Preconditioners for Contact Problems in Elasticity

Nedoma, Jiří
1998

INSTITUTE OF COMPUTER SCIENCE

ACADEMY OF SCIENCES OF THE CZECH REPUBLIC

Preconditioners for contact problems in elasticity

Jiří Nedoma, Zdeněk Kestřánek

Technical report No. 739

February 1998

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic
phone: 66053911 fax: (+4202) 8585789
e-mail: zdenda@uivt.cas.cz

INSTITUTE OF COMPUTER SCIENCE

ACADEMY OF SCIENCES OF THE CZECH REPUBLIC

Preconditioners for contact problems in elasticity

Jiří Nedoma, Zdeněk Kestřánek [1]

Technical report No. 739
February 1998

Abstract

In this report we suggest a modification of preconditioned algorithms for the quadratic programming problem which arises from the discretization of contact problem. This approach takes fully the advantage the property of corresponding projection so that this projection itself is not affected by the preconditioning matrix. The performance of the proposed method is shown on three geomechanical models, the behaviour of which can be used in further simulations of high level radioactive waste repositories. Firstly, the mathematical-geodynamical model of Himalaya is analysed, as it may give the initial information for other models, i.e. service line tunnels for hydroelectric power stations situated in rock massifs with active deep faults and the model of open pit cooper mine.

Keywords
Geomechanics, FEM, Mathematical programming

[1]Supported by Grant COPERNICUS CP940820 - HIPERGEOS of the European Comission and Grant of Ministry of Education, Youth and Sport of the Czech Republic OK-158(1996)

1 Introduction

We show the behaviour of some preconditioners on geodynamical models which can be simulated as contact elasticity problems. To ensure the unique solution for models, the boundary conditions and external loads have to be imposed properly. Some conditions concerning this requirement are given in Section 2. In Section 3, a solution algorithm is mentioned. The stress is laid on the preconditioning aspects. Section 4 presents the numerical results. The description of models, statistics, pictures and graphs are shown. The creation and convergence properties of a preconditioning matrix are quite surprising on contrary to the initial expectations.

2 Problem definition

Let Ω and $\partial\Omega$ denote the region and the boundary of the region, respectively. We suppose that the region consists of S bodies, $\Omega = \bigcup_{k=1}^{S} \Omega^k$. Let

$$\Gamma = \Gamma_u \cup \Gamma_\tau \cup \Gamma_c^{kl} \cup \Gamma_0 \ , \ 1 \le k < l \le S.$$

Let $v_n^k = v_i^k n_i, \ v_n^l = v_i^l n_i \ v_t^k = v_i^k t_i, \ v_t^l = v_i^l t_i$, where n_i are the components of unit outward normal to Γ_c^k, $n_i t_i = 0$. We introduce the set of virtual displacements by

$$V \equiv \{\mathbf{v} \in [H^1(\Omega)]^2 | \, \mathbf{v} = \mathbf{u}_0 \quad \text{on } \Gamma_u, \quad v_n = 0 \quad \text{on } \Gamma_0 \}, \tag{2.1}$$

where $\mathbf{u}_0 \in [H^1(\Omega)]^2$, and the set of all admissible displacements by

$$K \equiv \{\mathbf{v} \in V | \, v_n^k - v_n^l \le 0 \quad \text{on } \Gamma_c^{kl} \}. \tag{2.2}$$

For simplicity we assume $\mathbf{u}_0 \equiv 0$ on Γ_u. The transformation $\mathbf{v} = \mathbf{w} + \mathbf{u}_0$ is used for the general case. Let the potential energy functional be of the following form

$$\mathcal{L}(\mathbf{v}) \ = \ \mathcal{L}_0(\mathbf{v}) + j(\mathbf{v}) \ , \tag{2.3}$$

where

$$\mathcal{L}_0(\mathbf{v}) \ = \ \frac{1}{2} A(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) \ , \tag{2.4}$$

$$A(\mathbf{u},\mathbf{v}) = \int_\Omega c_{ijkm} e_{ij}(\mathbf{u}) e_{km}(\mathbf{v}) d\mathbf{x} \ , \tag{2.5}$$

$$L(\mathbf{v}) = \int_\Omega F_i v_i d\mathbf{x} + \int_{\Gamma_\tau} P_i v_i d\mathbf{s} \ , \tag{2.6}$$

$$j(\mathbf{v}) = \int_{\Gamma_c^{kl}} g^{kl} |v_t^k - v_t^l| d\mathbf{s} \ \text{ and} \tag{2.7}$$

$$\mathbf{F} \in [L^2(\Omega)]^2, \ \mathbf{P} \in [L^2(\Gamma_\tau)]^2, \ g^{kl} \in [L^\infty(\Gamma_c^{kl})]^2.$$

We suppose that there exists a constant $c_0 > 0$ such that

$$c_{ijkm}(x)e_{ij}e_{km} \geq c_0 e_{ij}e_{ij} \tag{2.8}$$

is valid for all sym. matrices e_{ij} and almost everywhere in Ω. Furthermore, let

$$R = \{\mathbf{v} \in V \mid \exists l \; 1 \leq l \leq S \; e_{ij}(\mathbf{v}) = 0 \text{ on } \Omega^l \}. \tag{2.9}$$

We solve the problem: find $\mathbf{u} \in K$ such that

$$\mathcal{L}(\mathbf{u}) \leq \mathcal{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K \; . \tag{2.10}$$

The necessary condition for the solution of (2.10) is given in the following theorem.

