left(\{\omega \in \Omega : U_{\rho,X}(\omega) \subset A\} \,/\, \{\omega \in \Omega : U_{\rho,X}(\omega) \neq \emptyset\}\right). \tag{9.2}$$

Even if we already mentioned a more general level of this definition, if compared with the combinatoric one, as it enables to define degrees of belief for at least some subsets of infinite basic space, till now we have rather limited ourselves to the case when S is finite with the aim to translate into the probabilistic framework the notions defined and the results achieved by the classical model of D.–S. theory developed over finite state spaces S and using the combinatoric computational rules. Crossing the borderlines of this classical finitistic model we have to realize, first of all, that if the set S is infinite, its power-set $\mathcal{P}(S)$ is uncountable. Hence, given a mapping $m = \mathcal{P}(S) \rightarrow \langle 0,1 \rangle$, we are not able to define, in general, the probability distribution (basic probability assignment) on $\mathcal{P}(\mathcal{P}(S))$ or on a nontrivial σ-field $\mathcal{S} \subset \mathcal{P}(\mathcal{P}(S))$ containing also systems of subsets of S of the kind $\mathcal{P}(T)$ for infinite subsets T of S. Remember, e. g., the Borel probability measure on the unit interval $\langle 0,1 \rangle$ of real numbers when the measure of this interval is one, even if $\langle 0,1 \rangle = \bigcup_{x \in \langle 0,1 \rangle} \{x\}$ is an uncountable union of disjoint sets (singletons), each of them possessing the zero measure. Hence, the combinatoric definition of belief function cannot be extended to the case of infinite spaces S.

When discussing the problem of measurability of the mapping $U_{\rho,X} : \Omega \rightarrow \mathcal{P}(S)$, and the resulting problem of (non)definability of the value $bel^*_{\rho,X}(T)$ for some $T \subset S$, the two extremal cases are perhaps worth being mentioned explicitly. If $\mathcal{S} = \{\emptyset, \mathcal{P}(S)\}$ is the minimal (the most rough) σ-field of systems of subsets of S, then every mapping $U_{\rho,X} : \Omega \rightarrow \mathcal{P}(S)$ is measurable no matter which the probability space $\langle \Omega, \mathcal{A}, P \rangle$ may be, but in this case only $bel^*_{\rho,X}(S)$ can be defined (its value being obviously 1). Let us recall that the empty set $\emptyset$ occurring in the definition $\mathcal{S} = \{\emptyset, \mathcal{P}(S)\}$ above, is the empty subset of $\mathcal{P}(S)$, not of S, so that neither $bel_{\rho,X}(\emptyset)$ need not be defined for the empty subset of S.

The reader not familiar with the foundations of measure theory and probability theory may perhaps ask, why not to simplify our model by considering only probability space $\langle \Omega, \mathcal{A}, P \rangle$ with $\mathcal{A} = \mathcal{P}(\Omega)$ and set-valued random variables U taking their values in the complete measurable space $\langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S)) \rangle$. Under these conditions, $\mathcal{P}(T) \in \mathcal{S}$ holds for each $T \subset S$ (such T is called *$\mathcal{S}$-regular*) and $bel^*_{\rho,X}(T)$ is defined. The reason for our not accepting these simplifications is that in such a case the scale of probability measures P and random variables U being at our disposal would be rather narrow. Consider, e. g., $\Omega = \langle 0,1 \rangle$ together with two probability measures, P_1 and P_2, defined on certain subsets of Ω. Namely, let $\Omega_0 = \{\omega_1, \omega_2, \ldots\}$ be a countable subset of Ω, let $P_1 : \Omega \rightarrow \langle 0,1 \rangle$ be such a mapping that $\sum_{i=1}^{\infty} P_1(\omega_i) = 1$. P_1 defines a probability measure on $\mathcal{P}(\Omega_0)$, setting simply $P_1(A) = \sum_{\omega_i \in A} P_1(\omega_i)$ for each $A \subset \Omega_0$. This definition can be immediately extended to each $A \subset \Omega$ setting $P_1(A) = 0$ for $A \subset \Omega - \Omega_0$, i. e., setting $P_1(A) = P_1(A \cap \Omega_0)$ for each $A \subset \Omega$. Hence, P_1 can be unambiguously extended to a probability measure on $\mathcal{P}(\Omega)$.

Let P_2 be defined on the class of semi-open subintervals of $\langle 0,1 \rangle$ in such a way that $P_2(\langle a,b)) = b - a$ for each such subinterval. This probability measure can be extended to the class $\text{Ł} \subset \mathcal{P}(\langle 0,1 \rangle)$ of Lebesgue sets (sets measurable in the Lebesgue sense), and it is a well-known fact that $\text{Ł} \neq \mathcal{P}(\langle 0,1 \rangle)$ so that there exists a set $D \subset \langle 0,1 \rangle$ which is not measurable in the Lebesgue sense (the axiom of choice plays a key role when proving the existence of such a set). On the other hand, the probability measure P_2 plays an important role in probability theory as it enables to formalize the notion of equiprobable random sample from the uncountable set $\langle 0,1 \rangle$. Let $\langle \Omega, \mathcal{A}, P \rangle = \langle \langle 0,1 \rangle, \text{Ł}, P_2 \rangle$ be the probability space over $\langle 0,1\rangle$ just defined, let $S = \langle 0,1\rangle$, let $U : \Omega \to \mathcal{P}(S)$ be defined by $U(\omega) = \{\omega\}$. Then, for $D \subset \langle 0,1\rangle$, D not measurable in the Lebesgue sense,

$$\begin{aligned} bel^*_U(D) &= P_2\left(\{\omega \in \Omega : U(\omega) \subset D\}\right) = P_2\left(\{\omega \in \Omega : \{\omega\} \subset D\}\right) = \\ &= P_2\left(\{\omega \in \Omega : \omega \in D\}\right) = P_2(D), \end{aligned} \tag{9.3}$$

and the last value is not defined, so that bel^*_U cannot be extended to whole $\mathcal{P}(S)$.

Besides these theoretical restrictions there may be also many practical reasons for which we cannot consider all subsets of S when defining belief functions. E. g., we are not able to distinguish two values of the set-valued random variable U, if they are, in a sense, close enough to each other, or when such a distinguishing is too time, space, or other expenses consuming. All these cases can be theoretically reflected when considering a relatively poor σ-field $\mathcal{S}$ in $\mathcal{P}(\mathcal{P}(S))$. The non-negligible remaining portion of idealization in this approach consists in our assumption that this class of subsets of $\mathcal{P}(S)$ is still a σ-field.

A special case of our approach to definitions of belief functions over infinite sets S is presented and investigated by J. Kohlas ([21]), when the support Ω of the basic probability space is supposed to be finite and the σ-field $\mathcal{A}$ is identified with $\mathcal{P}(\Omega)$. A generalization to infinite countable sets Ω is immediate and will be considered in our explanation. The probability measure on $\langle \Omega, \mathcal{P}(\Omega)\rangle$ is uniquely defined by a mapping $P : \Omega \to \langle 0,1\rangle$ such that $\sum_{\omega\in\Omega} P(\omega) = 1$. Under these simplifying conditions each mapping $U : \langle \Omega, \mathcal{P}(\Omega), P\rangle \to \langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S))\rangle$ is obviously measurable, hence, is a random variable. For each such a mapping U there exists a finite or countable system $\mathcal{A}_0(U)$ of subsets of S such that

$$\mathcal{A}_0(U) = \{A \subset S : \{\omega \in \Omega : U(\omega) = A\} \neq \emptyset\}. \tag{9.4}$$

So,

$$P\left(\{\omega \in \Omega : U(\omega) = A\}\right) = \sum\nolimits_{\omega : U(\omega) = A} P(\omega) = P\left(U^{-1}(A)\right) \geq 0, \tag{9.5}$$

if $A \in \mathcal{A}_0$, and $P(U^{-1}(A)) = P(\emptyset) = 0$, if $A \in \mathcal{P}(S) - \mathcal{A}_0$, hence, P looks like the degenerated probability measure on $\langle 0,1\rangle$ defined above. Consequently, for each $T \subset S$

$$\begin{aligned} bel^*_U(T) &= P\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(T) - \{\emptyset\}\}\right) = \\ &= \sum\nolimits_{\emptyset \neq A \subset T,\, A \in \mathcal{A}_0} P\left(U^{-1}(A)\right) \end{aligned} \tag{9.6}$$

is defined,

$$bel_U(T) = \frac{\sum_{\emptyset \neq A \subset T,\, A \in \mathcal{A}_0} P\left(U^{-1}(A)\right)}{\sum_{\emptyset \neq A \subset S,\, A \in \mathcal{A}_0} P\left(U^{-1}(A)\right)} \tag{9.7}$$

is defined, if there exists $A \subset S,\ A \in \mathcal{A}_0 - \{\emptyset\}$, such that $P(U^{-1}(A)) > 0$ holds, and the relations are analogous to those for the case when S is finite. Hence, the Kohlas' model outlines the domain when the combinatoric definitions of belief functions are immediately extendable to infinite sets S, so that some algorithms or other implementation results can be directly applied to this wider class of situations.

