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INSTITUTE OF COMPUTER SCIENCE

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Spectral Radius of Operators on l_∞

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Abstract

We study n by n matrices considered as linear operators on C^n equipped with the l^∞ norm $|x| = \max |x_j|$. A characterization of matrices A of norm $|A| = 1$ and spectral radius $r(A) = 1$ is given. The equality $|A| = r(A)$ is closely related to the combinatorial structure of the matrix A. The investigation of the combinatorial structure of these matrices is based on a new approach to the study of oriented graphs the principles of which are explained in detail.

Keywords

The present survey is concerned with contractions (linear operators of norm not exceeding one) on n-dimensional l^∞-spaces. The main problem is to characterize among all contractions those matrices whose spectral radius equals one. The description is given in terms of two different kinds of characteristics: one is concerned with the combinatorial structure, the other with properties of matrix character the interplay of these two characteristics will be examined in in detail.

In this section we intend to present the classical results on spectral radii for operators on C^n equipped with the l_∞ norm. If $x \in C^n$, $x = (x_1, \ldots, x_n)^T$ then $|x| = \max_i |x_i|$. A linear operator A on this space is represented by an n by n matrix (a_{ik}) such that

$$(Ax)_i = \sum_k a_{ik} x_k.$$

Its norm equals

$$|A| = \max_i \sum_k |a_{ik}|$$

We shall see that the combinatorial pattern of zeros and nonzeros in a matrix A plays an important role if A represents a linear operator on l_∞.

A matrix of type (m, n) is a complex valued mapping defined on the cartesian product $M \times N$ where M and N are two sets of indices of cardinalities m and n respectively. A square matrix A defined on $N \times N$ is said to be decomposable if there exists a nonvoid $H \subset N$ different from N such that $a_{ik} = 0$ for $i \in H$ and $k \in N \backslash H$. In other words, A is decomposable iff there exists a permutation matrix P such that $P^T AP$ has a nontrivial block decomposition of the form

$$P^T AP = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

with square blocks on the diagonal. A matrix is said to be indecomposable if it is not decomposable.

A linear operator on a Banach space is said to be a contraction if its norm does not exceed 1. In the following three propositions we shall consider contractions on an n-dimensional l_∞ space; they will be represented by matrices $A = (a_{ik})$ defined on $N \times N$ where $N = \{1, 2, \ldots, n\}$.

It is obvious that the study of matrices A satisfying $|A| = r(A) = 1$ may be reduced to the particular case of contractions with 1 in the spectrum. The statement of the results simplify confiderably and may easily be modified to cover the general case.

(1.1) *Suppose A is an indecomposable contraction and $Ax = x$, $x \neq 0$; then $|x_1| = \ldots = |x_n|$ and $\sum_j |a_{ij}| = 1$, for all i.*

Proof. Set $M = \{i; |x_i| = |x|\}$. If $i \in M$ $k \notin M$ and $a_{ik} \neq 0$ then

$$\begin{aligned} |x_i| &\leq \sum_j |a_{ij}||x_j| = \sum_{j \neq k} |a_{ij}||x_j| + |a_{ik}||x_k| \\ &\leq \sum_{j \neq k} |a_{ij}||x_i| + |a_{ik}||x_k| < \sum_j |a_{ij}||x_i| \leq |x_i| \end{aligned}$$

This contradiction shows that either $M = N$ or $a_{ik} = 0$ for $i \in M$ and $k \notin M$. The second possibility is a contradiction with the indecomposability of A. It follows that $|x_i| = |x|$ for all i; this implies, in its turn, that $\sum\limits_j |a_{pj}| = 1$ for all p. $\square$

As an immediate consequence we obtain a classical result of O. Taussky.

(1.2) *Suppose A is indecomposable and diagonally dominant in the following sense:*

$$|a_{ii}| \geq \sum_{k \neq i} |a_{ik}| \quad \textit{for all} \;\; i$$

and the inequality is strict for at least one index i. Then A is invertible.

Proof. It follows from the indecomposability of A that $a_{ii} \neq 0$ for all i. Accordingly, the matrix $D = \operatorname{diag} a_{ii}$ is invertible. Represent A in the form $A = D(I - B)$; then B is indecomposable and $\sum\limits_k |b_{ik}| \leq 1$ for all i, the inequality being strict for at least one i. If $Ax = 0$ for some $x \neq 0$ then $Bx = x$. This is impossible by (1.1). $\square$

Another consequence of (1.1) is the following result of L.Collatz.

