Basic Fuzzy Logic and BL-Algebras

Hájek, Petr
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Petr Hájek

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Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic
phone: (+4202) 688 4244 fax: (+4202) 8585789
e-mail: hajek@uivt.cas.cz

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Abstract

The relation of the basic fuzzy logic BL to continuous t-norms is studied and two additional axioms are formulated such that the extended logic is complete with respect to tautologies over all logics given by continuous t-norms.

Keywords

Basic fuzzy logic, continuous t-norms, residuated lattices.

0.1 Introduction.

Basic fuzzy logic BL, as developed and investigated in [3], is closely related to continuous t-norms; as summarized bellow, each continuous t-norm determines (1) a semantics of fuzzy propositional logic for which BL is sound, and (2) a particular linearly ordered BL-algebra, BL-algebras from a variety for which BL is sound and complete. Full treatment is found in [3]; bellow we summarize basic facts in Sections 1 - 3. At the end of Sect. 3 we formulate the main problem of completeness of BL with respect to BL-algebras given by continuous t-norms (t-algebras). In Sect. 4 we develop some algebra of linearly ordered BL-algebras. In Sect. 5 we exhibit two additional axioms (B1), (B2) and show soundness and completeness of BL + (B1) + (B2) for t-algebras. The problem whether BL proves (B1), (B2) remains open.

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0.2 Continuous t-norms

We recall some well-known facts on continuous t-norms.

A *t-norm* is a binary operation $*$ on $[0, 1]$
(i.e. $t: [0,1]^2 \rightarrow [0,1]$) satisfying the following conditions:

(i) $*$ is commutative and associative, i.e., for all $x, y, z \in [0, 1]$,

$$\begin{aligned} x * y &= y * x, \\ (x * y) * z &= x * (y * z), \end{aligned}$$

(ii) t is non-decreasing in both arguments, i.e.

$$\begin{aligned} x_1 \leq x_2 \quad &\text{implies} \quad x_1 * y \leq x_2 * y, \\ y_1 \leq y_2 \quad &\text{implies} \quad x * y_1 \leq x * y_2, \end{aligned}$$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$.

$*$ is a *continuous t-norm* if it is a t-norm and is a continuous mapping of $[0,1]^2$ into $[0, 1]$ (in the usual sense).

The following are our most important examples of continuous t-norms:

(i) *Lukasiewicz t-norm*: $x * y = \max(0, x + y - 1)$,

(ii) *Gödel t-norm:* $x * y = \min(x, y)$,

(iii) *Product t-norm:* $x * y = x.y$ (product of reals).

It is elementary to verify conditions (i)-(iii) above.

Let $*$ be a continuous t-norm. Then there is a unique operation $x \Rightarrow y$ satisfying, for all $x, y, z \in [0, 1]$, the condition $(x * z) \leq y$ iff $z \leq (x \Rightarrow y)$, namely $x \Rightarrow y = \max\{z \mid x * z \leq y\}$.

The operation $x \Rightarrow y$ is called the *residuum* of the t-norm.

The following operations are residua of the three t-norms above: $x \Rightarrow y = 1$ for $x \leq y$ and

(i) *Lukasiewicz implication*: $x \Rightarrow y = 1 - x + y$

(ii) *Gödel implication*: $x \Rightarrow y = y$

(iii) *Goguen implication*: $x \Rightarrow y = y/x$

for $x > y$ (residuum of product conjunction).

For each continuous t-norm the set E of all its idempotents is a closed subset of [0,1] and hence its complement is a union of a set $\mathcal{I}_{\text{open}}(E)$ of countably many non-overlapping open intervals. Let $[a,b] \in \mathcal{I}(E)$ iff $(a,b) \in \mathcal{I}_{\text{open}}(E)$ (the corresponding closed intervals, contact intervals of E). For $I \in \mathcal{I}(E)$ let $(* \restriction I)$ be the restriction of $*$ to I^2. The following theorem characterizes all continuous t-norms.

Theorem 1. If $*, E, \mathcal{I}(E)$ are as above, then

(i) for each $I \in \mathcal{I}(E)$, $(* \restriction I)$ is isomorphic either to the product t-norm (on [0,1]) or to Lukasiewicz's t-norm (on [0,1]).

(ii) If $x, y \in [0,1]$ are such that there is no $I \in \mathcal{I}(E)$ with $x, y \in I$, then $x * y = \min(x, y)$.