A σ-field $\mathcal{S} \subset \mathcal{P}(\mathcal{P}(S))$ is called *Dempster–Shafer complete* (DS-*complete*), if every $T \subset S$ is $\mathcal{S}$-regular, i. e., if $\mathcal{P}(T) \in \mathcal{S}$ holds for each $T \subset S$, so that $bel^*_{\rho,X}(T)$ can be defined for each $T \subset S$ supposing that $U_{\rho,X} : \langle \Omega, \mathcal{A}, P \rangle \to \langle \mathcal{P}(S), \mathcal{S} \rangle$ is a set-valued random variable. If $\mathcal{S}$ is DS-complete, then for each $\emptyset \neq \mathcal{S}_1 \subset \mathcal{P}(S)$,

$$\bigcap_{A \in \mathcal{S}_1} \mathcal{P}(A) = \mathcal{P}\left(\bigcap_{A \in \mathcal{S}_1} (A)\right) \in \mathcal{S}. \tag{9.8}$$

The following property of DS-complete σ-fields is perhaps less trivial.

Theorem 9.1. Let $\mathcal{S} \subset \mathcal{P}(\mathcal{P}(S))$ be a DS-complete σ-field, let $\mathcal{S}_1 \subset \mathcal{P}(S)$ be such that $\text{card}(\mathcal{S}_1) \leq \aleph_0$ and $\text{card}(A) \leq \aleph_0$ for each $A \in \mathcal{S}_1$. Then $\mathcal{S}_1 \in \mathcal{S}$. □

Proof. Being a σ-field, $\mathcal{S}$ is closed with respect to finite or countable unions so that the only we have to prove is that $\{A\} \in \mathcal{S}$ holds for each finite or countable $A \subset S$. Let A be such a subset of S. For each $i = 1, 2, \ldots$, the system $\mathcal{P}(A) \cap (\mathcal{P}(S) - \mathcal{P}(A - \{x_i\}))$ contains, if $x_i \in A$, just those subsets of A which are not subsets of $A - \{x_i\}$, i. e., just those subsets of A which contain x_i. By induction, the system

$$\mathcal{P}(A) \cap \bigcap_{i=1}^{n} (\mathcal{P}(S) - \mathcal{P}(A - \{x_i\})), \tag{9.9}$$

if $A = \{x_1, x_2, \ldots, x_n\}$ is finite, or the system

$$\mathcal{P}(A) \cap \bigcap_{i=1}^{\infty} (\mathcal{P}(S) - \mathcal{P}(A - \{x_i\})), \tag{9.10}$$

if $A = \{x_1, x_2, \ldots\}$ is infinite countable, contains just those subsets of A which contain all the elements $x_1, x_2, \ldots, x_n$ or $x_1, x_2, \ldots$ However, there is just one subset of A possessing this property, namely the set A itself, so that (9.9) or (9.10) defines just the singleton $\{A\}$. As $\mathcal{P}(A) \in \mathcal{S}$ and $\mathcal{P}(A - \{x_i\}) \in \mathcal{S}$ hold for each $x_i \in S$, the system of sets defined by (9.9) or (9.10) is also in $\mathcal{S}$. The assertion is proved. □

Till now, we have taken profit of the apparatus of the measure and probability theory in order to arrive at general enough definition of belief functions over infinite sets. In what follows, we shall try to overcross some restrictions, involved by this approach, using the ideas of *inner measure* and *outer measure* (cf. [15]) in order to generalize belief functions also to the subsets of S which are not $\mathcal{S}$-regular. In order to simplify our reasonings we shall still suppose, in the sequel, that the empty subset $\emptyset$ of S is $\mathcal{S}$-regular, i. e., that $\mathcal{P}(\emptyset) = \{\emptyset\} \in \mathcal{S}$, and that

$$P(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(\emptyset)\}) = P(\{\omega \in \Omega : U(\omega) = \emptyset\}) < 1 \tag{9.11}$$

holds, here and below $U(\omega)$ abbreviates $U_{\rho,X}(X(\omega))$ supposing that ρ and X are fixed in the given context. For each of the two functions, $bel_{\rho,X}(T) = bel(U, \mathcal{S})(T)$ (in order to introduce U and $\mathcal{S}$ as explicit parameters, and the corresponding plausibility function $pl(U, \mathcal{S})(T) = 1 - bel(U, \mathcal{S})(S - T)$, the four alternative ways of generalizations will be considered). We shall investigate only the case of normalized belief functions, the modifications for the non-normalized case can be easily obtained.

Definition 9.1. Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, let $\langle \mathcal{P}(S), \mathcal{S} \rangle$ be a measurable space over a nonempty set $\mathcal{S}$, let U be a set-valued (generalized) random variable defined on $\langle \Omega, \mathcal{A}, P \rangle$ and taking its values in $\langle \mathcal{P}(S), \mathcal{S} \rangle$, let $bel_U(T)$ and $pl_U(T)$ be defined, for each $\mathcal{S}$-regular $T \subset S$ and $\mathcal{S}$-regular $S - T \subset S$ by

$$bel_U(T) = P(\{\omega \in \Omega : U(\omega) \subset T\}/\{\omega \in \Omega : U(\omega \neq \emptyset)\}), \tag{9.12}$$

$$pl_U(T) = 1 - bel_U(S - T). \tag{9.13}$$

Set, for each $T \subset S$,

$$bel_+(U, \mathcal{S})(T) = \sup\{P(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}/ \{\omega \in \Omega : U(\omega) \neq \emptyset\}) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(T) - \{\emptyset\}\} \tag{9.14}$$

$$bel^+(U, \mathcal{S})(T) = \inf\{P(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}/ \{\omega \in \Omega : U(\omega) \neq \emptyset\}) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \supset \mathcal{P}(T) - \{\emptyset\}\}, \tag{9.15}$$

$$bel_{++}(U, \mathcal{S})(T) = \sup\{bel_U(R) : R \subset T\}, \tag{9.16}$$

$$bel^{++}(U, \mathcal{S})(T) = \inf\{bel_U(R) : T \subset R \subset S\}, \tag{9.17}$$

$$pl_i(U, \mathcal{S})(T) = 1 - bel_i(U, \mathcal{S})(S - T) \tag{9.18}$$

for each $i = {}_+, {}^+, {}_{++}, {}^{++}$. □

As can be easily observed, all the eight functions introduced above are defined for each $T \subset S$, as the set φ of real numbers to which the sup or inf operation is applied, is nonempty in all cases. For (9.14) we take $\mathcal{B} = \emptyset^*$ (the empty subset of $\mathcal{P}(S)$ which is always in $\mathcal{S}$ and which should not be confused with the empty subset $\emptyset$ of S or with the nonempty subset $\{\emptyset\}$ of $\mathcal{P}(S)$), so that $P(\{\omega \in \Omega : U(\omega) \in \emptyset^*\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) = 0$ is in φ, for (9.15) we take $\mathcal{B} = \mathcal{P}(S) \in \mathcal{S}$, so that $P(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(S)\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) = 1$ is in φ, for (9.16) the set φ contains $bel_U(\emptyset) = 0$ and for (9.17) φ contains $bel_U(S) = 1$. For all the variants of plausibility functions an analogous consideration obviously holds as well.

The following assertion proves that all the four generalizations of belief and plausibility functions agree with their original variants bel_U and pl_U, supposing that this original variant is defined.

Theorem 9.2. For each $T \subset S$,

$$bel_{++}(U, \mathcal{S})(T) \leq bel_+(U, \mathcal{S})(T), \tag{9.19}$$

$$pl_{++}(U, \mathcal{S})(T) \geq pl_+(U, \mathcal{S})(T), \tag{9.20}$$

$$bel^{++}(U, \mathcal{S})(T) \geq bel^+(U, \mathcal{S})(T), \tag{9.21}$$

$$pl^{++}(U, \mathcal{S})(T) \leq pl^+(U, \mathcal{S})(T). \tag{9.22}$$

If T is $\mathcal{S}$-regular, then

$$\begin{aligned} bel_{++}(U,\mathcal{S})(T) &= bel_{+}(U,\mathcal{S})(T) = bel^{+}(U,\mathcal{S})(T) = \\ &= bel^{++}(U,\mathcal{S})(T) = bel_U(T), \end{aligned} \tag{9.23}$$

if $S - T$ is $\mathcal{S}$-regular, then

$$\begin{aligned} pl_{++}(U,\mathcal{S})(T) &= pl_{+}(U,\mathcal{S})(T) = pl^{+}(U,\mathcal{S})(T) = \\ &= pl^{++}(U,\mathcal{S})(T) = pl_U(T). \end{aligned} \tag{9.24}$$

□

Proof. To abbreviate our notation, we shall omit the parameters U and $\mathcal{S}$ in functions defined by (9.14)–(9.18), if no misunderstating menaces. Let $T \subset S$, let $\varphi_T = \{\mathcal{B} : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(T) - \{\emptyset\}\}$, let $R \subset T$ be such that $bel_U(R) = P(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R) - \{\emptyset\}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\})$ is defined. Then $\mathcal{P}(R) - \{\emptyset\} \in \mathcal{S}$ and $R \subset T$ implies that $\mathcal{P}(R) - \{\emptyset\} \subset \mathcal{P}(T) - \{\emptyset\}$, hence, $\mathcal{P}(T) - \{\emptyset\} \in \varphi_T$. It follows immediately that

$$\begin{aligned} bel_{++}(T) &= \sup\{P(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R) - \{\emptyset\}\}/ \\ &\quad \{\omega \in \Omega : U(\omega) \neq \emptyset\}) : R \subset T\} \leq \\ &\leq \sup\{P(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) : \mathcal{B} \in \varphi_T\} \\ &= bel_{+}(T). \end{aligned} \tag{9.25}$$

The inequality $pl_{+}(T) \leq pl_{++}(T)$ follows immediately from (9.18) and (9.27). (9.21) and (9.22) can be proved in an analogous way, replacing dually sup by inf, $\subset$ by $\supset$, and $\leq$ by $\geq$.

