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1997

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Technical report No. V737

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Abstract

Trakhtenbrot theorem is shown to be valid for the three main fuzzy logics - Łukasiewicz, Gödel and product logic.

Keywords
fuzzy logic, undecidability, finite model theory

0.1 Introduction.

It follows from Gödel completeness theorem that the set of all tautologies of the classical (Boolean) predicate logic (denote this set by $TAUT^{Bool\forall}$) is recursively enumerable, i. e. Σ_1 and it is known that it is Σ_1-complete. Tautologies are formulas true in all models and it is crucial that both finite and infinite models are considered. Finite model theory (flourishing due to its obvious relevance for databases) uses the language of classical logic but admits only finite models. See [3]. Let $fTAUT^{Bool\forall}$ be the set of all formulas true in all finite models. Clearly, $TAUT^{Bool\forall} \subseteq fTAUT^{Bool\forall}$. Trakhtenbrot proved as early as in 1950 [7] that the set $fTAUT^{Bool\forall}$ is *not* recursively enumerable (hence not recursively axiomatizable); moreover, the set is Π_1-complete. Due to the properties of classical negation it follows that the set $fSAT^{Bool\forall}$ of all formulas ϕ true at least in one finite model is Σ_1-complete. The fact that there is no recursive axiomatic system complete for tautologies of finite model theory means that deductive methods have only limited importance for database theory.

Fuzzy logic generalizes Boolean logic by introducing more than two truth values; typically the real unit interval $[0,1]$ serves as the ordered set of truth values (truth degrees). Let us stress that fuzzy logic can be developed rather far in the style of mathematical logic (see [2, 1]). On the other hand, there is a research in fuzzy databases [6]. Thus whether and in which form Trakhtenbrot theorem generalizes to fuzzy logic appears to be very natural. To answer this question is the main purpose of this paper. We shall investigate three important fuzzy predicate calculi having $[0,1]$ for their truth set - Łukasiewicz predicate logic Ł$\forall$, Gödel predicate logic $G\forall$ and product predicate logic $\Pi\forall$. Let $\mathcal{C}$ vary over Ł$, G, \Pi$, let $fTAUT^{\mathcal{C}}\forall$ be the set of all formulas true in the sense of $\mathcal{C}\forall$ in at least one finite model. Our main result is as follows:

Theorem. For $\mathcal{C}$ being Ł$, G, \Pi$, the set $f - TAUT^{\mathcal{C}\forall}$ is Π_1-complete and the set $fSAT^{\mathcal{C}}\forall$ is Σ_1-complete.

This will be proved in Sect. 3. Section 2 contains preliminaries on arithmetical hierarchy and fuzzy logic.

Acknowledgements This research has been partially supported by the COST action 15 (Many - valued logics for computer science applications.)

0.2 Preliminaries:

The reader is assumed to be familiar with basic properties of *recursive sets* (of natural numbers, words etc.), recursive relations and recursive functions. A set A is Σ_1 (or *recursively enumerable*) if there is a binary recursive relation R such that

$$A = \{n|(\exists m)R(m,n)\}.$$

A is Π_1 if there is a binary recursive relation R such that

$$A = \{n|(\forall m)R(m,n)\}.$$

Similarly, A is Σ_2 if for some ternary recursive relation R,

$$A = \{n|(\exists m)(\forall b)R(m,k,n)\}$$

etc. A is Σ_1-*complete* if it is Σ_1 and each Σ_1 - set B is recursively reducible to A, i. e. for some recursive function f,

$$B = \{n|f(n) \in A\}.$$

Similarly for Π_1-complete etc. A set is Δ_1 if it is both Σ_1 and Π_1. Recall that Δ_1 sets are exactly all recursive sets. See [5] for more information. We also assume that the reader knows basic notion of the theory of computational complexity, i. e. what it means that a set is in P (recognized by a deterministic Turing machine running in polynomial time) or in NP ($\cdots$ nondeterministic Turing machine $\cdots$). Here we deal with polynomial reducibility and NP-completeness as well as co-NP-completeness. See [4].