0.3 The basic many-valued logic

Fix a continuous t-norm $*$: you fix a propositional calculus (whose set of truth values is $[0,1]$): $*$ is the truth function of the (strong) conjunction &, the residuum $\Rightarrow$ of $*$ becomes the truth function of the implication. Further connectives are defined as follows:

$$\begin{aligned} \varphi \wedge \psi &\quad \text{is} \quad \varphi \& (\varphi \Rightarrow \psi), \\ \varphi \vee \psi &\quad \text{is} \quad ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg \varphi &\quad \text{is} \quad \varphi \to \bar{0}, \\ \varphi \equiv \psi &\quad \text{is} \quad (\varphi \to \psi) \& (\psi \to \varphi). \end{aligned}$$

An *evaluation of propositional variables* is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0,1]$.

This extends:

$$\begin{aligned} e(\bar{0}) &= 0, \\ e(\varphi \to \psi) &= (e(\varphi) \Rightarrow e(\psi)), \\ e(\varphi \& \psi) &= (e(\varphi) * e(\psi)). \end{aligned}$$

A formula φ is a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e.

The following formulas are axioms of the basic logic [3]:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \& \psi) \to \varphi$
(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$
(A4) $(\varphi \& (\varphi \to \psi) \to (\psi \& (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\bar{0} \to \varphi$

The *deduction rule* of BL is modus ponens. Given this, the notions of a *proof* and of a *provable formula* in BL are defined in the obvious way

All axioms of BL are 1-tautologies in each $PC(*)$. If φ and $\varphi \to \psi$ are 1-tautologies of $PC(*)$ then ψ is also a 1-tautology of $PC(*)$. Consequently, each formula provable in BL is a 1-tautology of each $PC(*)$.

Note that Lukasiewicz logic is the extension of BL by the axiom $\neg\neg\varphi \to \varphi_i$; Gödel logic is the extension of BL by the axiom $\varphi \to (\varphi \& \varphi)$. Finally, product logic is the extension of BL by the following two axioms:

$$\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi)),$$

$$\varphi \wedge \neg\varphi \to \bar{0}.$$

0.4 BL-algebras; a completeness theorem.

BL-algebras are algebras of the logic BL; their theory is developed in the style of related algebras and logics (as in [1, 2, 4]). Details are in [3].

A *BL-algebra* is an algebra

$$\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative, $1 * x = x$ for all x (thus $\mathbf{L}$ is a residuated lattice, and

(iii) the following conditions hold:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z.
(2) $x \cap y = x * (x \Rightarrow y)$
(3) $x \cup y = ((x \Rightarrow y) \Rightarrow y)) \cap ((y \Rightarrow x) \Rightarrow x)$
(4) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

Note that (3) is redundant. Define $(-)x = (x \Rightarrow 0)$.

The class of all BL-algebras is a primitive class of algebras (a variety).

MV-*algebras* are BL-algebras satisfying $(-)(-)x = x$. *Product algebras* are BL-algebras satisfying

$$x \cap (-)x = 0$$

$$(-)(-)z \Rightarrow ((x * z = y * z) \Rightarrow x = y) = 1.$$

Let **L** be a BL-algebra. An **L**-evaluation of propositional variables is any mapping e assigning to each propositional variable p an element $e(p)$ of **L**. This extends in the obvious may to an evaluation of all formulas using the operations on **L** as truth functions.

The logic BL is sound with respect to **L**-tautologies: if φ is provable in BL then φ is an **L**-tautology for each BL-algebra. More generally, if T is a theory over BL and T proves φ then, for each BL-algebra **L** and each **L**-evaluation e of propositional variables assigning the value 1 to all the axioms of T, $e(\varphi) = 1$.

Classes of provably equivalent formulas (w.r.t. a theory T) form a BL-algebra.

A usual theory of filters can be developed; in particular, prime filters correspond to linearly ordered factorizations [4] for a broader class of algebras and [2], [1] for a more narrow class of MV-algebras.)

Theorem 2. BL is complete, i.e. for each formula φ the following there things are equivalent:

(i) φ is provable in BL,

(ii) for each linearly ordered BL-algebra **L**, φ is an **L**-tautology;

(iii) for each BL-algebra **L**, φ is an **L**- tautology.