Let T be $\mathcal{S}$-regular, so that $\mathcal{P}(T) \in \mathcal{S}$ and $\mathcal{P}(T) - \{\emptyset\} \in \mathcal{S}$. Probability measure is monotonous with respect to the set inclusion, so that $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{S}$, $B_1 \subset B_2$, yields that

$$\begin{aligned} & P(\{\omega \in \Omega : U(\omega) \in B_1\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) \leq \\ \leq \;& P(\{\omega \in \Omega : U(\omega) \in B_2\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) \end{aligned} \tag{9.26}$$

holds. Consequently, for each $\mathcal{B} \in \varphi_T$,

$$\begin{aligned} & P(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) \leq \\ \leq \;& P(\{\omega \in \Omega : U(\omega) \in P(T) - \{\emptyset\}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) = bel_U(T), \end{aligned} \tag{9.27}$$

so that, as $\mathcal{P}(T) - \{\emptyset\} \in \varphi_T$ holds,

$$\begin{aligned} bel_{+}(T) &= \sup\{P(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) : \mathcal{B} \in \varphi_T\} = \\ &= bel_U(T). \end{aligned} \tag{9.28}$$

In a similar way:

$$P(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R) - \{\emptyset\}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) = bel_U(R), \tag{9.29}$$

$$P(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(T) - \{\emptyset\}\}/\{\omega \in \Omega : U(\omega) \neq \emptyset\}) = bel_U(T) \tag{9.30}$$

for each $\mathcal{S}$-regular $R \subset T \subset S$, so that

$$bel_{++}(T) = \sup\{bel_U(R) : R \subset T \subset S,\ R \text{ is } \mathcal{S}\text{-regular}\} = \\ = bel_U(T). \tag{9.31}$$

The proof for $bel^{+}(T)$ and $bel^{++}(T)$ is the same up to the dual replacements introduced above. Applying (9.23) to the subset $S - T \subset S$, (9.24) trivially follows. The assertion is proved. □

In what follows, we shall state and prove some inequalities expressing super-additivity or sub-additivity of various generalizations of belief and plausibility functions. The degree in which these inequalities agree with the inequalities holding for the original belief and plausibility functions, supposing that the latest ones are defined, will serve as an argument when choosing which among the generalizations in question is the most favourable one. Namely, in the next theorem we shall investigate the "lower" or "inner" generalizations bel_{+}, bel_{++}, pl_{+}, and pl_{++}.

Theorem 9.3. For each $T_1, T_2 \subset S$, $T_1 \cap T_2 = \emptyset$, the inequality

$$bel_{+}(U,\mathcal{S})(T_1) + bel_{+}(U,\mathcal{S})(T_2) \le bel_{+}(U,\mathcal{S})(T_1 \cup T_2) \tag{9.32}$$

holds. If, moreover, T_1, T_2 and $T_1 \cup T_2$ are $\mathcal{S}$-regular, then the inequality

$$bel_U(T_1) + bel_U(T_2) \le bel_U(T_1 \cup T_2) \tag{9.33}$$

holds. For each $T_1, T_2 \subset S$ the inequalities

$$pl_{+}(U,\mathcal{S})(T_1) + pl_{+}(U,\mathcal{S})(T_2) \ge pl_{+}(U,\mathcal{S})(T_1 \cup T_2), \tag{9.34}$$

$$pl_{++}(U,\mathcal{S})(T_1) + pl_{++}(U,\mathcal{S})(T_2) \ge pl_{++}(U,\mathcal{S})(T_1 \cup T_2) \tag{9.35}$$

hold. If, moreover, $S - T_1$ and $S - T_2$ are $\mathcal{S}$-regular, then the inequality

$$pl_U(T_1) + pl_U(T_2) \ge pl_U(T_1 \cup T_2) \tag{9.36}$$

holds. □

Proof. Parameters U, $\mathcal{S}$ are omitted as in the proof of Theorem 9.2; to abbreviate our notation more substantially we shall denote by $P^{\emptyset}$ the conditional probability measure defined by P and by the condition $\{\omega \in \Omega : U(\omega) \neq \emptyset\}$. Hence, we shall write $P^{\emptyset}(A)$ instead of $P(A/\{\omega \in \Omega : U(\omega) \neq \emptyset\})$ for each $A \subset \Omega$, $A \in \mathcal{A}$.

Let $T_1, T_2 \subset S$, $T_1 \cap T_2 = \emptyset$, set, for both $i = 1, 2$

$$\varphi_i = \left\{P^{\emptyset}(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}) : \mathcal{B} \in \mathcal{S},\ \mathcal{B} \subset \mathcal{P}(T_i) - \{\emptyset\}\right\}, \tag{9.37}$$

so that $bel_{+}(T_i) = \sup \varphi_i$ for both $i = 1, 2$. As we have already proved, $\varphi_i \neq \emptyset$, $i = 1, 2$, and we can choose $a_1 \in \varphi_1$, $a_2 \in \varphi_2$. Hence, there exist $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{S}$ such that $\mathcal{B}_i \subset \mathcal{P}(T_i) - \{\emptyset\}$ and $P^{\emptyset}(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_i\}) = a_i$ for $i = 1, 2$. But,

$\mathcal{P}(R_1) \cap \mathcal{P}(R_2) = \mathcal{P}(R_1 \cap R_2)$ for each R_1, $R_2 \subset S$, so that $T_1 \cap T_2 = \emptyset$ implies that $\mathcal{P}(T_1 \cap T_2) = \mathcal{P}(T_1) \cap \mathcal{P}(T_2) = \mathcal{P}(\emptyset) = \{\emptyset\}$, consequently, $(\mathcal{P}(T_1) - \{\emptyset\}) \cap (\mathcal{P}(T_2) - \{\emptyset\}) = \emptyset^*$ (let us recall that $\emptyset^*$ denotes the empty subset of $\mathcal{P}(S)$ or: in $\mathcal{P}(\mathcal{P}(S))$). So, $\mathcal{B}_i \subset S$ and $\mathcal{B}_i \subset \mathcal{P}(T_i) - \{\emptyset\}$, $i = 1, 2$, yields that $\mathcal{B}_1 \cup \mathcal{B}_2 \subset \mathcal{S}$ and $\mathcal{B}_1 \cup \mathcal{B}_2 = \emptyset^*$, hence,

$$\begin{aligned} & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1 \cup \mathcal{B}_2\}\right) = \\ = \ & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1\}\right) + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_2\}\right) = a_1 + a_2. \end{aligned} \tag{9.38}$$

So, setting

$$\varphi_3 = \left\{ P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}\right) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(T_1 \cup T_2) - \{\emptyset\} \right\}, \tag{9.39}$$

we obtain immediately, that $bel_+(T_1 \cup T_2) = \sup \varphi_3$, and $a_1 + a_2 \in \varphi_3$. For each $\varepsilon > 0$ there are $a_i \geq \sup \varphi_i - (\varepsilon/2)$, $a_i \in \varphi_i$, $i = 1, 2$, so that $a_1 + a_2 \geq (\sup \varphi_1 + \sup \varphi_2) - \varepsilon$ and $a_1 + a_2 \in \varphi_3$. Consequently,

$$bel_+(T_1 \cup T_2) = \sup \varphi_3 \geq \sup \varphi_1 + \sup \varphi_2 = bel_+(T_1) + bel_+(T_2), \tag{9.40}$$

and (9.32) is proved. If, moreover, T_1, T_2, and $T_1 \cup T_2$ are $\mathcal{S}$-regular, then, due to Theorem 9.2, $bel_+(T_i) = bel_U(T_1)$, $i = 1, 2$, and $bel_+(T_1 \cup T_2) = bel_U(T_1 \cup T_2)$, so that (9.32) immediately follows.

For the case of plausibility functions, let us begin with the case when pl_U is defined. Let R_1, R_2 be any $\mathcal{S}$-regular subsets of S. Then $R_1 \cap R_2$ is $\mathcal{S}$-regular as well, as $\mathcal{P}(R_i) \in \mathcal{S}$, $i = 1, 2$, implies that $\mathcal{P}(R_1 \cap R_2) = \mathcal{P}(R_1) \cap \mathcal{P}(R_2) \in \mathcal{S}$. Then, $\mathcal{P}(R_1) - \mathcal{P}(R_1 \cap R_2)$, $\mathcal{P}(R_2) - \mathcal{P}(R_1 \cap R_2)$, and $\mathcal{P}(R_1 \cap R_2)$ are mutually disjoint subsets of $\mathcal{P}(S)$ and $\mathcal{P}(R_1 \cap R_2) \subset \mathcal{P}(R_i)$ holds for both $i = 1, 2$, so that

$$\begin{aligned} & 1 \geq P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_1) - \mathcal{P}(R_1 \cap R_2)\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_2) - \mathcal{P}(R_1 \cap R_2)\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_1) \cap \mathcal{P}(R_2)\}\right) = \\ = \ & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_1)\}\right) - P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_1 \cap R_2)\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_2)\}\right) - P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_1 \cap R_2)\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R_1 \cap R_2)\}\right) = \\ = \ & bel_U(R_1) + bel_U(R_2) - bel_U(R_1 \cap R_2). \end{aligned} \tag{9.41}$$

For $T_i = S - R_i$, $i = 1, 2$, we obtain that

$$\begin{aligned} & bel_U(S - T_1) + bel_U(S - T_2) - bel_U((S - T_1) \cap (S - T_2)) = \\ = \ & bel_U(S - T_1) + bel_U(S - T_2) - bel_U((S - (T_1 \cup T_2)) \leq 1, \end{aligned} \tag{9.42}$$

consequently,

$$bel_U(S - T_1) + bel_U(S - T_2) - 2 \leq bel_U((S - T_1 \cup T_2)) - 1, \tag{9.43}$$

$$(1 - bel_U(S - T_1)) + (1 - bel_U(S - T_2)) \geq 1 - bel_U\left((S - (T_1 \cup T_2)\right), \tag{9.44}$$

so that (9.18) implies that

$$pl_U(T_1) + pl_U(T_2) \geq pl_U(T_1 \cup T_2) \tag{9.45}$$

and (9.36) is proved.