(1.3) *Let A be indecomposable and suppose that $\sum\limits_k |a_{ik}| \leq 1$ for all i; if $\sum\limits_k |a_{ik}| < 1$ for at least one index i, then $r(A) < 1$.*

Proof. Suppose $Ax = \lambda x$ some $x \neq 0$. Let i be an index for which $|x_i| \geq |x_j|$ for all j. Then

$$|\lambda x_i| \leq \sum_j |a_{ij} x_j| \leq \sum_j |a_{ij}||x_i| \leq |x_i|$$

so that $|\lambda| \leq 1$. If $|\lambda| = 1$, the matrix $B = \lambda^{-1} A$ is indecomposable, $\sum\limits_k |b_{ik}| \leq 1$ for all i and $Bx = x$. It follows from (1.1) that $\sum\limits_k |a_{ik}| = \sum\limits_k |b_{ik}| = 1$ for all i, a contradiction. $\square$

Many of the problems to be treated in this chapter are of a combinatorial character. A natural tool for the study of these problems is the theory of graphs, in particular of oriented graphs. Since we shall only be dealing with oriented graphs in the sequel we shall use the term graph in the more restricted meaning of oriented graph.

The approach to graph theory to be used in the following investigations is based on an idea pointed out in [6], [7], [8]. The idea consists in viewing a graph as a mapping whose range is contained in its domain of definition: the ensuing possibility to use iterates of the mapping brings not only a considerible technical simplification but also deeper insight into the connections between iterates of a matrix and of the corresponding graph.

A graph φ on a set N is an additive mapping of $\exp N$ into $\exp N$: φ satisfies the following two requirements: φ maps the empty set onto itself and $\varphi(A_1 \cup A_2) =$

$\varphi(A_1) \cup \varphi(A_2)$ for any two subsets A_1, A_2 of N. A path W of length k in φ is a sequence $i_0, \ldots, i_k$ such that $i_j \in \varphi(i_{j-1})$ for $j = 1, \ldots, k$. The vertices i_0 and i_k are called the initial and end point of the path W; the number k is called the length of W and will be denoted by $l(W)$. Clearly there exists a path of length k connecting the initial point i_0 to the end point i_k if and a only if $i_k \in \varphi^k(i_0)$. A cycle of length k in φ is a path of length k in φ whose end point coincides with the initial point. A simple cycle is one for which the vertices $i_0, \ldots, i_{k-i}$ are all distinct. There is a cycle of length k in φ passing through i if and only if $i \in \varphi^k(i)$.

Let φ be a graph in N. If φ^0 is understood as the identity mapping $\varphi^0(X) = X$, for all $X \subset N$, define the graph φ^∞ by setting

$$\varphi^\infty(A) = \bigcup_{j=0}^{\infty} \varphi^k(A)$$

so that $j \in \varphi^\infty(A)$ if and only if j may be reached by a path starting in A. Clearly $\varphi(\varphi^\infty(A)) \subset \varphi^\infty(A)$ for every A.

A graph φ on N is said to be irreducible if it has no nontrivial invariant set, more precisely, if there exists no nonvoid set $P \subset N$ different form N for which $\varphi(P) \subset P$.

If φ is irreducible then $\varphi^\infty(A) = N$ for each nonvoid A; in particular, given arbitrary $i, k \in N$, there is a path of length ≥ 1 in φ connecting i to k. Conversely, if $\varphi^\infty(A) = N$ for every nonvoid A then φ is irreducible.

Let φ be an indecomposable graph. The index of imprimitivity of the graph φ, denoted by $h(\varphi)$, is defined as the *gcd* of the lengths of all cycles in φ, in other words, $h(\varphi) = gcd$ of all k for which there exists x such that $x \in \varphi^k(x)$.