Note that we also get *strong completeness* (for provability in theories over BL.)

For completeness theorems of the three stronger logics (Łukasiewicz, Gödel, product) see [3].

Definition 1. (1) A t-algebra is a BL-algebra $([0,1], \cap, \cup, *, \Rightarrow, 0, 1)$ whose lattice part is the real interval $[0,1]$ with min and max and $*$ is a continuous t-norm (whereas $\Rightarrow$ is its residuum).

(2) A formula φ is a t-*tautology* if it is an **L**-tautology for each t-algebra **L**.

Problem. Clearly, each BL-provable formula is a t-tautology. Is the converse true? I.e. is each t-tautology provable in BL? We are not able to answer this question; but we develop below some theory of BL-algebras and exhibit two additional axioms (B1), (B2) such that each t-tautology is provable in BL + (B1) + (B2).

0.5 Structure of BL-chains

Clearly, saying "a BL-chain" we mean "a linearly ordered BL-algebra." We start with a lemma on arbitrary BL-algebras.

Lemma 1. Let $\mathbf{M}$ be a BL- algebra and let $x, y, u \in \mathbf{M}$.

(1) If $x > y$ then $y = x * (x \Rightarrow y)$.

(2) If $x \leq u \leq y$ and u is idempotent then $x * y = x$.

(3) If $x < u \leq y$ and u is idempotent then $y \Rightarrow x = x$ (caution: the first inequality is strict!).

Proof. (1) If $x > y$ then $y = x \cap y = x * (x \Rightarrow y)$.

(2) First, the statement is trivial for $x = u \leq y$: Then $u*y \geq u*u = u$ and obviously $u*y \leq u$. Thus assume $x < u \leq y$. Then $x*y \geq x*u = u*(u \Rightarrow x)*u = u*(u \Rightarrow x) = x$.

(3) On the one hand, $x \leq (y \Rightarrow x)$. On the other hand, if $x < z \leq u$ then $z * y = z > x$, thus $z > (y \Rightarrow x)$.

From now on, we shall investigate BL-chains.

Definition 2. Let $\mathbf{M}$ be a BL-chain and $u < v$ two idempotents. Let $[u, v]_{\mathbf{M}} = \{x \in M | u \leq x \leq v\}$; endow $[u, v]_{\mathbf{M}}$ with the ordering of $\mathbf{M}$ and put, for $x, y \in [u, v]_{\mathbf{M}}$,

$$x *' y = x * y,$$

$$x \Rightarrow' y = x \Rightarrow y \text{ if } x > y,$$

$$x \Rightarrow' y = v \text{ if } x \leq y.$$

Lemma 2. The structure $[u, v]_{\mathbf{M}}$ is a BL-algebra.

Proof. First observe that $[u, 1]_{\mathbf{M}}$ is a BL-subalgebra of $\mathbf{M}$ (with respect to $\cap, \cup, x, \Rightarrow$, 1 and u instead of 0; thus all axioms not mentioning 0 are true in $[u, 1]_{\mathbf{M}}$ and clearly u is the least element of $[u, 1]_{\mathbf{M}}$. Second, $[u, v]_{\mathbf{M}}$ is a homomorphic image of $[u, 1]_{\mathbf{M}}$ via the homomorphism f identical on $[u, v]_{\mathbf{M}}$ and sending all elements $x > v$ onto v. (The verification is left to the dilligent reader; we only show that $[u, 1]$ is closed under $\Rightarrow$. Indeed if $x > y \geq u$ then $u * x = u \leq y$, thus $u \leq x \Rightarrow y$.)

Definition 3. Let $\mathbf{M_1}, \mathbf{M_2}$ be two BL-chains. Taking possibly isomorphic copies assume that $1_{M_1} = 0_{M_2}$ and the rests of M_1, M_2 are disjoint (i.e. $(M_1 - \{1_{M_1}\}) \cap (M_2 - \{0_{M_2}\}0 = \emptyset)$. Let $\mathbf{M}_1 \bigoplus \mathbf{M}_2$ be the structure whose universe is $M_1 \cup M_2$,
$x \leq y$ if $(x, y \in M_1$ and $x \leq_1 y)$,
or $(x, y \in M_2$ and $x \leq_2 y)$,
or $(x \in M_1$ and $y \in M_2)$.
Furthermore, $x * y = x *_i y$ for $x, y \in M_i$, $x * y = x$ for $x \in M_1$ and $y \in M_2$; for $x \leq y, (x \Rightarrow y) = 1_{M_2}$; for $x > y$ we put $(x \Rightarrow y) = (x \Rightarrow_i y)$ if $x, y \in M_i$ and put $(x \Rightarrow y) = y$ for $x \in M_2$ and $y \in M_1 - M_2$.