For arbitrary R_1, $R_2 \subset S$, take $\mathcal{B}_1 \in \mathcal{S}$, $\mathcal{B}_1 \subset \mathcal{P}(R_1) - \{\emptyset\}$, and take $\mathcal{B}_2 \in \mathcal{S}$, $\mathcal{B}_2 \subset \mathcal{P}(R_2) - \{\emptyset\}$, then, setting $\mathcal{B}_3 = \mathcal{B}_1 \cap \mathcal{B}_2$, we obtain immediately that $\mathcal{B}_3 \in \mathcal{S}$ and $\mathcal{B}_3 \subset (\mathcal{P}(R_1) \cap \mathcal{P}(R_2)) - \{\emptyset\} = \mathcal{P}(R_1 \cap R_2) - \{\emptyset\}$. Moreover,

$$\begin{aligned} & 1 \geq P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1 \cup \mathcal{B}_2\}\right) = \\ = \; & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in (\mathcal{B}_1 - \mathcal{B}_3) \cup (\mathcal{B}_2 - \mathcal{B}_3) \cup \mathcal{B}_3\}\right) \\ = \; & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1\}\right) - P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_3\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_2\}\right) - P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_3\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_3\}\right) = \\ = \; & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1\}\right) + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_2\}\right) - \\ & \quad - P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1 \cap \mathcal{B}_2\}\right) \geq \\ \geq \; & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_1\}\right) + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}_2\}\right) - \\ & \quad - \sup\left\{P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}\right) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(R_1 \cap R_2) - \{\emptyset\}\right\}. \end{aligned} \tag{9.46}$$

Being valid for each $\mathcal{B}_1$, $\mathcal{B}_2$ possessing the properties in question, (9.46) holds also for the supremum value, hence,

$$\begin{aligned} & \sup\left\{P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}\right) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(R_1) - \{\emptyset\}\right\} + \\ & \quad + \sup\left\{P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}\right) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(R_2) - \{\emptyset\}\right\} - \\ & \quad - \sup\left\{P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}\right) : \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(R_1 \cap R_2) - \{\emptyset\}\right\} = \\ = \; & bel_{+}(R_1) + bel_{+}(R_2) - bel_{+}(R_1 \cap R_2) \leq 1 \end{aligned} \tag{9.47}$$

holds. Consequently,

$$bel_{+}(S - T_1) + bel_{+}(S - T_2) - bel_{+}(S - (T_1 \cup T_2)) \leq 1 \tag{9.48}$$

and we obtain, analogously to (9.43) and (9.44), that

$$pl_{+}(T_1) + pl_{+}(T_2) \geq pl_{+}(T_1 \cup T_2) \tag{9.49}$$

and (9.34) is proved.

Finally, for pl_{++} the proof will be similar. Let R_1, $R_2 \subset S$ be arbitrary, let $H_1 \subset R_1$ be such that $bel_U(H_1)$ is defined, let $H_2 \subset R_2$ be such that $bel_U(H_2)$ is defined. Then $\mathcal{P}(H_i) \in \mathcal{S}$ for both $i = 1, 2$, $H_1 \cap H_2 \subset R_1 \cap R_2$, and $\mathcal{P}(H_1 \cap H_2) = \mathcal{P}(H_1) \cap \mathcal{P}(H_2)$ as well as $\mathcal{P}(H_1 \cap H_2) - \{\emptyset\}$ are in $\mathcal{S}$. Consequently,

$$\begin{aligned} & 1 \geq P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H_1) \cup P(H_2) - \{\emptyset\}\}\right) = \\ = \; & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H_1) - \{\emptyset\}\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H_2) - \{\emptyset\}\}\right) - \\ & -P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H_1 \cap H_2) - \{\emptyset\}\}\right) \geq \\ \geq \; & P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H_1) - \{\emptyset\}\}\right) + \\ & \quad + P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H_2) - \{\emptyset\}\}\right) - \\ & \quad - \sup\left\{P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(H) - \{\emptyset\}\}\right) : \mathcal{P}(H) \in \mathcal{S}, H \subset R_1 \cap R_2\right\}. \end{aligned} \tag{9.50}$$

Applying the supremum operation to the first two summands we obtain that

$$\begin{aligned} & \sup\left\{P^{\emptyset}\left(\{\omega\in\Omega : U(\omega)\in\mathcal{P}(H)-\{\emptyset\}\}\right) : \mathcal{P}(H)\in\mathcal{S},\, H\subset R_1\right\} + \\ & \quad + \sup\left\{P^{\emptyset}\left(\{\omega\in\Omega : U(\omega)\in\mathcal{P}(H)-\{\emptyset\}\}\right) : \mathcal{P}(H)\in\mathcal{S},\, H\subset R_2\right\} - \\ & \quad - \sup\left\{P^{\emptyset}\left(\{\omega\in\Omega : U(\omega)\in\mathcal{P}(H)-\{\emptyset\}\}\right) : \mathcal{P}(H)\in\mathcal{S},\, H\subset R_1\cap R_2\right\} = \\ = & \sup\{bel_U(H) : H\subset R_1\} + \sup\{bel_U(H) : H\subset R_2\} - \\ & \quad - \sup\{bel_U(H) : H\subset R_1\cap R_2\} = \\ = & \; bel_{++}(R_1) + bel_{++}(R_2) - bel_{++}(R_1\cap R_2) \leq 1. \end{aligned} \tag{9.51}$$

In the same way as above we obtain that

$$pl_{++}(T_1) + pl_{++}(T_2) \geq pl_{++}(T_1\cup T_2). \tag{9.52}$$

Hence, (9.35) and Theorem 9.3 as a whole are proved. □

It is perhaps worth mentioning explicitly, that an analogy of (9.32) for bel_{++} does not hold. Let $\emptyset \neq T_1, T_2 \subset S$ be such that $T_1 \cap T_2 = \emptyset$, let $\mathcal{S}$ be the minimal σ-field in $\mathcal{P}(\mathcal{P}(S))$ containing $\mathcal{P}(T_1)$ and $\mathcal{P}(T_2)$, so that $bel_U(T_1)$ and $bel_U(T_2)$ are defined, let $0 < bel_U(T_i)$ for both $i = 1, 2$. Then $\mathcal{S}$ is just the set of all finite unions of nonempty sets from $\mathcal{S}_0$ (including the empty union which defines the empty set $\emptyset^*$), where

$$\begin{aligned} \mathcal{S}_0 \;=\; & \{a\cap b\cap c : a,\, b,\, c\in\{\mathcal{P}(\emptyset),\, \mathcal{P}(S)-\mathcal{P}(\emptyset),\, \mathcal{P}(T_1),\, \mathcal{P}(S)-\mathcal{P}(T_1) \\ & \mathcal{P}(T_2),\, \mathcal{P}(S)-\mathcal{P}(T_2)\}. \end{aligned} \tag{9.53}$$

Or, evidently, a countable union of finite unions of nonempty sets from $\mathcal{S}_0$ reduces to a finite union of nonempty sets from $\mathcal{S}_0$, and a complement of a finite union B of nonempty sets from $\mathcal{S}_0$ can be defined by the finite union of just those nonempty elements of $\mathcal{S}_0$ which do not occur in the finite union B. Let us prove that if $T \subset S$, $T \neq S$ is such that $T \neq T_1$, $T \neq T_2$, and $T_1 \subset T$ or $T_2 \subset T$, then $\mathcal{P}(T)$ does not belong to $\mathcal{S}$.

Suppose that $S \neq T \supset T_1$, $T \neq T_1$ (hence, $T \neq T_2$), as the case when $T \supset T_2$, $T \neq T_2$ is quite analogous. So, $\mathcal{P}(S) \neq \mathcal{P}(T) \supset \mathcal{P}(T_1)$, $\mathcal{P}(T) \neq \mathcal{P}(T_1)$, hence, if $\mathcal{P}(T) \in \mathcal{S}$, then $\mathcal{P}(T) = \mathcal{P}(T_1) \cup B$ for $B = \mathcal{P}(T) - \mathcal{P}(T_1) = \mathcal{P}(T) \cap (\mathcal{P}(S) - \mathcal{P}(T_1))$, so that $B \in \mathcal{S}$. An exhaustive examination of all sets in $\mathcal{S}$ proves that the only sets in $\mathcal{S}$ which are proper subsets of $\mathcal{P}(S)$ and which contain $\mathcal{P}(T_1)$ as their own proper subset are $\mathcal{P}(T_1) \cup \mathcal{P}(T_2)$, $\mathcal{P}(T_1) \cup (\mathcal{P}(S) - \mathcal{P}(T_2))$, and $\mathcal{P}(T_1) \cup ((\mathcal{P}(S) - \mathcal{P}(T_1)) \cap (\mathcal{P}(S) - \mathcal{P}(T_2))) = (\mathcal{P}(S) - \mathcal{P}(T_2)) \cup \mathcal{P}(\emptyset)$. The equality $\mathcal{P}(T) = \mathcal{P}(T_1) \cup \mathcal{P}(T_2)$ cannot hold for no matter which $T \subset S$, as $T_1, T_2 \in \mathcal{P}(T)$ implies that $T_1 \cup T_2 \in \mathcal{P}(T)$, but $T_1, T_2 \in \mathcal{P}(T_1) \cup \mathcal{P}(T_2)$ does not imply $T_1 \cup T_2 \in \mathcal{P}(T_1) \cup \mathcal{P}(T_2)$ for $T_1, T_2 \neq \emptyset$. The equalities $\mathcal{P}(T) = \mathcal{P}(T_1) \cup (\mathcal{P}(S) - \mathcal{P}(T_2))$ or $\mathcal{P}(T) = (\mathcal{P}(S) - \mathcal{P}(T_2)) \cup \mathcal{P}(\emptyset)$ cannot hold as well, as $S \notin \mathcal{P}(T)$ for $S \neq T$, but $S \in \mathcal{P}(S) - \mathcal{P}(T_2)$, as $T_2 \neq S$.