THEOREM 2.1. Let the solution (2.10) exist. Then $-L(\mathbf{z}) + j(\mathbf{z}) \geq 0 \; \forall \; \mathbf{z} \in K \cap R$.
Proof: see e.g. [11]. □

The next theorem [5] gives a sufficient condition for (2.10).
THEOREM 2.2. Let $-L(\mathbf{z}) + j(\mathbf{z}) > 0 \quad \forall \; \mathbf{z} \in K \cap R - \{0\}$. Then there exists a solution $\mathbf{u}$ of (2.10). If $\tilde{\mathbf{u}}$ is another solution of (2.10), then $\mathbf{u} - \tilde{\mathbf{u}} \in R$.

The last theorem shows the condition which ensures the uniqueness of the solution.
THEOREM 2.3. Let $|L(\mathbf{z})| > j(\mathbf{z}) \quad \forall \; \mathbf{z} \in V \cap R - \{0\}$. Then there exists at most one solution of (2.10).
Proof: We suppose that two solutions $\mathbf{u}_1, \; \mathbf{u}_2$ exist. Then we can write

$$A(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) - L(\mathbf{u}_2 - \mathbf{u}_1) + j(\mathbf{u}_2) - j(\mathbf{u}_1) \;\; \geq \;\; 0 \; , \tag{2.11}$$

$$A(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - L(\mathbf{u}_1 - \mathbf{u}_2) + j(\mathbf{u}_1) - j(\mathbf{u}_2) \;\; \geq \;\; 0 \; . \tag{2.12}$$

From (2.11) and (2.12) it follows $A(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \;\leq\; 0$ therefore using (2.8), $\mathbf{u}_1 - \mathbf{u}_2 \;\in\; R$. Let $\mathbf{u}_2 = \mathbf{u}_1 + \mathbf{z}$ for some $\mathbf{z} \in R$. From the equality

$$\mathcal{L}_0(\mathbf{u}_2 - \mathbf{z}) \; = \mathcal{L}_0(\mathbf{u}_1) \; = \mathcal{L}_0(\mathbf{u}_1 + \mathbf{z}) \tag{2.13}$$

and from the convexity of $j(\mathbf{v})$, we deduce

$$\exists \; \mathbf{z} \in V \cap R \;\; : \quad |L(\mathbf{z})| \leq j(\mathbf{z}) \, , \tag{2.14}$$

which is the contradiction to the assumption. □

3 Algorithms

3.1 Solution strategy

In the discrete form, the problem (2.10) leads to the problem :
Find $x \in K_d$ such that

$$\tilde{f}(x) \leq \tilde{f}(y) \quad \forall \, y \in K_d, \tag{3.1}$$

where

$$\tilde{f}(y) = \frac{1}{2} y^T C y - y^T d + \sup_{|\mu_i| \le 1} (y^T G^T \mu),$$

$$K_d = \{ y \in R^N | Ay \le 0 \}.$$

The global stiffness matrix C is of the type $N \times N$, block diagonal. Every block of it is sparse, symmetric, positive semidefinite matrix and corresponds to just one body of the model. In the coercive case C is positive definite, in the semi-coercive case [6, 11], it is positive semidefinite. The constraint matrix A is of the type $M \times N$, $M \ll N$; we assume its rows to be linearly independent. The friction matrix G is of type $P \times N$.

THEOREM 3.1.1. Let $z^T d \le 0 \quad \forall z \in Ker\ C - \{0\},\ Az \le 0$. Then there exists a solution of the problem (3.1). If $\tilde{x}$ is another solution of (3.1), then $\tilde{x} = x + z$ for some $z \in Ker\ C$.
Proof: see [5]. □

Since the functional $\tilde{f}(y)$ is nondifferentiable, it is convenient to transform the problem (3.1) to a saddle point problem. By these means, the problem is in certain sense "linearized". Two saddle point formulations are available. We will use the first one in this report only. The latter can lead to more optimal method in certain cases [9, 10], however, it is not very robust.
Let us define

$$\bar{L} = \{ \mu \in R^P \,|\, |\mu_i| \le 1,\ i = 1, \ldots, P \}$$

and

$$\bar{f}(y, \mu) = \frac{1}{2} y^T C y - y^T d + y^T G^T \mu.$$

Then we consider the problem:
Find $(x, \lambda) \in K_d \times \bar{L}$ such that

$$\bar{f}(x, \mu) \le \bar{f}(x, \lambda) \le \bar{f}(y, \lambda) \quad \forall\, (y, \mu) \in K_d \times \bar{L}. \tag{3.2}$$

The Uzawa algorithm [3, 4, 8] is used for (3.2).