Now we recall some basic facts on fuzzy logics. A logic with the truth set $[0,1]$ is given by the choice of *truth functions* determining the truth value of a compound formula from the truth values of its components. In [1] the reader may find some theory of continuous t-norms as possible truth functions for the *conjunctions*, their residua as truth functions of implication and the corresponding truth functions of negation. we shall not need this; we shall only need three particular choices. (They are extremely outstanding choices.) Here they are:

Lukasiewicz (L):

$$x * y = \max(0, x+y-1); \tag{0.1}$$
$$x \Rightarrow y = 1 \text{ for } x \leq y, \tag{0.2}$$
$$x \Rightarrow y = 1 - x + y \text{ for } x \geq y; \tag{0.3}$$
$$(-)x = 1 - x \tag{0.4}$$

Gödel (G):

$$x * y = \min(x, y) \tag{0.5}$$
$$x \Rightarrow y = 1 \text{ for } x \leq y, \tag{0.6}$$
$$x \Rightarrow y = y \text{ for } x > y; \tag{0.7}$$
$$(-)0 = 1, \tag{0.8}$$
$$(-)x = 0 \text{ for } x > 0. \tag{0.9}$$

Product (Π):

$$x * y = x \cdot y \text{ (usual multiplication)} \tag{0.10}$$
$$x \Rightarrow y = 1 \text{ for } x \leq y, \tag{0.11}$$
$$x \Rightarrow y = y/x \text{ for } x > y; \tag{0.12}$$
$$(-x) \text{ as in Gödel .} \tag{0.13}$$

The corresponding propositional logic has formulas built from propositional variable, the constant $\bar{0}$ and connectives $\&, \rightarrow$. Negation, the min-conjunction and the max-disjunction are defined as follows:

$$\neg\varphi \text{ is } \varphi \rightarrow \bar{0},$$

$$\varphi \wedge \psi \text{ is } \varphi\&(\varphi \rightarrow \psi),$$

$$\varphi \vee \psi \text{ is } ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi).$$

Each evaluation e of propositional variables by elements of $(0,1)$ extends uniquely to the evaluation $e_{\mathcal{C}}(\varphi)$ of each formula φ using the truth function of $\mathcal{C}$ (e being Ł, G, Π). $TAUT^{\mathcal{C}}$ is the set of all φ such that $e_{\mathcal{C}}(\varphi) = 1$ for at least one e.

Fact: For $\mathcal{C}$ being Ł, $G, \Pi, TAUT^{\mathcal{C}}$ is co-NP complete and $SAT^{\mathcal{C}}$ is NP - complete. (See [1] for details.)

The *predicate calculus* $\mathcal{C}\forall$ has a language consisting of predicates (each having a positive natural arity). (Here we disregard object constants.)

Atomic formulas have the form $P(x_1, \ldots, x_n)$ where P is an n-ary predicate and $x_1, \ldots, x_n$ are *object variables.* If φ, ψ are formulas then $\varphi\&\psi, \varphi \rightarrow \psi, (\forall x)\varphi, (\exists x)\varphi$ are formulas; $\bar{0}$ is a formula.

A *model* has the form $\mathbf{M} = (M, (r_P)_{P\,predicate})$ where $M \neq \emptyset$ is a set and $r_P : M^{ar(P)} \rightarrow [0,1]$ is a fuzzy relation M of arity equal to the arity of P. An *evaluation of variables* is a mapping $v : Var \rightarrow M$ (Var being the set of object variables. The truth value of a formula φ (over $\mathcal{C}$) given by $\mathbf{M}, v$ is defined inductively in Tarski's style, i.e.

$$\|P(x_1, \ldots, x_n)\|^{\mathcal{C}}_{\mathbf{M},v} = r_P(v(x_1), \ldots, v(x_n)),$$

$$\|\varphi\&\psi\|^{\mathcal{C}}_{\mathbf{M},v} = \|\varphi\|^{\mathcal{C}}_{\mathbf{M},v} * \|\psi\|^{\mathcal{C}}_{\mathbf{M},v}, \text{analogous for} \rightarrow, \Rightarrow;$$

$$\|(\forall x)\varphi\|^{\mathcal{C}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathcal{C}}_{\mathbf{M},w} | v \equiv_x w\},$$

Analogously for $\exists, \sup$ (Note that $v \equiv_x w$ mean that v coincides with w for all arguments except possibly x.) $TAUT^{\mathcal{C}\forall}$ is the set of all formulas φ such that $\|\varphi\|^{\mathcal{C}}_{\mathbf{M},v} = 1$ for all $\mathbf{M}, v$; $SAT^{\mathcal{C}\forall}$ is the set of all φ such that $\|\varphi\|^{\mathcal{C}}_{\mathbf{M},v} = 1$ for some $\mathbf{M}, v$.