Consequently, T_1 and T_2 are the only proper subsets of S for which, with respect to the given $\mathcal{S}$, bel_U is defined, so that

$$\begin{aligned} & bel_{++}(U,\mathcal{S})\,(T_1\cup T_2) = \sup\{bel_U(T) : T\subset T_1\cup T_2\} = \\ = \; & \max\{bel_U(T_1),\, bel_U(T_2)\} < bel_{++}(T_1) + bel_{++}(T_2), \end{aligned} \tag{9.54}$$

as $bel_{++}(T_i) > 0$ holds for both $i = 1, 2$. Hence, an analogy of (9.32) for bel_{++} does not hold.

As far as the approximations of belief and plausibility functions indexed by upper crosses are concerned, we can easily prove that no of the relations (9.32), (9.34), and (9.35) hold when bel_+ is replaced by bel^+, pl_+ by pl^+, and bel_{++} by pl^{++}. The counter-examples can be obtained as follows.

Let $\emptyset \neq T_1, T_2 \subset S$ be such that $T_1 \cap T_2 = \emptyset$, let $\mathcal{S}_1 = \{\emptyset^*, \mathcal{P}(\emptyset), \mathcal{P}(S), \mathcal{P}(S) - \mathcal{P}(\emptyset)\}$ be the minimal σ-field in $\mathcal{P}(\mathcal{P}(S))$ generated by $\mathcal{P}(\emptyset)$. Hence, setting $T_3 = T_1 \cup T_2$, for each $i = 1, 2, 3$ the only sets in $\mathcal{S}_1$ containing $\mathcal{P}(T_i) - \mathcal{P}(\emptyset) = \mathcal{P}(T_i) - \{\emptyset\}$ are $\mathcal{P}(S) - \mathcal{P}(\emptyset)$ and $\mathcal{P}(S)$, so that, again for each $i = 1, 2, 3$,

$$\begin{aligned} & bel^+(U, \mathcal{S}_1)(T_i) = \\ = \; & \inf\left\{P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}) : \mathcal{B} \in \mathcal{S}_1, \mathcal{B} \supset \mathcal{P}(T_i) - \{\emptyset\}\right\} = \\ = \; & P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(S) - \{\emptyset\}\}) = 1, \end{aligned} \tag{9.55}$$

so that $bel^+(U, \mathcal{S}_1)(T_1) + bel^+(U, \mathcal{S}_1)(T_2) = 1 + 1 = 2 > 1 = bel^+(U, \mathcal{S}_1)(T_1 \cup T_2)$, and an assertion analogous to (9.32) does not hold.

Let $\emptyset \neq R_1, R_2 \subset S$, let $R = R_1 \cap R_2$, let $R \neq R_1$, $R \neq R_2$, let $\mathcal{S}_2$ be the minimal σ-field in $\mathcal{P}(\mathcal{P}(S))$ generated by $\mathcal{P}(\emptyset)$ and $\mathcal{P}(R)$, hence

$$\begin{aligned} \mathcal{S}_2 \; = \; & \{\emptyset^*, \mathcal{P}(\emptyset), \mathcal{P}(R), \mathcal{P}(S) - \mathcal{P}(\emptyset), \mathcal{P}(S) - \mathcal{P}(R), \\ & \mathcal{P}(R) - \mathcal{P}(\emptyset), (\mathcal{P}(S) - \mathcal{P}(R)) \cup \mathcal{P}(\emptyset)\}, \end{aligned} \tag{9.56}$$

as can be easily verified checking that $\mathcal{P}(S) = A \in \mathcal{S}_2$ and $A \cup B \in \mathcal{S}_2$ holds for each $A, B \in \mathcal{S}_2$. Let

$$0 < P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R) - \mathcal{P}(\emptyset)\}) = \alpha < 1 \tag{9.57}$$

hold, consequently, $\mathcal{P}(R) - \mathcal{P}(\emptyset) \neq \emptyset^*$, so that $R = \emptyset$. Then, for both $i = 1, 2$, the only sets in $\mathcal{S}_2$ containing $\mathcal{P}(R_1) - \{\emptyset\}$ are $\mathcal{P}(S) - \{\emptyset\}$ and $\mathcal{P}(S)$, so that

$$\begin{aligned} & bel^+(U, \mathcal{S}_2)(R_i) = \\ = \; & \inf\left\{P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}) : \mathcal{B} \in \mathcal{S}_2, \mathcal{B} \supset \mathcal{P}(R_i) - \{\emptyset\}\right\} = \\ = \; & P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(S) - \mathcal{P}(\emptyset)\}) = 1. \end{aligned} \tag{9.58}$$

However,

$$\begin{aligned} & bel^+(U, \mathcal{S}_2)(R) = bel^+(U, \mathcal{S}_2)(R_1 \cap R_2) = \\ = \; & \inf\left\{P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{B}\}) : \mathcal{B} \in \mathcal{S}_2, \mathcal{B} \supset \mathcal{P}(R) - \{\emptyset\}\right\} = \\ = \; & P^\emptyset(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R) - \mathcal{P}(\emptyset)\}) = \alpha, \end{aligned} \tag{9.59}$$

as $\mathcal{P}(R) = \mathcal{P}(\emptyset) \in \mathcal{S}_2$. Consequently,

$$\begin{aligned} & bel^+(U, \mathcal{S}_2)(R_1) + bel^+(U, \mathcal{S}_2)(R_2) - bel^+(U, \mathcal{S}_2)(R_1 \cap R_2) = \\ = \; & 1 + 1 - \alpha > 1. \end{aligned} \tag{9.60}$$

Using the same σ-field $\mathcal{S}_2$ we can easily see that the only $T \subset S$ for which $bel_U(T)$ is defined are $T = \emptyset$, $T = R_1 \cap R_2$, and $T = S$. So, for both $i = 1, 2$,

$$bel^{++}(U, \mathcal{S}_2)\,(R_i) = \inf\{bel_U(H) : R_i \subset H \subset S\} = bel_U(S) = 1, \tag{9.61}$$

but

$$\begin{aligned} & bel^{++}(U, \mathcal{S}_2)\,(R_1 \cap R_2) = \\ = \;& \inf\{bel_U(H) : R_1 \cap R_2 \subset H \subset S\}\} = bel_U(R) = \\ = \;& P^{\emptyset}\left(\{\omega \in \Omega : U(\omega) \in \mathcal{P}(R) - \mathcal{P}(\emptyset)\}\right) = \alpha, \end{aligned} \tag{9.62}$$

so that, again,

$$bel^{++}(U, \mathcal{S}_2)\,(R_1) + bel^{++}(U, \mathcal{S}_2)\,(R_2) - bel^{++}(U, \mathcal{S}_2)\,(R_1 \cap R_2) > 1 \tag{9.63}$$

holds. Using the same way of reasoning and computation as above, when proving (9.34) and (9.35), and setting $R_i = S - T_i$ in (9.60), we obtain that

$$\begin{aligned} & bel^{+}(U, \mathcal{S}_2)\,(S - T_1) + bel^{+}(U, \mathcal{S}_2)\,(S - T_2) - \\ & -bel^{+}(U, \mathcal{S}_2)\,((S - T_1) \cap (S - T_2)) > 1, \end{aligned} \tag{9.64}$$

$$\begin{aligned} & (1 - bel^{+}(U, \mathcal{S}_2)\,(S - T_1)) + (1 - bel^{+}(U, \mathcal{S}_2)\,(S - T_2)) < \\ < \;& 1 - bel^{+}(U, \mathcal{S}_2)\,(S - (T_1 \cup T_2)), \end{aligned}$$

hence,

$$pl^{+}(U, \mathcal{S}_1)\,(T_1) + pl^{+}(U, \mathcal{S}_2)\,(T_2) < pl^{+}(U, \mathcal{S}_2)\,(T_1 \cup T_2). \tag{9.65}$$

Replacing $+$ by $++$ in (9.64) we obtain an inequality for pl^{++} analogous to (9.65), so that neither (9.34) for pl^{+} nor (9.35) for pl^{++} hold.

Combining together and reconsidering the results of the last chapter we can see that among the four alternatives how to generalize belief and plausibility functions for those subsets of S which are not $\mathcal{S}$-regular, only the function bel_{+} conserves the property (9.32) typical for the original belief function bel_U. In order to support our idea to consider bel_{+} and pl_{+} as only reasonable extensions of bel_U and pl_U to whole $\mathcal{P}(S)$, we shall investigate, in the rest of this chapter, the dependence of belief and plausibility functions, and their generalizations defined above, on the σ-field $\mathcal{S}$ of subsets of $\mathcal{P}(S)$ taken as discernible sets of values of the set-valued random variable U.