3.2 The Preconditioning

The minimization part of the Uzawa algorithm is the most expensive step. We use, similarly to [1, 7], the Preconditioned Conjugate Projected Gradient Method for solving this part. We assume the form $W = EE^T$, $C = W + R$, E lower triangular, for the preconditioning matrix W. A Diagonal, SOR and incomplete factorization preconditioning matrices are used. Although in essence our models do not seem to be very different, an interesting behaviour occurs in the incomplete factorization. For the first model, the incomplete factorization is not successfull, even not the version with the diagonal modification [7]. We have to use a similar idea to [2] before calculating corresponding square roots to overcome this problem. Moreover, the factorization is successfull for the third model which is created, unlike the first one, as (weakly) semi-coercive. The problem of the efficient calculation of projected gradient is solved in previous report [7]. We mention the final version of the algorithm only.

$SUBROUTINE\ PCG-2(J', x, E^T, f')$
$r^0 = f'$ { from previous iteration }
$g^0 = (I - P_{J'})E^{-1}(I - P_{J'})r^0$
$v^0 = -(I - P_{J'})E^{-T}g^0$

for $k = 0, 1, \ldots$

$\alpha^k = (g^k, g^k)/(v^k, Cv^k)$
$\overline{\alpha}^k = \min_{\mathcal{M}} -\frac{(a_i, x^k)}{(a_i, v^k)}$
$IF\ (\overline{\alpha}^k < \alpha^k)\ THEN$
$x = x^k + \overline{\alpha}^k v^k$
$f' = r^k + \overline{\alpha}^k C v^k$ { and return }
$ELSE$
$x^{k+1} = x^k + \alpha^k v^k$
$r^{k+1} = r^k + \alpha^k C v^k$
$g^{k+1} = (I - P_{J'})E^{-1}(I - P_{J'})r^{k+1}$
$\beta^{k+1} = (g^{k+1}, g^{k+1})/(g^k, g^k)$
$v^{k+1} = -(I - P_{J'})E^{-T}g^{k+1} + \beta^{k+1}v^k$
$ENDIF$

Comparing with original algorithms [7], we have to calculate the projection three times during one iteration. This projection is, however, much simpler to calculate than the projection appearing in those versions. It is obvious, that the version $PCG-2$ is much more efficient concerning time aspect.

4 Results

Three models are analyzed in this section: The Subduction of Himalaya Plate, the Giri Hydroelectric Project (service line tunnels for hydroelectric power stations) and the model of cooper mine. The location of the last two models is determined by vertex 98 (Fig. 4.1) in the first model. The resulting displacement in this vertex within certain time interval (we use 5 years) may serve as the boundary condition for the tunnel and mine models. We use the Digital Visual Fortran 5.0 compiler for Windows95/NT platform.

4.1 Subduction of Himalaya Plate

This model corresponds to Figures 4.1-4.10.

Material parameters: 27 regions with different material parameters.

Boundary conditions: Zero Dirichlet condition on all sides except the side $55-96$ and the top side. Prescribed displacements on the side $96-55$ which correspond to the motion of the litospheric plate ($0.3e4[m]$) in given time interval ($0.6e5$ years). Contact boundaries: $98-17-14$ and $55-17$.

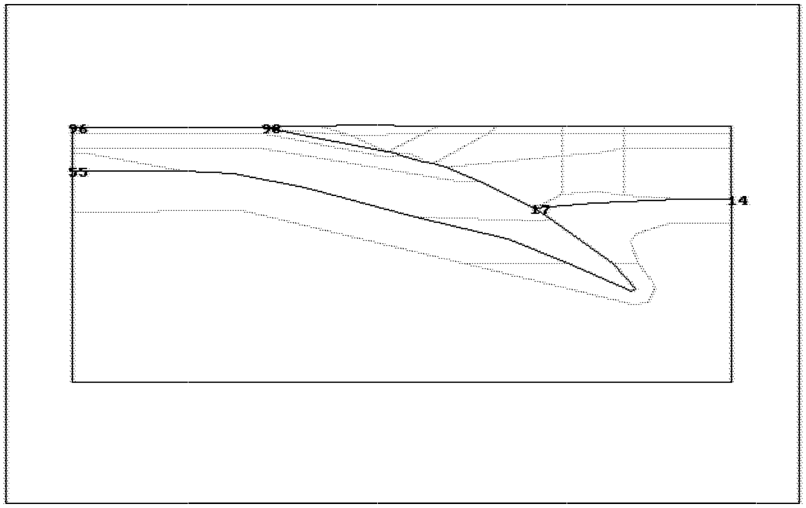

Figure 4.1: Subduction of Himalaya plate - model

Discretization statistics: 487 nodes, 774 elements, 922 unknowns, 6 119 nonzero entries in the stiffness matrix, 216 nonzero entries in the constraint matrix.