Lemma 3. $M = \mathbf{M}_1 \bigoplus \mathbf{M}_2$ is a BL-chain with $0_M = 0_{M_1}$, $1_M = 1_{M_2}$ and $1_{M_1} = 0_{M_2}$ being a non-extremal idempotent.

Proof. By checking. Let us check divisibility and residuation. $\min(x, y) = x*(x \Rightarrow y)$ is evident if $x, y \in M_i$ for $i = 1$ or $i = 2$. Assume $x \in M_2 - M_1, y \in M_1 - M_2$; then $x * (x \Rightarrow y) = x * y = y$ by definition.

Residuation: we check $x * z \le y$ iff $z \le (x \Rightarrow y)$. Again the only non-trivial case is $x \in M_2 - M_1$, $y \in M_1 - M_2$. Then we have to prove $x * z \le y$ iff $z \le y$. Let $x * z \le y$, then $z \in M_1$, $z = x * z \le y$.

We generalize the above definition as follows:

Definition 4. Let $(I, \le)$ be a chain with a least element 0 and a largest element 1. For each $\alpha \in I$, let α^+ be the upper neighbour of α, if it exists, i.e. $\alpha^+ = \beta$ iff $\alpha < \beta$ and there is no γ such that $\alpha < \gamma < \beta$. Otherwise $\alpha^+ = \alpha$. Let$\{\mathbf{M}_\alpha | \alpha \in I\}$ be a system of BL-chains such that if $\mathbf{M}_\alpha$ has the least element α and largest element α^+ (thus if $\alpha = \alpha^+$ then $\mathbf{M}_\alpha$ is the one-element BL-algebra). Assume that for $\alpha \ne \beta$ the non-extremal elements of M_α are disjoint from the non-extremal elements of M_β. Let $\bigoplus_{\alpha\in I} \mathbf{M}_\alpha$ be the structure defined as follows:

The domain is $\bigcup_{\alpha\in I} M_\alpha$; for $x \in M_\alpha$, $y \in M_\beta$ we put $x \le y$ iff $\alpha < \beta$ or $[\alpha = \beta$ and $x \le_\alpha y]$.
$x * y = x *_\alpha y$ for $x, y \in M_\alpha$,
$x * y = \min(x, y)$ for $x \in M_\alpha, y \in M_\beta, \alpha \ne \beta$;
$x \Rightarrow y = 1$ if $x \le y$;
$x \Rightarrow y = x \Rightarrow_\alpha y$ if $x > y$ and $x, y \in M_\alpha$;
$x \Rightarrow y = y$ if $x \in M_\beta - M_\alpha, y \in M_\alpha - M_\beta$ and $\alpha < \beta$.

Lemma 4. Under the above notation, $\mathbf{M} = \bigoplus_{\alpha\in I} \mathbf{M}_\alpha$ is a BL-chain; for each α, $\mathbf{M}_\alpha = [\alpha, \alpha^+]_\mathbf{M}$.

Proof as above

Definition 5. Let $\mathbf{M}$ be a BL-chain. A pair $X, Y \subseteq M$ is a *cut* in $\mathbf{M}$ if
(i) $X \cup Y = M$
(ii) $x \in X$ and $y \in Y$ implies $x \le y$ for each x, y,
(iii) Y is closed under $*$
(iv) for each $x \in X$, $y \in Y$, $x * y = x$.

Lemma 5. Let X, Y be a cut. Then X is also closed under $*$; for $x \in X - Y$, $y \in Y - X$ we have $(y \Rightarrow x) = x$.

Proof. We prove the last thing, the other ones being evident. If $X \cap Y = \{d\}$ then d is an idempotent by (iv) and $(y \Rightarrow x) = x$ follows by Lemma 1 above. If $X \cap Y = \emptyset$ then evidently $z * y = x$ iff $z = y$, thus $(y \Rightarrow x) = y$.