Now let φ be irreducible and let $h = h(\varphi)$. It follows from the irreducibility of φ that, given $i, k \in N$, there exists a positive integer m such that $k \in \varphi^m(i)$. Let us show now that all exponents m with this property belong to the same class modulo h. To see that take a positive integer w such that $i \in \varphi^w(k)$; it follows that $i \in \varphi^w(k) \subset \varphi^{w+m}(i)$. Thus $w + m$ is divisible by h and m belongs to the class of $-w$ modulo h. We shall denote this class by $d(i, k)$.

In this manner d is defined on $N \times N$, its values are residue classes modulo h; we shall see that it possesses some of the properties of a distance. The following proposition exhibits three different ways of describing d.

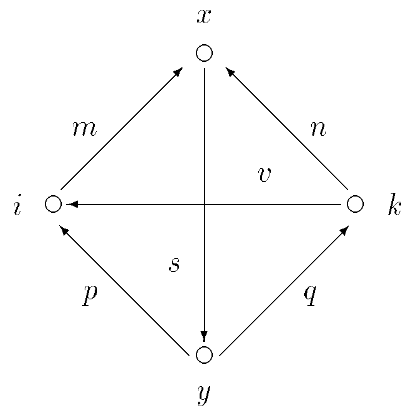
It could be described as *oriented* distance: clearly $d(x, x) = 0$ for every x, it is additive, $d(i, k) = d(i, s) + d(s, k)$ for all triples i, s, k; it is not symmetric, however; indeed, $d(k, i) = -d(i, k)$.

(2.1) *Proposition. Let φ be an irreducible graph. Let $i, k \in N$ be given. Consider the following sets of integers*

$$\begin{aligned} S_1 &= \{d;\ d \geq 0 \ \text{ such that } \ i \in \varphi^d(k)\} \\ S_2 &= \{n - m, \ \text{ such that } \ \varphi^m(i) \cap \varphi^n(k) \neq 0\} \\ S_3 &= \{q - p, \ \text{ such that } \ \varphi^{-p}(i) \cap \varphi^{-q}(k) \neq 0\} \end{aligned}$$

There exists a class of integers modulo h which contains all three sets S_1, S_2, S_3. This class will be denoted by $d(i, k)$.

Proof. It suffices to prove the last two statements, the first set being a subset of the second. Fix an integer v such that $i \in \varphi^v(k)$. Furthermore, consider an $x \in \varphi^m(i) \cap \varphi^n(k)$ and a $y \in \varphi^{-p}(i) \cap \varphi^{-q}(k)$. Let s be an integer such that $y \in \varphi^s(x)$.



We have the following relations modulo h

$$\begin{aligned} p + m + s &= 0 \\ q + n + s &= 0 \\ q + v + m + s &= 0 \end{aligned}$$

The first two yield $p - q = n - m$. Combining the last two we obtain $n - m = v$. $\square$

This "distance" is additive in the following sense: $d(i,k) = d(i,p) + d(p,k)$; in particular $d(k,i) = -d(i,k)$. It follows that the relation E on N defined by pEq iff $d(p,q) = 0$ is an equivalence relation.

Given any $A \subset N$ we write $E(A)$ for the set of all x such that aEx for some $a \in A$. Clearly $E(\varphi A) = \varphi(E(A))$.

(2.2) *Proposition. Let φ be irreducible and let $h = h(\varphi) > 1$. Let B be an arbitrary class of the equivalence E, $E(p)$ say. For any integer k*

$$\varphi^k(B) = \{x;\ d(p,x) = k \bmod h\},$$

in particular $\varphi^h(B) = B$. Furthermore,

$$B \cup \varphi(B) \cup \ldots \cup \varphi^{h-1}(B)$$

is the decomposition of N into classes of the equivalence E.

Proof. Consider the set

$$H = B \cup \varphi(B) \cup \ldots \cup \varphi^{h-1}(B)$$

Since $\varphi(H) \subset H$ it follows that $H = N$. If the intersection $\varphi^r(B) \cap \varphi^s(B)$ is nonvoid for some $0 \leq r \leq s \leq h-1$, we have $d(p,x) = r$, $d(p,x) = s$ for some element x of this intersection. Then $0 \leq s - r < h$ and $s - r = 0 \bmod h$ so that $r = s$. The sets $\varphi^j(B)$ are thus disjoint. $\square$

If φ is irreducible and if $h(\varphi) > 1$, the preceding proposition shows that φ^h is reducible; indeed, any class of the equivalence E is φ^h invariant. Furthermore, φ^d is easily seen to be reducible for any nontrivial divisor of h. In this manner $h(\varphi) > 1$ implies the reducibility of some iterate of φ. The following proposition shows that this property of the iterates of φ is characteristic for $h(\varphi) > 1$.