Theorem 9.4. Let $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{P}(\mathcal{P}(S))$ be two nonempty σ-fields of subsets of $\mathcal{P}(S)$, let $U : \langle \Omega, \mathcal{A}, P \rangle \to \langle \mathcal{P}(S), \mathcal{S}_2 \rangle$ be a set-valued $\mathcal{S}_2$-measurable random variable. Then, for each $T \subset S$,

$$bel_{+}(U, \mathcal{S}_1)\,(T) \leq bel_{+}(U, \mathcal{S}_2)\,(T), \tag{9.66}$$

$$bel^{+}(U, \mathcal{S}_1)\,(T) \geq bel^{+}(U, \mathcal{S}_2)\,(T), \tag{9.67}$$

$$bel_{++}(U, \mathcal{S}_1)\,(T) \leq bel_{++}(U, \mathcal{S}_2)\,(T), \tag{9.68}$$

$$bel^{++}(U, \mathcal{S}_1)\,(T) \geq bel^{++}(U, \mathcal{S}_2)\,(T), \tag{9.69}$$

$$pl_{+}(U,\mathcal{S}_1)(T) \geq pl_{+}(U,\mathcal{S}_2)(T), \tag{9.70}$$
$$pl^{+}(U,\mathcal{S}_1)(T) \leq pl^{+}(U,\mathcal{S}_2)(T), \tag{9.71}$$
$$pl_{++}(U,\mathcal{S}_1)(T) \geq pl_{++}(U,\mathcal{S}_2)(T), \tag{9.72}$$
$$pl^{++}(U,\mathcal{S}_1)(T) \leq pl^{++}(U,\mathcal{S}_2)(T). \tag{9.73}$$

If T is $\mathcal{S}_1$-regular, then
$$bel_{U,\mathcal{S}_1}(T) = bel_{U,\mathcal{S}_2}(T), \tag{9.74}$$
if $S - T$ is $\mathcal{S}_1$-regular, then
$$pl_{U,\mathcal{S}_1}(T) = pl_{U,\mathcal{S}_2}(T). \tag{9.75}$$
□

Proof. As can be easily seen, if U is an $\mathcal{S}_2$-measurable mapping which takes Ω into $\mathcal{P}(S)$, then U is also $\mathcal{S}_1$-measurable, moreover, $\mathcal{S}_1 \subset \mathcal{S}_2$ implies that each $\mathcal{S}_1$-regular subset of S is also $\mathcal{S}_2$-regular. Define, for each $i = 1,2$, the following subsets of the unit interval $\langle 0,1\rangle$ of real numbers:

$$\varphi_{+,i}(T) = \left\{P^{\emptyset}(\{\omega\in\Omega : U(\omega)\in\mathcal{A}\}) : \mathcal{A}\in\mathcal{S}_i,\ \mathcal{A}\subset\mathcal{P}(T)-\{\emptyset\}\right\}, \tag{9.76}$$
$$\varphi_{i}^{+}(T) = \left\{P^{\emptyset}(\{\omega\in\Omega : U(\omega)\in\mathcal{A}\}) : \mathcal{A}\in\mathcal{S}_i,\ \mathcal{A}\supset\mathcal{P}(T)-\{\emptyset\}\right\}. \tag{9.77}$$

As $\{\emptyset,\mathcal{P}(S)\}\subset\mathcal{S}_1\subset\mathcal{S}_2$ holds, all the four sets are nonempty and $\varphi_{+,1}(T)\subset\varphi_{+,2}(T)$, $\varphi_1^{+}(T)\subset\varphi_2^{+}(T)$, so that

$$bel_{+}(U,\mathcal{S}_1)(T) = \sup\varphi_{+,1}(T) \leq \sup\varphi_{+,2}(T) = bel_{+}(U,\mathcal{S}_2)(T), \tag{9.78}$$
$$bel^{+}(U,\mathcal{S}_1)(T) = \inf\varphi_1^{+}(T) \geq \inf\varphi_2^{+}(T) = bel^{+}(U,\mathcal{S}_2)(T), \tag{9.79}$$

hence, (9.66) and (9.67) hold. Similarly, set

$$\psi_{+,i}(T) = \{bel_U(R) : R\subset T,\ \mathcal{P}(R)\in\mathcal{S}_i\}, \tag{9.80}$$
$$\psi_i^{+}(T) = \{bel_U(R) : T\subset R\subset S,\ \mathcal{P}(R)\in\mathcal{S}_i\}, \tag{9.81}$$

for both $i = 1,2$. Then, again, all the four sets are nonempty, $\psi_{+,1}(T)\subset\psi_{+,2}(T)$ and $\psi_1^{+}(T)\subset\psi_2^{+}(T)$ hold, consequently

$$bel_{++}(U,\mathcal{S}_1)(T) = \sup\psi_{+,1}(T) \leq \sup\psi_{+,2}(T) = bel_{++}(U,\mathcal{S}_2)(T), \tag{9.82}$$
$$bel^{++}(U,\mathcal{S}_2)(T) = \inf\psi_1^{+}(T) \geq \inf\psi_2^{+}(T) = bel^{++}(U,\mathcal{S}_2)(T), \tag{9.83}$$

so that (9.68) and (9.69) hold as well.

For $j = +$ or $j = ++$,

$$pl_j(U,\mathcal{S}_1)(T) = 1 - bel_j(U,\mathcal{S}_1)(S-T) \leq \tag{9.84}$$
$$\leq 1 - bel_j(U,\mathcal{S}_2)(S-T) = pl_j(U,\mathcal{S}_2)(T),$$

when applying (9.78) or (9.82) to the subset $S - T$ of S, and

$$pl^j(U,\mathcal{S}_1)(T) = 1 - bel^j(U,\mathcal{S}_1)(S-T) \leq \tag{9.85}$$
$$\leq 1 - bel^j(U,\mathcal{S}_2)(S-T) = pl^j(U,\mathcal{S}_2)(T),$$

applying (9.79) or (9.83) instead of (9.78) or (9.82). Hence, (9.70), (9.71), (9.72), and (9.73) are proved. If T is $\mathcal{S}_1$-regular, it is also $\mathcal{S}_2$-regular, i. e. $\mathcal{P}(T)\in\mathcal{S}_1\subset\mathcal{S}_2$, so that $bel_{U,\mathcal{S}_1}(T) = P^{\emptyset}(\{\omega\in\Omega : U(\omega)\in\mathcal{P}(T)\}) = bel_{U,\mathcal{S}_2}(T)$, if $S - T$ is $\mathcal{S}_1$-regular, then obviously $pl_{U,\mathcal{S}_1}(T) = pl_{U,\mathcal{S}_2}(T)$, hence, (9.74), (9.75), and Theorem 9.4 as a whole are proved. □

10 Belief and Plausibility Functions Defined by Boolean Combinations of Set–Valued Random Variables

As we remember, the role of one of the basic building stones in our definition of belief and plausibility functions over infinite sets S was played by a set-valued random variable U, defined on the abstract probability space $\langle \Omega, \mathcal{A}, P \rangle$ and taking its values in a measurable space $\langle \mathcal{P}(S), \mathcal{S} \rangle$ over the power-set $\mathcal{P}(S)$ of all subsets of S. Having at hand two or more such set-valued random variables, an immediate idea arises to define new set-valued random variables, applying boolean set-theoretical operations to the values of the original variables. Namely, let $\mathcal{U}$ be a nonempty set of random variables defined on $\langle \Omega, \mathcal{A}, P \rangle$ and taking their values in $\langle \mathcal{P}(S), \mathcal{S} \rangle$, let $U \in \mathcal{U}$. We may define set-valued mappings $\bigcap \mathcal{U}$, $\bigcup \mathcal{U}$ and $S - U$ setting, for each $\omega \in \Omega$,

$$\begin{aligned} \left(\bigcap \mathcal{U}\right)(\omega) &= \bigcap \{U(\omega) : \mathcal{U} \in \mathcal{U}\}, \\ \left(\bigcup \mathcal{U}\right)(\omega) &= \bigcup \{U(\omega) : \mathcal{U} \in \mathcal{U}\}, \\ (S - U)(\omega) &= S - U(\omega). \end{aligned} \tag{10.1}$$

Consequently, for each $\mathcal{A}_0 \subset \mathcal{P}(S)$, $\mathcal{A}_0 \in \mathcal{S}$,

$$\{\omega \in \Omega : (S - U)(\omega) \in \mathcal{A}_0\} = \{\omega \in \Omega : S - U(\omega) \in \mathcal{A}_0\}, \tag{10.2}$$

$$\{\omega \in \Omega : \left(\bigcap \mathcal{U}\right)(\omega) \in \mathcal{A}_0\} = \left\{\omega \in \Omega : \bigcap_{U \in \mathcal{U}} U(\omega) \in \mathcal{A}_0\right\}, \tag{10.3}$$

$$\{\omega \in \Omega : (\bigcup \mathcal{U})(\omega) \in \mathcal{A}_0\} \; = \; \left\{\omega \in \Omega : \bigcup_{U \in \mathcal{U}} U(\omega) \in \mathcal{A}_0\right\}, \qquad (10.4)$$

so that these subsets of Ω need not be, in general, in $\mathcal{A}$, hence, $\bigcap \mathcal{U}$, $\bigcup \mathcal{U}$ and $S - U$ need not be, in general, random variables over $\langle \Omega, \mathcal{A}, P\rangle$. For $\mathcal{U} = \{U_1, U_2\}$, ($\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$, resp.) we write $U_1 \cap U_2$ and $U_1 \cup U_2$ ($U_1 \cap U_2 \cap \cdots \cap U_n$, $U_1 \cup U_2 \cup \cdots \cup U_m, \bigcup_{i=1}^n U_i$, $\bigcap_{i=1}^n U_i$, resp.) instead of $\bigcap \mathcal{U}$ and $\bigcup \mathcal{U}$.