Examples (1) Let $*$ be a continuous t-norm on $[0, 1]$ and let I be the set of its idempotents. For each $\alpha \in I$, let $\mathbf{M}_\alpha = [\alpha, \alpha^+]_\mathbf{M}$ where $\mathbf{M}$ is the t-algebra given by $*$. By the representation theorem for continuous t-norms, $\mathbf{M} = \bigoplus_{\alpha\in I} \mathbf{M}_\alpha$; each $\mathbf{M}_\alpha$ is either isomorphic to the standard MV-chain $[0, 1]_Ł$ or the standard product algebra $[0, 1]_\Pi$ or is a singleton.

(2) Let $I = \{0, 1, 2\}$, $0 < 1 < 2$; let $\mathbf{M}_0$ be the standard MV-algebra $\mathbf{M}_Ł$ on $[0, 1]$ and $\mathbf{M}_1$ the standard product algebra $\mathbf{M}_\Pi$ linearly shifted to $[1, 2]$ (thus $x * y = 1 + (x-1)(y-1)$). Let $\mathbf{M}_{01} = \mathbf{M}_0 \bigoplus_I \mathbf{M}_1$ be as defined above. M_{01} has exactly one non-

extremal idempotent 1; let $\mathbf{M} = \mathbf{M}_{01} - \{1\}$. $\mathbf{M}$ has no non-extremal idempotents and does not satisfy cancellation: for $0 < z < 1, 1 < x < y < 2$ we get $0 < x * z = y * z = z$, but $x \neq y$.

Definition 6. A BL-chain $\mathbf{M}$ is *saturated* if for each cut X, Y there is an idempotent d such that $x \in X$ and $y \in Y$ implies $x \leq d \leq y$.

Theorem 3: Each BL-chain $\mathbf{M}$ can be isomorphically embedded into a saturated BL-chain.

Proof. For a given cut X, Y in $\mathbf{M}$ such that there is no idempotent d separating X, Y (i.e. such that $x \in X$ and $y \in Y$ imply $x \leq d \leq y$) we extend $\mathbf{M}$ by such an idempotent $d = d_{XY}$ and define $x < d$ iff $x \in X, d < y$ iff $y \in Y$; $x * d = x$, for $x \in X$, $y * d = d$ for $y \in Y$. Furthermore, $x \Rightarrow d = 1$ and $d \Rightarrow x = x$ for $x \in X$, $y \Rightarrow d = d$ and $d \Rightarrow y = 1$ for $y \in Y$. The resulting algebra $\mathbf{M}' = \mathbf{M} \cup \{d\}$ is a BL-chain. Let us check associativity: let $x \in X, y \in Y$, thus $(x * d) * y = x * y, x * (d * y) = x * d = x$, thus using $x * y = x$ we get $x * y = x$ for $x \in X$ and $y \in Y$. Similarly for other axioms; we just check residuation for $y \Rightarrow d$, $y \in Y$. Indeed, $y * z \leq d$ iff $z \leq d$, thus $d = \max\{z | y * z \leq d\}$.

Now observe that you may add all the new idempotents at once (for all cuts) and that the old structure $\mathbf{M}$ is dense in the new structure $\mathbf{M}^{\infty}$: for any two new idempotents $d < d'$ there is an $x \in M, d < x < d'$. Thus there emerge no new (non-separated) cuts.

Definition 7. $\mathbf{M}$ is *reducible* if there are $\mathbf{M}_1, \mathbf{M}_2$ each having at least two elements and such that $\mathbf{M} = \mathbf{M}_1 \bigoplus \mathbf{M}_2$. $\mathbf{M}$ is *weakly reducible* if there are $\mathbf{M}_1, \mathbf{M}_2$ and an embedding f of M into $\mathbf{M}_1 \bigoplus \mathbf{M}_2$ such that both $f(M_1)$ and $f(M_2)$ have at least two elements.

Theorem 4. Each saturated BL-chain $\mathbf{M}$ is an $\bigoplus$-sum of an ordered system of saturated irreducible BL-chains.

In more details, $\mathbf{M} = \bigoplus_{\alpha \in I} [\alpha, \alpha^+]_{\mathbf{M}}$ where I is the set of idempotents of $\mathbf{M}$.

Proof obvious from the preceding.