(2.3) *Suppose that φ is irreducible. Then φ^r is irreducible if and only if $(h,r) = 1$.*

Proof. It suffices to show that $(h,r) = 1$ implies φ^r irreducible, in other words, to prove the following assertion: given any $x, y \in N$, there exists a nonnegative integer b such that $y \in \varphi^{br}(x)$. Consider simple cycles $C_1, \ldots, C_s$ in the graph φ such that h is the *gcd* of their lengths $d_1, \ldots, d_s$. Since φ is irreducible there exists a path connecting x to y which intersects each of the cycles $C_1, \ldots, C_s$. Let m be the length of this path. Since $1 = (r,h) = (r, d_1, \ldots, d_s)$ there exist integers $y_0, y_1, \ldots, y_s$ such that $m = ry_0 + \sum d_j y_j$. Set $A = rd_1 \ldots d_s$ and define $A_0 = Ar^{-1}$, $A_j = Ad_j^{-1}$. Let $x_1, \ldots, x_j$ be negative integers for which $y_j + x_j A_j < 0$. Set $x_0 = -\sum_1^s x_j$. Then

$$\begin{aligned} r(y_0 + x_0 A_0) + \sum_1^s d_j(y_j + x_j A_j) &= \\ = ry_0 + \sum_1^s d_j y_j + \sum_0^s x_k A &= m \end{aligned}$$

Since m is positive the integer $b = y_0 + x_0 A_0$ is positive. In this manner we have obtained a relation of the form

$$m + \sum_1^s d_j v_j = br$$

with positive b and $v_1, \ldots, v_s$. Since $y \in \varphi^m(x)$ and the v_j are positive we also have $y \in \varphi^{m'}(x)$ where $m' = m + \sum d_j v_j$. Since m' is a positive multiple of r, the irreducibility of φ^r is proved.

(2.4) *Let φ be irreducible; if φ^r is reducible for some $r > 1$ then the smallest integer s for which φ^s is reducible equals $d = (h,s)$. In particular s is a divisor of h.*

Proof. It follows from the preceding proposition that $h > 1$. Let s be the smallest exponent for which φ^s is reducible. Set $d = (h,s)$. Since φ^s is reducible, $d > 1$ by

(2.3). Since φ^d is reducible, we have $s \le d$. Since s is a multiple of d it follows that $s = d$ whence $s|h$. □

Now we are ready to describe the combinatorial structure of l_∞ contractions with spectral radius one. We begin by proving a simple proposition about the solvability of a system of equations $xl_s = B_s$ where l_s are given integers and B_s given numbers. The meaning of the proposition will become evident in the sequel.

(3.1) *Let $B_1, \dots B_n$ be given real numbers and let $l_1, \dots l_n$ be given positive integers. Denote by h the greatest common divisor of the l_j so that there exist integers m_j such that $h = \sum l_j m_j$*

1° suppose there exists an x which satisfies

$$xl_s = B_s \quad \text{for} \ \ s = 1, 2, \dots, n \tag{0.1}$$

then

$$x = \frac{1}{h} \sum B_s m_s$$

and the following implication holds:

$$\textit{whenever} \quad \sum l_s y_s = 0 \quad \textit{then} \quad \sum B_s y_s = 0 \tag{0.2}$$

2° if the implication (0.2) holds then $x = \frac{1}{h} \sum B_s m_s$ is the unique solution of the system of equations (0.1)

Proof. Suppose first that x is a solution of the system of equations (0.1). Then $xh = \sum x l_s m_s = \sum B_s m_s$. Furthermore, if $\sum l_s y_s = 0$ then $0 = x \sum l_s y_s = \sum B_s y_s$.