Convergence statistics: 916 iterations of the Uzawa algorithm, one CG iteration (in CPU seconds): 4.44 (No P.), 4.51 (Diag.), 3.39 (SOR), 3.68 (ILL).

4.2 Giri Hydroelectric Project

This model corresponds to Figures 4.11-4.15.

Material parameters: 2 regions with different material parameters.

Boundary conditions: Prescribed displacements on vertical sides ($8.0e{-}2[m]$, $0.0e+0[m]$). Pressure $7.5e+6[Pa]$ on horizontal sides, caused by weight of the rock cover. Contact boundaries: $14-2$ and $7-11$.

Discretization statistics: 409 nodes, 677 elements, 738 unknowns, 4 983 nonzero entries in the stiffness matrix, 56 nonzero entries in the constraint matrix.

Convergence statistics: 382 iterations of the Uzawa algorithm, one CG iteration (in CPU seconds): 1.76 (No P.), 2.14 (Diag.), 1.54 (SOR), 1.42 (ILL).

4.3 Cooper Mine

This model corresponds to Figures 4.16-4.18.

Material parameters: 2 regions with different material parameters.

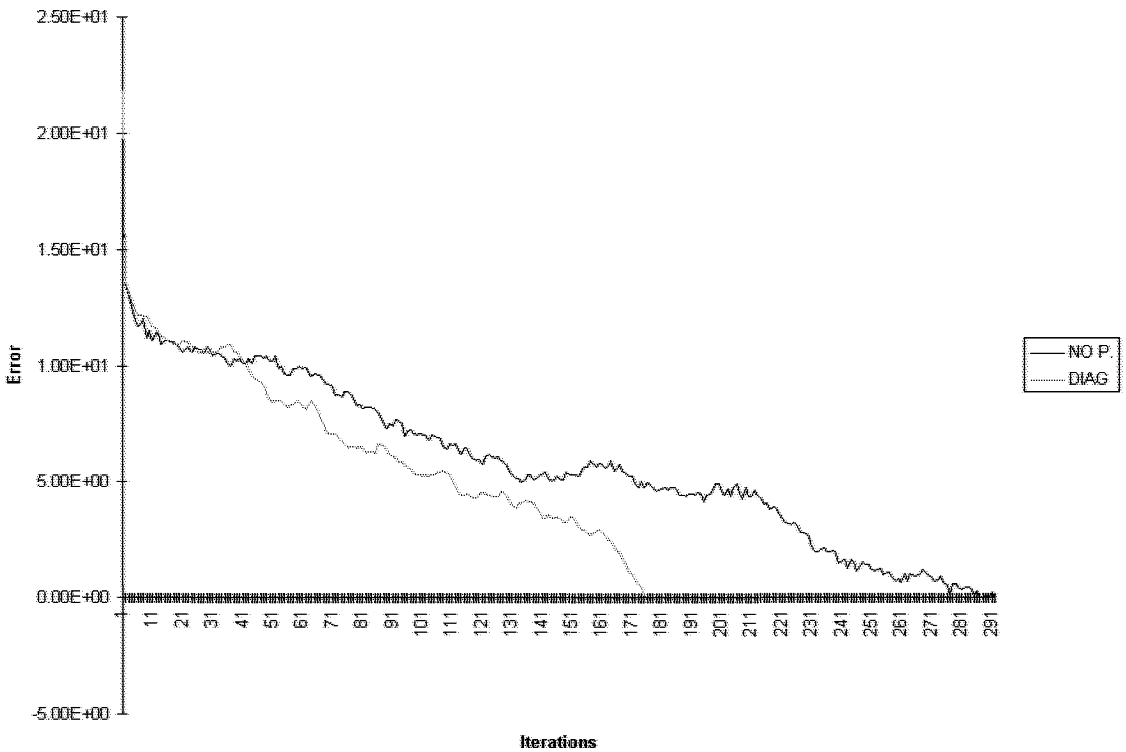

Figure 4.2: Subduction of Himalaya plate - diagonal p. convergence

Boundary conditions: Prescribed displacements ($8.0e-2[m]$, $0.0e+0[m]$) on the side $14-16$. Pressure $3.34e+4[Pa]$ on top sides ($14-13$ and $4-3$). Contact boundary: $16-9$.

Discretization statistics: 410 nodes, 657 elements, 706 unknowns, 4 661 nonzero entries in the stiffness matrix, 82 nonzero entries in the constraint matrix.

Convergence statistics: 4 627 iterations of the Uzawa algorithm, one CG iteration (in CPU seconds): 1.32 (No P.), 1.64 (Diag.), 1.05 (SOR), 0.93 (ILL).