0.6 On the problem of axiomatizing t-algebras

The problem if BL is complete with respect to t-algebras reduces to the problem if each non-degenerated[1] irreducible saturated BL-chain is either an MV-algebra or a product algebra. Recall that each MV-chain is locally embeddable into $[0,1]_{\text{Ł}}$ and each product chain (linearly ordered product algebra) is locally embeddable into $[0,1]_{\Pi}$. This means that for each finite subset X of an MV-chain M there is a finite $Y \subseteq [0,1]$ and a

[1] Having at least two elements.

bijection $f : X \to Y$ such that for all $x, y, z \in X$, $x *_M y = z$ iff $f(x) *_{\text{Ł}} f(y) = f(z)$, the same for $\Rightarrow$, and $x \leq_M y$ iff $f(x) \leq f(y)$.)

To solve positively our problem it would be enough to show that each BL-chain is locally embeddable into a t-algebra, which in turn reduces, due to the theorems of our last section, to the above question on irreducible saturated BL-chains. Indeed, given a BL-chain $\mathbf{M}$ and a finite set $X \subseteq M$, you may assume $\mathbf{M}$ to be saturated (by embedding into a bigger algebra) and is a $\bigoplus$-sum of finitely many irreducible intervals $[\alpha, \alpha^+]$ (by deleting unnecessary factors). If our question has a positive answer you might associate to each $[\alpha, \alpha^+]$ a copy of $[0,1]_{\text{Ł}}$ or $[0,1]_{\Pi}$ and a local embedding of $X \cap [\alpha, \alpha^+]$ into it; thus you might compose a t-norm such that your X is locally embedabble into the corresponding t-algebra.

This still remains open, on the other hand, one may look for some t-tautologies as possible new axioms defining a subvariety of BL-algebras, leaving the question open if these formulas are BL-provable.

We shall show that it suffices to add two axioms to get the desired completeness.

Definition 8. Let $\mathbf{M}$ be a BL-chain, let $x, y, z \in \mathbf{M}$. The triple (x, y, z) is *pathological* if $x < z < y$, $x * y = x$, $x * z < x$ and $z * y < z$.

Lemma 6. (1) If (x, y, z) is pathological then x, y, z are from the same component $[\alpha, \alpha^+]$ for some idempotent α.

(2) If $\mathbf{M}$ is t-algebra then $\mathbf{M}$ has no pathological triples.

Proof. (1) By Lemma 1 (2), there is no idempotent between x, z and no between z, x.

(2) By (1), $x, y, z \in [\alpha, \alpha^+]$ for some $\alpha \in [0,1]$. Since $\mathbf{M}$ is a t-algebra, $[\alpha, \alpha^+]$ is isomorphic to $[0,1]_{\text{Ł}}$ or $[0,1]_{\Pi}$. Obviously, $x * z < x$ implies $x > \alpha$, and $z * y < z$ implies $y < \alpha^+$. but then $x * y < x$ (verify easily for $0 < x < y < 1$ in $[0,1]_{\text{Ł}}$, $[0,1]_{\Pi}$).

Corollary 1. If $\mathbf{M}$ has a pathological triple then it is not locally embeddable into any t-algebra.

Lemma 7. Let $\mathbf{M}$ be saturated, irreducible and without pathological triples. Then $x * y = x$ implies $x = 0$ or $y = 1$.

Proof Assume $x * y = x, x > 0, y < 1$. For each z, let $z \in X_0$ iff $z * y = z$, and $z \in Y$ iff $z * x = x$. Clearly $x \in X_0$ and $y \in Y$; and each z belongs either to X_0 or to Y. Y is closed under $*$ (evident) and $x \notin Y$ due to irreducibility. Put $X = M - Y$; we prove that for $u \in X$ and $v \in Y$, we have $u * v = u$.

First assume $u \leq x < v$. Then $u * v = x * (x \Rightarrow u) * v = x * (x \Rightarrow u) = u$ (since $v \in Y$).

Let $x < u < v$: We have $u * x < x$ (since $u \notin Y$), $x * v = x$ and hence $u * v = u$, otherwise (x, u, v) would be a pathological triple. This contradicts irreducibility of $\mathbf{M}$.

Lemma 8. A saturated irreducible BL-chain without pathological triples satisfies the following cancellation: if $x * z = y * z > 0$ then $x = y$.