On the other hand, suppose the implication (0.2) holds. We intend to show that $x = \frac{1}{h} \sum B_j m_j$ is a solution of the system (0.1). To see that, choose an index p and consider the difference $xl_p - B_p$. We have

$$h(xl_p - B_p) = \sum B_j m_j l_p - B_p \sum l_j m_j = \sum B_s y_s$$

where $y_j = m_j l_p$ for $j \neq p$ and $y_p = m_p l_p - \sum l_j m_j$. Since

$$\sum l_s y_s = \sum_{s \neq p} l_s m_s l_p + l_p(m_p l_p - h) = 0$$

the implication (0.2) yields $\sum B_s y_s = 0$ so that $xl_p - B_p = 0$. Since p was arbitrary, this completes the proof. □

The combinatorial structure of the pattern of nonzero entries of a matrix may be investigated by examining the graph of the matrix. We begin by assigning to each matrix on $N \times N$ a graph on N.

Let A be a matrix on $N \times N$ and consider the corresponding graph φ:

$$\text{if } A \subset N \text{ then } k \in \varphi(A) \text{ iff } a_{ik} \neq 0 \text{ for some } i \in A.$$

Given $i \in N$ and $k \in \varphi(i)$ set

$$\omega(i,k) = \frac{a_{ik}}{|a_{ik}|}$$

To each path $W = (i_0, \ldots, i_k)$ in φ we assign a number $\omega(W)$ by the formula

$$\omega(W) = \prod_{j=1}^{k} \omega(i_{j-1}, i_j)$$

(3.2) *Let φ be an irreducible graph on N, G a group and let ω be a mapping assigning to each path W in φ an element $\omega(W)$ of G such that*

$$\omega(W_1 W_2) = \omega(W_1)\omega(W_2)$$

if the endpoint of W_1 coincides with the initial point of W_2. Then these are equivalent

1° there exists a mapping $f : N \to G$ such that

$$\omega(W) = f(i)^{-1} f(k)$$

if $W = (i, \ldots, k)$

2° $\omega(C) = 1$ for each cycle in the graph φ

Proof. It suffices to prove the implication $2° \to 1°$. Fix a vertex $p \in N$. For each $i \in N$ fix a path W_{pi} from p to i and set $f(i) = \omega(W_{pi})$. To prove that $\omega(W) = f(i)^{-1} f(k)$ for each path W with initial point i and end point k it suffices to fix a path W^* from k to p and to observe that both $W_{pi} W W^*$ and $W^* W_{pk}$ are cycles in φ.

Now we are able to resume our study of linear operators on the n-dimensional l_∞ space. We shall use a classical notion. An n by n matrix $M = (m_{ik})$ is said to be *stochastic* if

$$\begin{aligned} m_{ik} &\geq 0 \qquad \text{for all } \; i, k \\ \sum_k m_{ik} &= 1 \qquad \text{for every } \; i \end{aligned}$$

(3.3) *Let A be an indecomposable contraction. Then these are equivalent*

1° there exists a nonzero vector x such that $Ax = x$

2° there exists a unique (up to scalar multiples) nonzero vector x with $Ax = x$

3° $\sum_k |a_{ik}| = 1$ for every i and $\omega(C) = 1$ for every cycle in the graph of A

4° there exists a diagonal matrix D with diagonal elements of modulus one such that $D^{-1}AD$ is stochastic

5° there exists a diagonal matrix D such that $D^{-1}AD$ is stochastic

Proof. Suppose first that x is a nonzero vector and that $Ax = x$. According to (1.1) $\sum_k |a_{ik}| = 1$ for all i and the moduli of the x_j are all equal. It follows that

$$\frac{a_{ik}x_k}{|a_{ik}x_k|} = \frac{x_i}{|x_i|}$$

for every i, k for which $a_{ik} \neq 0$. Since $|x_i| = |x_k|$ this implies $\omega_{ik} = \frac{a_{ik}}{|a_{ik}|} = \frac{x_i}{x_k}$. Fix an index p and set $f(j) = \frac{x_p}{x_j}$ for $j = 1, \ldots, n$. If $a_{ik} \neq 0$ then $\omega_{ik} = \frac{x_i}{x_k} = f(i)^{-1}f(k)$.