Fact. $TAUT^{G\forall}$ is Σ_1 complete; $TAUT^{\text{Ł}\forall}$ is Π_2 complete; $TAUT^{\Pi\forall}$ is Π_2-hard (i. e. each Π_2 set is reducible to $TAUT^{\Pi\forall}$; if the latter set is itself Π_2 is unknown.) See [1].

*

In the rest of the preliminaries we shall elaborate a technique of coding formulas of predicate logic by some formulas of propositional logic and finite models by some evaluation of propositional variables.

Definition. Let $\mathbf{M} = (M, (r_{P_i})_{i=1}^k)$ be a finite model, let M have n elements. For each predicate P_i of arity s we introduce n^s propositional variables $p_{ij_1\ldots j_s}$ where $j_1, \ldots j_s \in \{1, \ldots, n\}$ (assume $M = \{1, \ldots, n\}$). Define an evaluation $e_{\mathbf{M}}$ of these propositional variables by setting $e_{\mathbf{M}}(p_{ij_1,\ldots j_s}) = r_{P_i}(j_1, \ldots j_s)$ (i. e. the truth value of $p_{ij_1\ldots j_s}$ is the degree in which $(j_1, \ldots j_s)$ is in the relation r_{P_i}).

Investigate formulas of predicate logic with free variable substituted by elements of M. For each such object φ we define its translation $\varphi^{*,n}$ as follows:
$(P_i(j_1,\dots,j_s))^* = p_{ij_1,\dots j_s}$; $(\varphi\&\psi)^* = \varphi^*\&\psi^*$; analogously $\rightarrow$;
$(\bar{0})^* = \bar{0}$; $((\forall x)\varphi(x))^* = \bigwedge_{j=1}^{n}\varphi^*(j), ((\exists x)\varphi(x))^* = \bigvee_{j=1}^{n}\varphi^*(j)$.

Note that if φ is as assumed (free variables replaced by elements of M) then $\|\varphi\|_{\mathbf{M}}$ has the obvious meaning $\|\varphi\|_{\mathbf{M},v}$ where v just assigns to each free variable the corresponding element of M (and otherwise arbitrary).

Lemma. For each finite $\mathbf{M}$ of cardinality n and φ as above,

$$\|\varphi\|_{\mathbf{M}} = e_{\mathbf{M}}(\varphi^{*,n}).$$

Proof obvious by induction on φ observing that on a finite domain $\forall$ reduces to a finite $\wedge$-conjunction and analogously $\exists, v$.

Note that $\varphi^{*,n}$ is a recursive function of φ and n, the language $P_1,\dots,P_k$ being given.

0.3 The results

Lemma. For $\mathcal{C}$ being Ł$, G, \Pi$, the set $fSAT^{\mathcal{C}\forall}$ is Σ_1 and the set $fTAUT^{\mathcal{C}\forall}$ is Π_1.

Proof. Let φ vary over closed formulas. Recall that the sets $SAT^{\mathcal{C}}, TAUT^{\mathcal{C}}$ of propositional formulas are of low computational complexity (co-NP, NP) and hence recursive. Now

$$\varphi \in fSAT^{\mathcal{C}\forall} \text{ iff } (\exists n)(\varphi^{*,n} \in SAT^{\mathcal{C}}),$$

$$\varphi \in fTAUT^{\mathcal{C}\forall} \text{ iff } (\forall n)(\varphi^{*,n} \in TAUT^{\mathcal{C}}).$$

This proves the lemma.

Theorem. For $\mathcal{C}$ as above, $fSAT^{\mathcal{C}\forall}$ is Σ_1-complete and $fTAUT^{\mathcal{C}\forall}$ is Π_1-complete.

Proof. For $\mathcal{C}$ being G or Π the proof is easy using the double negation interpretation: let $\varphi^{\neg\neg}$ results from φ by attaching double negation to end atomic formula. Then (cf. [1] 6.2.8, 6.3.1)

$$\varphi \in fSAT^{Bool\forall} \text{ iff } \varphi^{\neg\neg} \in fSAT^{\mathcal{C}\forall}$$

$$\varphi \in fTAUT^{Bool\forall} \text{ iff } \varphi^{\neg\neg} \in fTAUT^{\mathcal{C}\forall}$$

Thus the complete sets on the left hand side are recursively reducible to the corresponding sets on the right hand side showing their respective completeness.