Proof. Let $x * z = y * z > 0$ and $x < y$; then $x = y * (y \Rightarrow x)\,(= \min(x, y))$. Then $y * z = (y * z) * (y \Rightarrow x)$ and $(y \Rightarrow x) < 1$, which gives $y * z = 0$ by the preceding lemma.

Corollary 2. A saturated irreducible BL-chain without pathological triples and without non-trivial zero divisors (i. e. $x, y > 0$ implies $x * y > 0$) is a product algebra.

Proof. See [3] 1.6.9 and 4.1.8. Indeed, if $0 < x \leq y$ then $x = y * (y \Rightarrow x)$ and $(y \Rightarrow x)$ the unique element u satisfying $x = y * u$, thus we have positive subtraction. Moreover, $x \geq x * y$ for all x, y.

Theorem 5. (1) The following formula (B1)

$$(\varphi \to (\varphi \& \chi)) \vee (\chi \to \varphi) \vee (\psi \to \chi) \vee ((\varphi \& \psi) \to (\varphi \& \psi)^2) \vee$$
$$\vee[(\varphi \to (\varphi \& \psi)) \to (\chi \to (\chi \& \psi))]$$

is a t-tautology.

(2) If (B1) is true in a BL-chain $\mathbf{M}$ then $\mathbf{M}$ has no pathological triples.

Proof. We have to prove the folowing claim $(*)$: IF $x * z < x$, $x < z$, $z < y$, $(x*y)^2 < x*y$ THEN $x \Rightarrow (x*y) \leq z \Rightarrow (z*y)$. Note that if $(*)$ is true for all x, y, z from a BL-chain $\mathbf{M}$ then, in particular, IF $x * z < x$, $x < z$, $z < y$, $(x * y)^2 < (x * y)$ AND $x = x * y$ THEN $z = z * y$.

Now observe that $x * z < x$, $x < z < y$ AND $x = x * y$ implies that $(x * y)$ is not an idempotent: if $x * y = u = u^2$ then $x = x * y \leq z * y \leq z$, thus $x \leq u \leq z$ and $u = u^2$, hence $x * z = x$, a contradiction. Thus we get IF $x * z < x, x < z < y$ AND $x * y = x$ THEN $z = z * y$, hence IF $x < z < y$ AND $x * y = x$ THEN $(x * z = x$ OR $z * y = z)$, hence $\mathbf{M}$ has no pathological triples. This proves (2).

Thus let us verify $(*)$ in any t-algebra. Let $x * z < x$, $(z * y)$ non-idempotent and $x < z < y$. Then x, z are from the same component $[\alpha, \alpha^+]$. If z, y are separated by an idempotent then $z * y = z$, hence $z \Rightarrow (z * y) = 1$. Otherwise also y is in $[\alpha, \alpha^+]$. Now $[\alpha, \alpha^+]$ is isomorphic either to $[0,1]_{Ł}$ or $[0,1]_{\Pi}$.

Case 1: Ł. We have to prove $1 - x + x * y \leq 1 - z + z * y$, i. e. $z - x \leq (z * y - x * y)$. Recall $x < z$ and $x * y > 0$ (since $x * y$ is not idempotent). Thus $z * y - x * y = z + y - 1 - x - y + 1 = z - x$.

Case 2: Π. The assumption $x * z < x$ implies $x > 0$, hence $y, z, x \cdot y, z \cdot y > 0$. We have to prove $(x \Rightarrow x * y) \leq z \Rightarrow (z * y)$; we even prove equality. Indeed, $x \Rightarrow x * y = (x \cdot y)/x = y = (z \cdot y)/z = z \Rightarrow y)$.

This completes the proof of the whole theorem.

Definition 9. $N(y, z)$ stands for $(y \Rightarrow z) \Rightarrow z$.

Lemma 9. The identity $N(y, z) = y$ holds
(1) in $[0,1]_{Ł}$ for all $y \geq z$,
(2) in $[0,1]_{\Pi}$ for all $y \geq z > 0$.

Proof. (1) Assuming $y \geq z$,

$$(y \Rightarrow z) \Rightarrow z = (1 - y + z) \Rightarrow z = 1 - 1 + y - z + z = y.$$

(2) Assuming $y \geq z > 0$,

$$((y \Rightarrow z) \Rightarrow z) = (z/y) \Rightarrow z = z/(z/y) = y.$$

Corollary 3. Each t-algebra satisfies the following:
IF $x^2 < (x \cap y)$ AND $x * y < (x \cap y)$ THEN $N(y, x^2) = y$.