On the other hand, suppose that condition 3° is satisfied. It follows from (3.2) that there exists a function f defined on N with values on the unit circle such that $\omega_{ik} = f(i)^{-1}f(k)$. It follows that

$$a_{ik}f(k)^{-1} = |a_{ik}|\omega_{ik}f(k)^{-1} = |a_{ik}|f(i)^{-1}$$

for every pair i, k for which $a_{ik} \neq 0$. Let D be the diagonal matrix with $f(1)^{-1}, \ldots, f(n)^{-1}$ on the diagonal. Then

$$(D^{-1}AD)_{ik} = f(i)a_{ik}f(k)^{-1} = |a_{ik}|$$

whence 4° follows.

If 5° is satisfied, consider the vector $e = (1, 1, \ldots, 1)^T$. Setting $x = De$ we obtain $Ax = x$.

If A is indecomposable the linear space $\mathrm{Ker}(A - 1)$ cannot contain two linearly independent vectors, since, in that case, it would also contain a nonzero vector with at least one coordinate zero and this is impossible by $(1, 1)$. The proof is complete. $\square$

(3.4) *Suppose A indecomposable, $|A| \leq 1$. Then the following conditions are equivalent*

> *1°* $r(A) = 1$
>
> *2°* $\sum_k |a_{ik}| = 1$ *for every i and the following implication holds:*
> *if $C_1, \ldots C_p$ are the simple cycles in A and if $a_1, \ldots a_p$ are integers such that $\sum l(C_s)a_s = 0$ then $\prod \omega(C_s)^{a_s} = 1$*
>
> *If these conditions are satisfied then a number λ of modulus one belongs to the spectrum of A if and only if $\lambda^h = \prod \omega(C_s)^{m_s}$ where $h = h(\varphi)$ and the m_s satisfy $h = \sum l(C_s)m_s$.*

Proof. Suppose that $r(A) = 1$, let λ be an eigenvalue of A of modulus 1 and x a nonzero vector with $Ax = \lambda x$. Set $B = \lambda^{-1}A$. Then $Bx = x$. It follows from (1.1) that $\sum_k |b_{ik}| = 1$ for every i so that $\sum_k |a_{ik}| = 1$ for every i as well. Furthermore, if $\omega_B(i, k) = \frac{b_{ik}}{|b_{ik}|}$ then $\omega_B(C) = 1$ for every cycle in B but this is equivalent to the identity $\omega_A(C) = \lambda^{l(C)}$ for every cycle in A. Suppose that $a_1, \ldots, a_p$ are integers such that $\sum l(C_s)a_s = 0$. It follows that $\prod \omega_A(C_s)^{a_s} = \lambda^{\Sigma l(C_s)a_s} = 1$.

In particular, if $h = \sum l(C_s)m_s$ then

$$\lambda^h = \prod \lambda^{l(C_s)m_s} = \prod \omega_A(C_s)^{m_s}$$

Conversely, suppose condition 2° is satisfied. Denote by h the greatest common divisor of the lengths l_j. Then $h = \sum l_s m_s$ for suitable m_s and $l_j = hq_j$ for suitable q_j. Since $h = \sum l_j m_j = \sum hq_j m_j$ we have $\sum m_j q_j = 1$.

Now consider a λ of modulus one which satisfies the equation

$$\lambda^h = \prod \omega_A(C_s)^{m_s}$$

and let us prove that it is an eigenvalue of A, or, in other words, that the matrix $B = \lambda^{-1}A$ possesses a nonzero fixed vector. Since $|\lambda| = 1$ we have $\sum_k |b_{ik}| = 1$ for every i. According to (3.3) it suffices to show that $\omega_B(C_t) = 1$ for every t, in other words, that $\frac{\omega_A(C_t)}{\lambda^{l_t}} = 1$ for each t. For brevity, we now write ω_p for $\omega_A(C_p)$. Given a fixed t, we have

$$\frac{\lambda^{l_t}}{\omega_t} = \frac{\lambda^{hq_t}}{\omega_t} = \frac{1}{\omega_t}\left(\prod \omega_s^{m_s}\right)^{q_t} = \omega_t^{m_t q_t - 1} \prod_{s \neq t} \omega_s^{m_s q_t}$$

In view of the implication sub 2° this product will equal one if we show that

$$l_t(m_t q_t - 1) + \sum_{s \neq t} l_s m_s q_t = 0;$$

this sum equals

$$-l_t + q_t \sum l_s m_s = -l_t + q_t \sum l_s m_s = -l_t + q_t h = 0 \qquad \square$$

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