Definition 10.1. Random variable $U : \langle \Omega, \mathcal{A}, P\rangle \rightarrow \langle \mathcal{P}(S), \mathcal{S}\rangle$ is called *weakly consistent*, if the empty subset $\emptyset$ of S is $\mathcal{S}$-regular with respect to U and if $P(\{\omega \in \Omega : U(\omega) = \emptyset\})$ $= 0$. U is called *strictly consistent*, if

$$\bigcap \{A : A \subset S, \; \{A\} \in \mathcal{S}, \; P(\{\omega \in \Omega : U(\omega) = A\}) > 0\} \neq \emptyset, \qquad (10.5)$$

i.e. if there exists $s_0 \in S$ such that $s_0 \in A$ for each $A \subset S$ possessing the property that $P(\{\omega \in \Omega : U(\omega) = A\})$ is defined and positive. Let $\mathcal{U}$ be a nonempty system of random variables defined on $\langle \Omega, \mathcal{A}, P\rangle$ and taking their values in $\langle \mathcal{P}(S), \mathcal{S}\rangle$. Then random variables in $\mathcal{U}$ are called *mutually weakly consistent*, if $\emptyset$ is $\mathcal{S}$-regular with respect to $\bigcap \mathcal{U}$, i.e., if $\{\omega \in \Omega : \bigcap_{U \in \mathcal{U}} U(\omega) = \emptyset\} \in \mathcal{A}$, and the probability of this subset of Ω equals 0. Random variables in $\mathcal{U}$ are called *mutually strictly consistent*, if

$$\bigcap_{U \in \mathcal{U}} [\bigcap \{A : A \subset S, \{A\} \in \mathcal{S}, \; P(\{\omega \in \Omega : U(\omega) = A\}) > 0\}] \neq \emptyset, \qquad (10.6)$$

i.e., if there exists $s_0 \in S$ such that $s_0 \in A$ holds for each $A \subset S$, $\{A\} \in \mathcal{S}$, possessing the property that $P(\{\omega \in \Omega : U(\omega) = A\})$ is positive for at least one $U \in \mathcal{U}$. □

Definition 10.2. Let $\mathcal{U}$ be as in Definition 10.1. Random variables in $\mathcal{U}$ are called *mutually statistically independent*, if for each n finite, $n \leq \text{card}\,\mathcal{U}$, each $\{U_1, U_2, \ldots, U_n\} \in \mathcal{U}$, and each $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\} \subset \mathcal{S}$, the equality

$$P\left(\bigcap_{i=1}^n \{\omega \in \Omega : U_i(\omega) \in \mathcal{B}_i\}\right) = \prod_{i=1}^n P\left(\{\omega \in \Omega : U_i(\omega) \in \mathcal{B}_i\}\right) \qquad (10.7)$$

holds. □

In what follows, we shall investigate, how belief and plausibility functions defined by composed set-valued random variables can be expressed through belief and plausibility functions defined by the original set-valued random variables.

Theorem 10.1. Let $\mathcal{U}$ be a nonempty system of mutually strictly consistent and mutually statistically independent set-valued random variables defined on $\langle \Omega, \mathcal{A}, P\rangle$ and taking their values in $\langle \mathcal{P}(S), \mathcal{S}\rangle$, let $T \subset S$ be $\mathcal{S}$-regular with respect to each $U \in \mathcal{U}$ as well as with respect to $\bigcup \mathcal{U}_0$ for each finite subset $\mathcal{U}_0 \subset \mathcal{U}$, let each such $\bigcup \mathcal{U}_0$ be a random variable. Then, for each nonempty and finite $\mathcal{U}_0 \subset \mathcal{U}$,

$$bel_{\mathcal{U}_0}(T) = bel_{\bigcup_{i=1}^n U_i}(T) = \prod_{U \in \mathcal{U}_0} bel_U(T). \qquad (10.8)$$

□

Proof. Let $T \subset S$ satisfy the conditions of Theorem 10,1, let $\mathcal{U}_0 = \{U_1, U_2, \dots, U_n\} \subset \mathcal{U}$. Then

$$\begin{aligned} & bel_{\mathcal{U}_0}(T) = bel_{\bigcup_{i=1}^n U_i}(T) = \\ = \ & P\left(\{\omega \in \Omega : \emptyset \neq \textstyle\bigcup_{i=1}^n U_i(\omega) \subset T\} / \{\omega \in \Omega : \emptyset \neq \textstyle\bigcup_{i=1}^n U_i(\omega)\}\right). \end{aligned} \tag{10.9}$$

Here, for each $U \in \mathcal{U}$,

$$P\left(\{\omega \in \Omega : \textstyle\bigcup_{U \in \mathcal{U}} U(\omega) = \emptyset\}\right) \leq P\left(\{\omega \in \Omega : U(\omega) = \emptyset\}\right) = 0, \tag{10.10}$$

as in the opposite case

$$\emptyset \neq \{A : A \subset S, \{A\} \subset \mathcal{S}, \ P\left(\{\omega \in \Omega : U(\omega) = A\}\right) > 0\}, \tag{10.11}$$

but this contradicts (10.5). Hence, the conditioning event in (10.8) possesses the probability one and can be avoided from further considerations and computations. So,

$$\begin{aligned} bel_{\mathcal{U}_0}(T) &= P\left(\{\omega \in \Omega : \textstyle\bigcup_{i=1}^n U_i(\omega) \subset T\}\right) = \\ &= P\left(\textstyle\bigcap_{i=1}^n \{\omega \in \Omega : U_i(\omega) \subset T\}\right), \end{aligned} \tag{10.12}$$

as $\bigcup_{i=1}^n U_i(\omega) \subset T$ holds iff $U_i(\omega) \subset T$ holds for each $i \leq n$. Random variables in $\mathcal{U}$ are supposed to be mutually statistically independent, so that, taking $\mathcal{B}_i = \mathcal{P}(T)$ for each $i \leq n$ and applying Definition 10.2, we obtain that

$$\begin{aligned} & bel_{\bigcup \mathcal{U}_0}(T) = P\left(\textstyle\bigcup_{i=1}^n \{\omega \in \Omega : U_i(\omega) \subset T\}\right) = \\ = \ & P\left(\textstyle\bigcup_{i=1}^n \{\omega \in \Omega : U_i(\omega) \in \mathcal{P}(T)\}\right) = \\ = \ & \textstyle\prod_{i=1}^n P\left(\{\omega \in \Omega : U_i(\omega) \in \mathcal{P}(T)\} / \{\omega \in \Omega : U_i(\omega) \neq \emptyset\}\right) = \\ = \ & \textstyle\prod_{i=1}^n P\left(\{\omega \in \Omega : \emptyset \neq U_i(\omega) \subset T\} / \{\omega \in \Omega : U_i(\omega) \neq \emptyset\}\right) = \\ = \ & \textstyle\prod_{i=1}^n bel_{U_i}(T) = \prod_{U \in \mathcal{U}_0} bel_U(T), \end{aligned} \tag{10.13}$$

as T is supposed to be $\mathcal{S}$-regular for each $U \in \mathcal{U}$ and $P\left(\{\omega \in \Omega : U(\omega) \neq \emptyset\}\right) = 1$ follows from (10.10). The assertion is proved. □

Unfortunately, (10.8) does not hold for the approximation bel_+ of the belief function bel, as the following example demonstrates.

Let $\emptyset =\neq A_1, \ A_2 \subset S$ be two subsets of S such that $A_1 \neq A_1 \cup A_2, \ A_2 \neq A_1 \cup A_2 \neq S$. Let $\mathcal{S} = \{\emptyset^*, \{\emptyset\}, \{A_1 \cup A_2\}, \{\emptyset, \ A_1 \cup A_2\}, \mathcal{P}(S) = \{\emptyset, \ A_1 \cup A_2\}, \mathcal{P}(S) - \{A_1 \cup A_2\}, \mathcal{P}(S) - \{\emptyset\}, \mathcal{P}(S)\}$ be a subset of $\mathcal{P}(\mathcal{P}(S))$, here $\emptyset^*$ is the empty subset of $\mathcal{P}(S)$ and $\emptyset$ is the empty subset of S. As can be easily verified, for each $\mathcal{B}_1, \ \mathcal{B}_2 \in \mathcal{S}, \ \mathcal{P}(S) - \mathcal{B}_1$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ are in $\mathcal{S}$, so that $\mathcal{S}$ is a field and, due to its finiteness, also a σ-field in $\mathcal{P}(\mathcal{P}(S))$. Let $U_1, \ U_2$ be two mappings defined on the probability space $\langle \Omega, \mathcal{A}, P \rangle$, taking their values in $\mathcal{P}(S)$ and such that $U_i(\omega) = A_i$ for each $\omega \in \Omega$ and for both $i = 1, \ 2$. Then, for each $\mathcal{B} \in \mathcal{S}, \ \{\omega \in \Omega : U_i(\omega) \in \mathcal{B}\} = \Omega$, if $A_i \in \mathcal{B}$, and $\{\omega \in \Omega : U_i(\omega) \in \mathcal{B}\} = \emptyset$, if $A_i \notin \mathcal{B}$, here $\emptyset$ denotes the empty subset of Ω. As $\{\emptyset, \Omega\} \subset \mathcal{A}$, both $U_1, \ U_2$ are random variables.

Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{S}$. Then

$$\begin{aligned} & P(\{\omega \in \Omega : U_1(\omega) \in \mathcal{B}_1, U_2(\omega) \in \mathcal{B}_2\}) = 1 \\ = \ & P(\{\omega \in \Omega : U_1(\omega) \in \mathcal{B}_1\})\ P(\{\omega \in \Omega : U_2(\omega) \in \mathcal{B}_2\}) \end{aligned} \tag{10.14}$$

iff $A_1 \in \mathcal{B}_1$ and $A_2 \in \mathcal{B}_2$, otherwise

$$\begin{aligned} & P(\{\omega \in \Omega : U_1(\omega) \in \mathcal{B}_1, U_2(\omega) \in \mathcal{B}_2\}) = 0 \\ = \ & P(\{\omega \in \Omega : U_1(\omega) \in \mathcal{B}_1\})\ P(\{\omega \in \Omega : U_2(\omega) \in \mathcal{B}_2\}), \end{aligned} \tag{10.15}$$

so that the random variables U_1, U_2 are mutually statistically independent.

For the mapping $U_1 \cup U_2 : \Omega \to \mathcal{P}(S)$ we obviously have $(U_1 \cup U_2)(\omega) = U_1(\omega) \cup U_2(\omega) = A_1 \cup A_2$ for each $\omega \in \Omega$, so that $U_1 \cup U_2$ is a random variable taking $\langle \Omega, \mathcal{A}, P \rangle$ into $\langle \mathcal{P}(S), \mathcal{S} \rangle$.

According to the definition of bel_+, for both $i = 1, 2$,

$$\begin{aligned} & bel_+(U_i, \mathcal{S})(A_1 \cup A_2) = \\ = \ & \sup \{ P(\{\omega \in \Omega : U_i(\omega) \in \mathcal{B}\} / \{\omega \in \Omega : U_i(\omega) \neq \emptyset\}) \\ & \quad \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(A_1 \cup A_2) - \{\emptyset\}\}. \end{aligned} \tag{10.16}$$

As $A_1 \cup A_2 \neq S$, the only sets in $\mathcal{S}$, which are subsets of $\mathcal{P}(A_1 \cup A_2) - \{\emptyset\}$ are $\emptyset^*$ and $\{A_1 \cup A_2\}$. However,

$$P(\{\omega \in \Omega : U_i(\omega) \in \emptyset^*\} / \{\omega \in \Omega : U_i(\omega) \neq \emptyset\}) = 0 \tag{10.17}$$

(trivially), and

$$\begin{aligned} & P(\{\omega \in \Omega : U_i(\omega) \in \{A_1 \cup A_2\} / \ \{\omega \in \Omega : U_i(\omega) \neq \emptyset\}) = \\ = \ & P(\{\omega \in \Omega : U_i(\omega) = A_1 \cup A_2\}) = 0 \end{aligned} \tag{10.18}$$

for both $i = 1, 2$, so that

$$bel_+(U_1, \mathcal{S})(A_1 \cup A_2) = bel_+(U_2, \mathcal{S})(A_1 \cup A_2) = 0. \tag{10.19}$$

However, again by the definition of bel_+,

$$\begin{aligned} & bel_+(U_1 \cup U_2, \mathcal{S})(A_1 \cup A_2) = \\ = \ & \sup \{ P(\{\omega \in \Omega : (U_1 \cup U_2)(\omega) \in \mathcal{B}\} / \{\omega \in \Omega : (U_1 \cup U_2)(\omega) \neq \emptyset\}) : \\ & \quad \mathcal{B} \in \mathcal{S}, \mathcal{B} \subset \mathcal{P}(A_1 \cup A_2) - \{\emptyset\}\}. \end{aligned} \tag{10.20}$$

As above, the only subsets of $\mathcal{P}(A_1 \cup A_2) - \{\emptyset\}$, which are in $\mathcal{S}$, are $\emptyset^*$ and $\{A_1 \cup A_2\}$. Now,

$$\begin{aligned} & P(\{\omega \in \Omega : (U_1 \cup U_2)(\omega) \in \{A_1 \cup A_2\} / \ \{\omega \in \Omega : (U_1 \cup U_2)(\omega) \neq \emptyset\}) = \\ = \ & P(\{\omega \in \Omega : U_1(\omega) \cup U_2(\omega) = A_1 \cup A_2\}) = 1, \end{aligned} \tag{10.21}$$

so that $bel_+(U_1 \cup U_2, \mathcal{S})(A_1 \cup A_2) = 1 \neq bel_+(U_1, \mathcal{S})(A_1 \cup A_2) \cdot bel_+(U_2, \mathcal{S})(A_1 \cup A_2)$. Hence, (10.8) does not hold for bel_+.

An explicit and easy to process expression for $bel_{\bigcap\mathcal{U}}$ can be obtained when S and $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ are both finite sets, $\mathcal{S} = \mathcal{P}(\mathcal{P}(S))$, and random variables $U_1, U_2, \ldots, U_n$ are mutually statistically independent. Under these conditions, $P(\{\omega \in \Omega : U_i(\omega) = A\}) = P(\{\omega \in \Omega : U_i(\omega) \in \{A\}\})$ is defined for each $A \subset S$ and each $1 \le i \le n$. Moreover, for each $A_1, A_2, \ldots, A_n \subset S$,

$$\begin{aligned} & P(\{\omega \in \Omega : U_1(\omega) = A_1,\, U_2(\omega) = A_2, \ldots, U_n(\omega) = A_n\}) = \qquad (10.22) \\ = \ & P(\{\omega \in \Omega : U_1(\omega) \in \{A_1\},\, U_2(\omega) \in \{A_2\}, \ldots, U_n(\omega) \in \{A\}_n\}) = \\ = \ & P\left(\bigcap_{i=1}^n \{\omega \in \Omega : U_i(\omega) \in \{A_i\}\}\right) = \\ = \ & \prod_{i=1}^n P(\{\omega \in \Omega : U_i(\omega) \in \{A_i\}\}). \end{aligned}$$

Set, for each T, $\emptyset \ne T \subset S$, $\mathcal{P}^{\emptyset}(T) \subset (\mathcal{P}(S))^n$ in such a way that

$$\mathcal{P}^{\emptyset}(T) = \left\{\langle C_1, C_2, \ldots, C_n\rangle : C_i \subset S,\ i = 1, 2, \ldots, n,\ \emptyset \ne \bigcap_{i=1}^n C_i \subset T\right\}, \qquad (10.23)$$

then, by the definition of bel and due to the fact that S and $\mathcal{P}(S)$ are finite,

$$\begin{aligned} bel_{\bigcap_{i=1}^n U_i}(T) = & \frac{P\left(\{\omega \in \Omega : \left(\bigcap_{i=1}^n U_i\right)(\omega) \in \mathcal{P}(T) - \{\emptyset\}\}\right)}{P\left(\{\omega \in \Omega : \left(\bigcap_{i=1}^n U_i\right)(\omega) \in \mathcal{P}(S) - \{\emptyset\}\}\right)} = \qquad (10.24) \\ = \ & \frac{\sum_{A \in \mathcal{P}(T) - \{\emptyset\}} P\left(\{\omega \in \Omega : \left(\bigcap_{i=1}^n\right)(\omega) = A\}\right)}{\sum_{A \in \mathcal{P}(S) - \{\emptyset\}} P\left(\{\omega \in \Omega : \left(\bigcap_{i=1}^n\right)(\omega) = A\}\right)} = \\ = \ & \frac{\sum_{\langle C_1, \ldots, C_n\rangle \in \mathcal{P}^{\emptyset}(T)} \prod_{i=1}^n P(\{\omega \in \Omega : U_i(\omega) = C_i\})}{\sum_{\langle C_1, \ldots, C_n\rangle \in \mathcal{P}^{\emptyset}(S)} \prod_{i=1}^n P(\{\omega \in \Omega : U_i(\omega) = C_i\})} = \\ = \ & \frac{\sum_{\langle C_1, \ldots, C_n\rangle \in \mathcal{P}^{\emptyset}(T)} \prod_{i=1}^n m_i(C_i)}{\sum_{\langle C_1, \ldots, C_n\rangle \in \mathcal{P}^{\emptyset}(S)} \prod_{i=1}^n m_i(C_i)}, \end{aligned}$$

where m_i is the basic probability assignment defined by, or related to, the random variable U_i, i.e., $m_i(A) = P(\{\omega \in \Omega : U_i(\omega) = A\})$ for each $A \subset S$ and each $1 \le i \le n$. If random variables $U_1, U_2, \ldots, U_n$ are mutually weakly consistent, then (10.24) obviously reduces to

$$bel_{\bigcap_{i=1}^n U_i}(T) = \sum_{\langle C_1, \ldots, C_n\rangle \in \mathcal{P}^{\emptyset}(T)} \prod_{i=1}^n m_i(C_i). \qquad (10.25)$$

Both the formulas (10.24) and (10.25) are nothing else than the well-known and above also discussed and analyzed Dempster combination rule with the only difference concerning the notation of the resulting belief functions. Hence, under the conditions imposed above to random variables $U_1, U_2, \ldots, U_n$, the relation

$$bel_{U_1} \oplus bel_{U_2} \oplus \cdots \oplus bel_{U_n} = bel_{\bigcap_{i=1}^n U_i} \qquad (10.26)$$

holds. In other words said, the belief function, resulting when applying Dempster combination rule to belief functions defined by statistically independent set-valued random variables, is also defined by the intersection of the particular random variables. The dual relation between $bel_{\bigcup_{i=1}^n} U_i$ and the dual Dempster combination rule $\tilde{\oplus}$ was investigated in Chapter 6 above or in [28]. As the conditional belief functions are defined as a particular case of Dempster combination rule, they can be also obtained when applying boolean operations to set-valued random variables. Consequently, also the intuitive interpretations of belief functions corresponding to various boolean compositions of particular set-valued random variables can be identified with those considered when analyzing the Dempster combination rule and its dual variant.

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