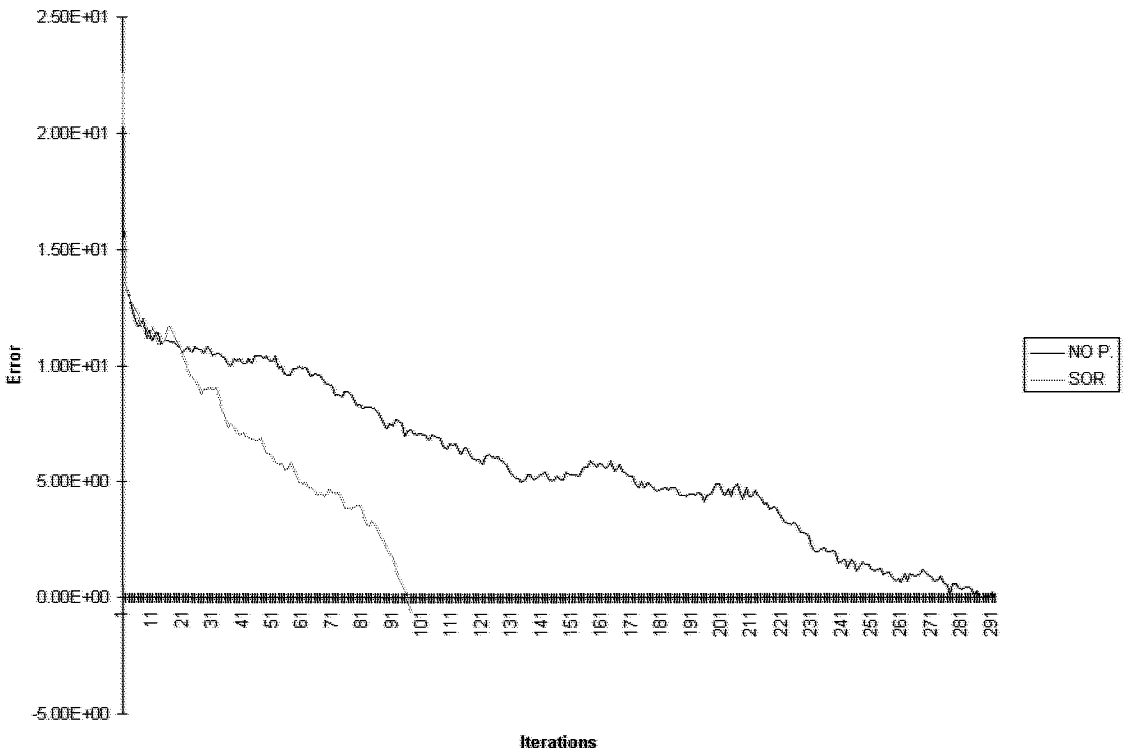

Figure 4.3: Subduction of Himalaya plate - SOR p. convergence

Figure 4.4: Subduction of Himalaya plate - ILL p. convergence

Figure 4.5: Subduction of Himalaya plate - deformations

Figure 4.6: Subduction of Himalaya plate - deformations, detail

Figure 4.7: Subduction of Himalaya plate - horizontal stresss

Figure 4.8: Subduction of Himalaya plate - vertical stresses

Figure 4.9: Subd. of Himalaya plate - Displacements difference on contact b.

Figure 4.10: Subd. of Himalaya plate - Stresses on contact b.