Proof. The assumption guarantees that x, y are from the same component, $y > x^2$ and $x > x^2$. Thus if $[\alpha, \alpha^+]$ is the component in question it is either isomorphic to $[0,1]_{Ł}$ or it is isomorphic to $[0,1]_\Pi$ and, in this latter case, $x^2 > 0$. Hence the result follows by the preceding lemma.

Theorem 6. Let (B2) be the formula

$$((\varphi \wedge \psi) \rightarrow \varphi^2) \vee ((\varphi \wedge \psi) \rightarrow (\varphi \& \psi)) \vee [((\psi \rightarrow \varphi^2) \rightarrow \varphi^2) \rightarrow \psi]$$

(1) (B2) is a t-tautology.

(2) Each irreducible saturated BL-chain satisfying (B1) and (B2) and having a non-trivial zero divisor ($x > 0$, $x^2 = 0$) is an MV-algebra.

Proof. We have to verify for each t-algebra:
IF $x^2 < \min(x, y)$ AND $x * y < \min(x, y)$ THEN $N(y, x^2) = y$. But this is just the corollary above.

To prove (2) observe that if ($x > 0$, $x^2 = 0$) and $0 < y < 1$ then $x*y < \min(x, y)$ due to a lemma above and the condition from the Corollary verifies $N(y, 0) = y$ (and obviously, $N(0,0) = 0$ and $N(0,1) = 1$). Thus the algebra in question satisfies $(-)(-)x = x$ and hence is an MV-algebra.

Completeness theorem 7. Let $BL^\sharp$ be the logic BL + (B1) + (B2), let φ be any formula.

$BL^\sharp \vdash \varphi$ iff φ is a t-tautology, i. e. φ is a tautology over any t-algebra. Thus $BL^\sharp$ is the logic of continuous t-norms.

Proof. Soundness is clear. Assume $BL^\sharp \not\vdash \varphi$. Then, by the completeness for BL, there is a BL-chain $\mathbf{M}$ satisfying (B1) and (B2) and an evaluation e of propositional variables in M such that $e_{\mathbf{M}}(\varphi) < 1_{\mathbf{M}}$. Let X be a finite subset of M containing $0_{\mathbf{M}}, 1_{\mathbf{M}}$ and values of all subformulas ψ of φ under $e_{\mathbf{M}}$. Assume $\mathbf{M}$ saturated, $\mathbf{M} = \bigoplus_{i=1}^n [\alpha_i, \alpha_i^+]$ let $0 = u_1 < \ldots < u_{n+1} = 1$ be rationals from the unit interval and let $X_i = X \cap [\alpha_i, \alpha_i^+]$. Define $u_i^+ = u_{i+1}$ for $i < n+1$. Construct a t-norm $*'$ whose restriction to $[u_i, u_i^+]$ is isomorphic to $[0,1]_{Ł}$ if $[\alpha_i, \alpha_i^+]$ is an MV-algebra; else to $[0,1]_\Pi$ if $[\alpha_i, \alpha_i^+]$ is a product algebra; For $i < n$ let f_i be a mapping of X_i into $[u_i, u_i^+]$ which is a partial isomorphism with respect to the operation $*$ of $\mathbf{M}$ and the operation $*'$ on $[0,1]$; let $f = \bigcup f_i$. Then f is a partial isomorphism of X into $[0,1]$ with respect to $(*, *')$ and hence also with respect to the corresponding implication $\Rightarrow, \Rightarrow'$. Let $\mathbf{M}'$ be the t-algebra given by $*'$ on $[0,1]$.

Let $e'(p_i) = f(e(p_i))$; then obviously, $e'_{\mathbf{M}'}(\psi) = f(e_{\mathbf{M}}(\psi))$ for each subformula ψ of φ and hence $e'_{\mathbf{M}'}(\varphi) < 1$; φ is not a t-tautology. This completes the proof.

Remark. It follows that BL-algebras satisfying all t-tautologies form a variety - the variety given by axioms of $BL^\sharp$. The problem remains if (B1), (B2) are provable in BL; on the other hand, each of the logics Ł, G, Π proves (B1), (B2). (Exercise: find corresponding proofs!) A possibly more easy problem is: simplify (B1), (B2) - admittedly they are not too inviting.

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