For $\mathcal{C} =$ Ł we must do more work. Let $Crisp(P_i)$ be the formula $(\forall\mathbf{x})(P(\mathbf{x})\vee\neg P(\mathbf{x}))$. It is easy to show that $\varphi \in fSAT^{Bool\forall}$ iff $\bigwedge_i Crisp(P_i) \wedge \varphi$ is in $fSAT^{\text{Ł}\forall}$. (This argument also works for G, Π as an alternative to the above proof.) Thus again we have a recursive reduction.

For $fTAUT^{\text{Ł}\forall}$ we proceed as follows (using a method of Ragaz, cf. [1] 6.3.6 - 6.3.9.: for each closed ψ, $\psi \in fSAT^{Bool\forall}$ iff $\bigwedge Crisp^2(P_i) \wedge \psi^2$ is positively finitely satisfiable, i. e. iff there is a finite model $\mathbf{M}$ such that $\|\psi^2\|_{\mathbf{M}} > 0$. (Cf. [1] 6.2.13.) Here one has to assume that ψ is *classical* in the sense that the only connectives used are $\wedge, \vee, \neg$. Thus for each classical φ,

$\varphi \in fTAUT^{Bool}$ iff $\neg\varphi \notin fSAT^{Bool}$ iff $\bigwedge_i Crisp^2(P_i) \wedge (\neg\varphi)^2$ is not finitely positively satisfiable in Ł∀, iff the formula

$$\bigvee_i (\exists \mathbf{x}) 2(P_i \wedge \neg P_i) \vee 2\varphi$$

is in $fTAUT^{Ł\forall}$. Thus we have reduced the Π_1-complete set $fTAUT^{Bool}$ to $fTAUT^{Ł\forall}$. Here of course α^2 is $\alpha \& \alpha$, 2α is $\alpha \underline{\vee} \alpha$, $\underline{\vee}$ being the strong disjunction of Lukasiewicz logic. This completes the proof.

0.4 Appendix.

It is of some interest to observe that $\mathcal{C}\forall$ has the *rational model property*:

Claim. For $\mathcal{C}$ being $Ł, G, \Pi$, the following holds:

(1) If there is a finite $\mathbf{M}$ with $\|\varphi\|^{\mathcal{C}}_{\mathbf{M}} = 1$ then there is a rational-valued model $\mathbf{M}'$ (of the same cardinality) with $\|\varphi\|^{\mathcal{C}}_{M'} = 1$.

(2) the same with < 1 instead of $= 1$.

(Note that this gives an alternative proof of $fSAT^{\mathcal{C}} \in \Sigma_1, fTAUT^{\mathcal{C}} \in \Pi_1$.)

Proof. Due to our representation (Sect. 2), it is enough to show for each *propositional* formula φ that if $e_{\mathcal{C}}(\varphi) = 1$ then for some rational-valued e', $e'_{\mathcal{C}}(\varphi) = 1$ and the same for < 1. First, this is easy for G since if $0 < z_1 < \ldots < z_n < 1$ contain all the values $e(p_i)$ involved and $0 < r_1 < \ldots < r_n < 1$ are rationals then one easily gets an *isomorphism* of $[0, 1]$ with respect to Gödel connectives moving z_i to r_i. For Ł and < 1 the claim follows immediately from the continuity of truth functions; for Ł and $= 1$ we use [1] 3.3.17. Finally, investigate Π and recall the transformation φ^I [1] 6.2.2 such that, for each I, φ^I does not contain $\bar{0}$ or φ^I is $\bar{0}$; and for each e such that $e(p_i) = 0$ iff $i \in I, e_{\Pi}(\varphi) = e_{\Pi}(\varphi^I)$. For "$< 1$" observe that for positive values $x_i = e(p_i)\,(i \notin I)$, the value $e_{\Pi}(\varphi_I)$ is continuous (and positive) in $x_i\,(i \notin I)$. (If φ_I is $\bar{0}$ then there is nothing to prove.)

Finally for "$= 1$" observe that if e is such that $e_{\Pi}(\varphi) = 1$ then for some *boolean* e', $e'_{\Pi}(\varphi) = 1$ ($e'(p_i) = 0$ if $e(p_i) = 0$, $e'(p_i) = 1$ otherwise). This completes the proof.

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