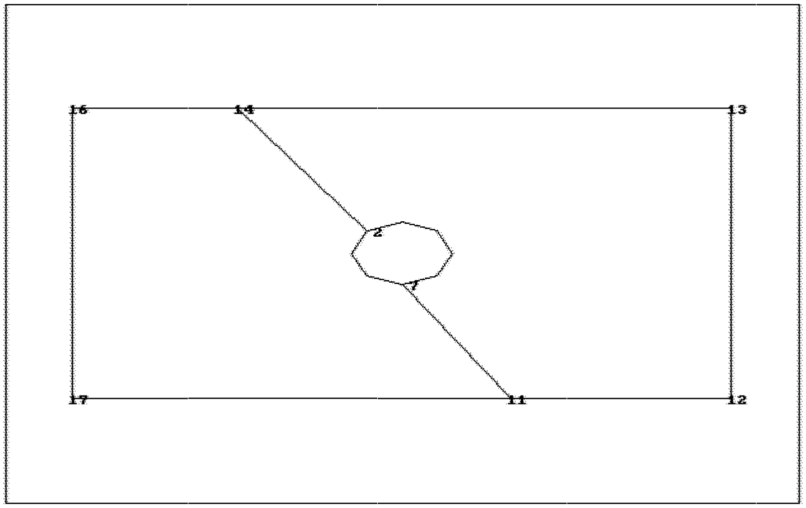

Figure 4.11: Giri Hydroelectric Project - model

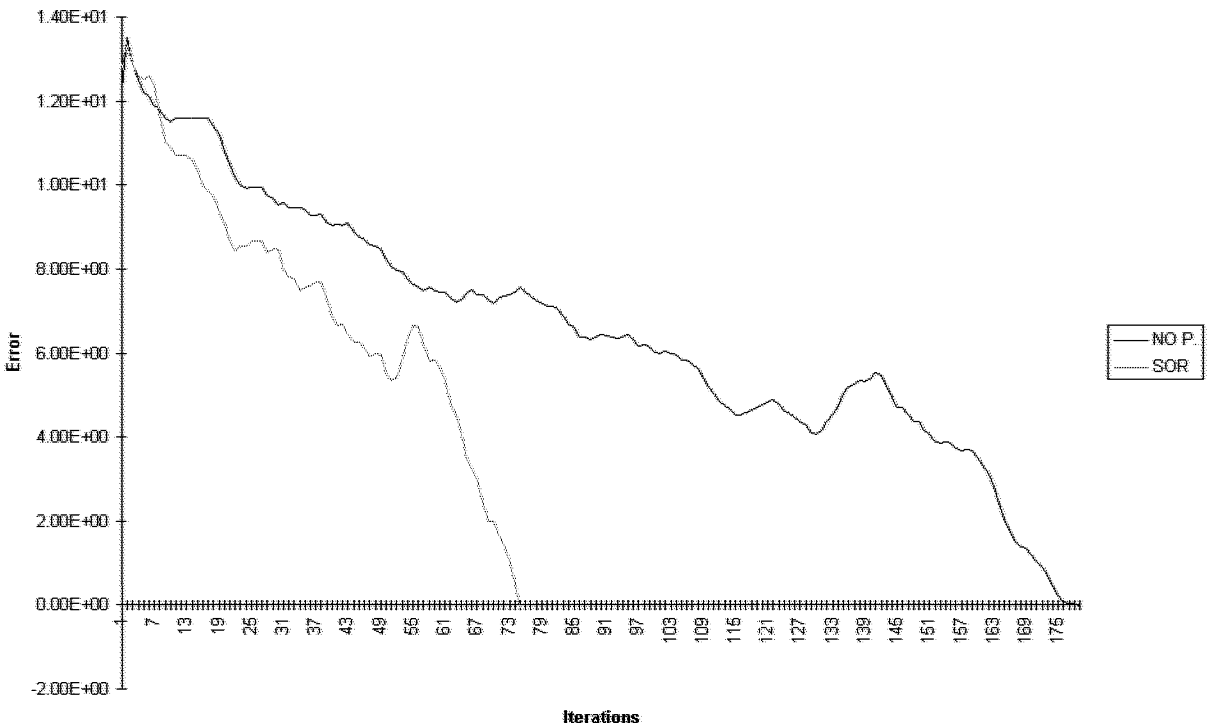

Figure 4.12: Giri Hydroelectric Project - SOR p. convergence

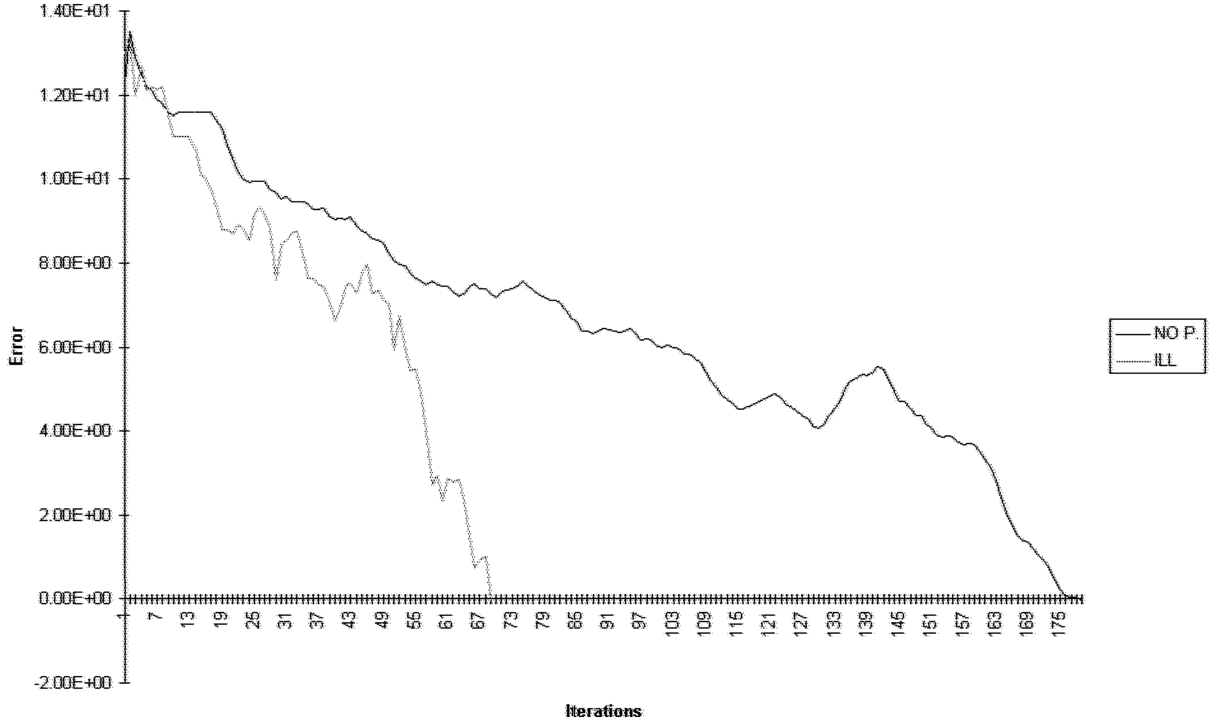

Figure 4.13: Giri Hydroelectric Project - ILL p. convergence

Figure 4.14: Giri Hydroelectric Project - horizontal stresses

Figure 4.15: Giri Hydroelectric Project - vertical stresses

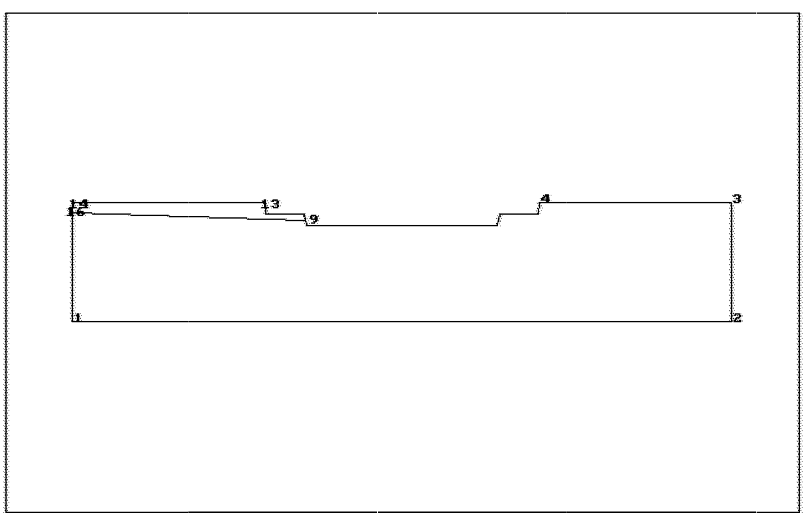

Figure 4.16: Cooper mine - model

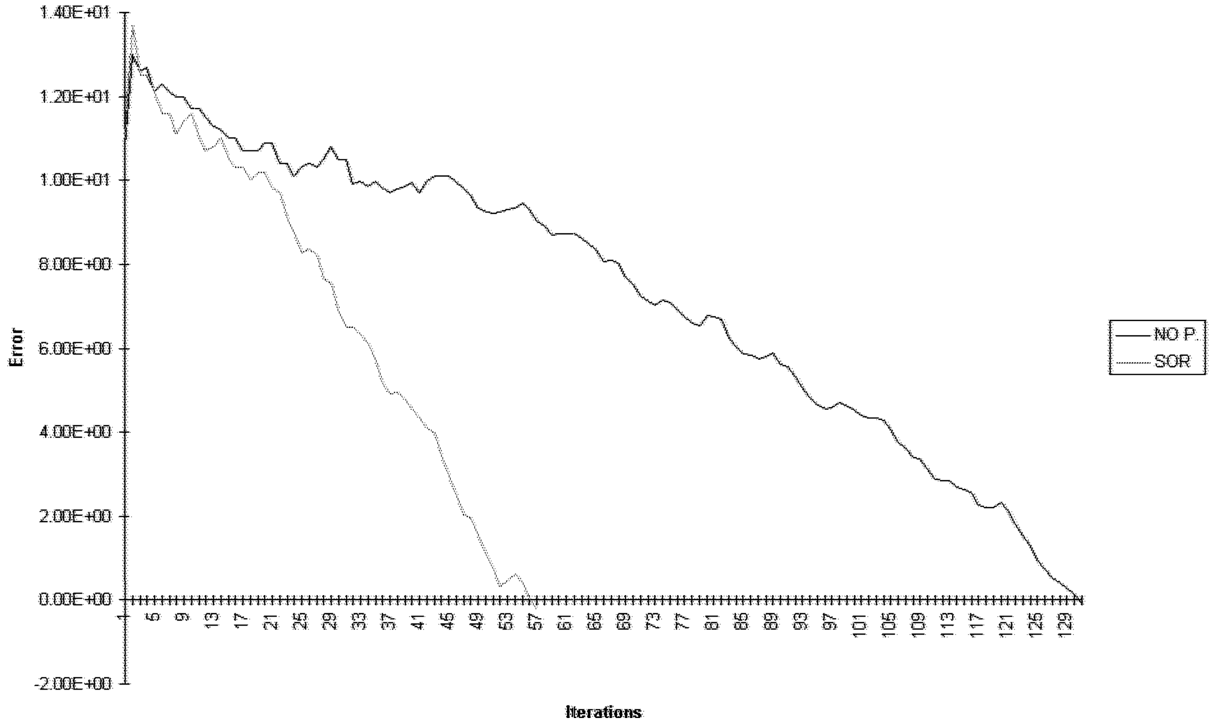

Figure 4.17: Cooper mine - SOR p. convergence

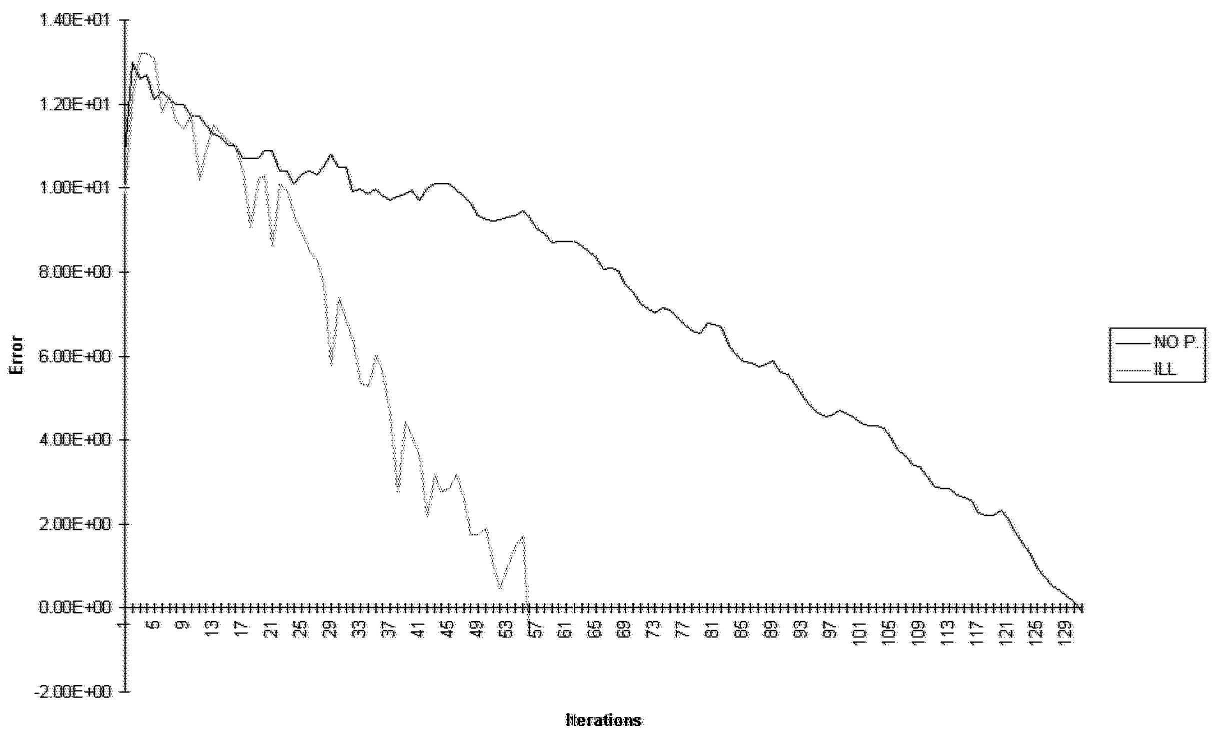

Figure 4.18: Cooper mine - ILL p. convergence

Bibliography

[1] O. Axelsson, V.A. Barker, *Finite Element Solution of Boundary Value Problems. Theory and Computation* (Academic Press, New York, London, 1984).

[2] J. R. Bunch, B.N. Parlett, Direct Methods for Solving Symmetric Indefinite Systems of Linear Equations, *SIAM J.Num.Anal.* 8 (1971) 639-655.

[3] J. Horák, H. Netuka, Numerical realization of contact problem with friction - semicoercive case, in: *Proc.Int.Conf. Mathematical Methods in Engineering* (Czech.Sci.Techn.Soc. ŠKODA Concern, Plzeň, 1991) 147-152.

[4] I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek, *Solution of variational inequalities in mechanics* (Alfa, Bratislava, 1982).

[5] D. Janovská, *Numerical solution of contact problems* (Dissertation, MFF UK, Praha, 1980).

[6] Z. Kestřánek, *Numerical Analysis of the Contact Problem. Comparison of Methods for Finding the Approximate Solution* (Techn. Rep. V-648, ICS AS CR, Praha, 1995).

[7] Z. Kestřánek, J. Nedoma, *The Conjugate Projected Gradient Method - Numerical Tests and Results* (Techn. Rep. V-677, ICS AS CR, Praha, 1996).

[8] Z. Kestřánek and J. Nedoma, Numerical simulation of a bridge on a non-stable slope. Comparison of results based on contact problem without and with friction, in: Z.Rakowski, Ed., *Proc.Int.Conf. Geomechanics'96* (A.A.Balkema, Rotterdam, 1997) 169-174.

[9] Z. Kestřánek, Numerical Modelling of Some Particular Cases in Contact Problem, submitted to M. Feistauer, Ed., *Proc.Int.Conf. NMICM'97* (Praha, 1997).

[10] Z. Kestřánek, J. Nedoma, *FEC - A code for contact problems in thermoelasticity with friction* (Technical Report V-740, ICS AS CR, Prague, 1998) 12p.

[11] J. Nedoma, Finite element analysis of contact problems in thermoelasticity. The semi-coercive case, *J.Comp.Appl.Math.* 50 (1994) 411-423.