Product of Matrices and P-Class

Hakl, František
1997

INSTITUTE OF COMPUTER SCIENCE

ACADEMY OF SCIENCES OF THE CZECH REPUBLIC

Product of Matrices and P-Class

František Hakl

Technical report No. 731

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic
phone: (+4202) 66 05 34 40 fax: (+4202) 85 85 789
e-mail: hakl@uivt.cas.cz

INSTITUTE OF COMPUTER SCIENCE

ACADEMY OF SCIENCES OF THE CZECH REPUBLIC

Product of Matrices and P-Class

František Hakl[1]

Technical report No. 731

Abstract

Necessary and sufficient conditions are given for following properties of two square matrices: 1) product of matrices $\mathbf{A}\cdot\mathbf{B}$ is in the class P, 2) product $\mathbf{A}\cdot\mathbf{D}\cdot\mathbf{B}$ is in the class P for all positive diagonal matrices $\mathbf{D}$. We use a notion of W-property, which is together with a W_0-property significant to investigate properties mentioned above.

Keywords
M–matrices, interval matrices, P–matrices

[1]This work was supported by the Grant Agency of the Czech Republic under grant 201/95/0979.

1 Introduction

A matrix $\mathbf{M} \in R^{n \times n}$ is called a P-matrix if every principal minor of $\mathbf{M}$ is positive. This notion, introduced by Fiedler and Pták [FP66], have numerous applications in diverse fields. The class of P-matrices is not closed under product of matrices. To study this problem we use [†]an extension of Wilson's concept [Wil71] which introduced a notion of W_0-pair of matrices to study certain equations arising in nonlinear DC-networks. We show that the following W-property of matrices is a very useful tool to study the relationship of matrix product and the class P. We extend the results proved by Johnson and Tsatsomeros [JT95] and show that necessary and sufficient condition to $\mathbf{A} \cdot \mathbf{B} \in P$ can be expressed using some properties of finite set of matrices.

Definition 1 *Let $\mathbf{A}_i$ denote the i-th column of a matrix $\mathbf{A}$. Then an $N \times N$ matrix $\mathbf{B}$ is called a representative matrix of a system of square matrices $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$ if*

$$(\ \forall i \in \{1, \ldots, N\}\ )\left(\mathbf{B}_i \in \left\{\mathbf{M}_{1i}, \ldots, \mathbf{M}_{ki}\right\}\right).$$

Definition 2 *A matrix system $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$ has the W-property if* $\det \mathbf{A} \cdot \det \mathbf{B} > 0$ *holds for every representative matrix $\mathbf{A}$ and $\mathbf{B}$ of $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$.*

Definition 3 *A matrix system $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$ has the W_0-property if there exists a representative matrix $\mathbf{A}$ such that* $\det \mathbf{A} \neq 0$ *and* $\det \mathbf{A} \cdot \det \mathbf{B} \geq 0$ *holds for every representative matrices $\mathbf{B}$ of $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$.*

In the sequel, we show that the question whether the matrix $\mathbf{A} \cdot \mathbf{B}$ is a P-matrix can be reduced to the problem of nonsingularity of a certain set of square matrices. We start with this simple result

Lemma 1 *Let $\mathbf{B}$ be a nonsingular $N \times N$ matrix and let $\mathbf{R}$ be an $N \times N$ matrix whose rows, except the j-th, are zero. Then $1 + (\mathbf{R}\mathbf{B}^{-1})_{jj} \leq 0$ if and only if there exists a $\gamma \in (0, 1]$ such that $\mathbf{B} + \gamma \mathbf{R}$ is singular.*

which is proven by Rohn in [Roh90].

The next two lemmas are based on Willson's results about the matrix pair which have been proven in [Wil71] using multilinearity of determinant and which have been reformulated in general case by Sznajder and Gowda [SG95] (Lemma 3).

Lemma 2 *For matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ the following two statements are equivalent:*

1. *$(\mathbf{M}_1, \mathbf{M}_2)$ has the W_0-property.*
2. $\det(\mathbf{M}_1\mathbf{D} + \mathbf{M}_2) \neq 0$ *for every diagonal matrix $\mathbf{D} > 0$.*

Lemma 3 *For matrix system $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$ the following statements are equivalent:*

[†]The author would like to thank Professor Miroslav Fiedler for his helpful comments.

1. $(\mathbf{M}_1, \ldots, \mathbf{M}_k)$ *has the W-property.*

2. *For arbitrary nonnegative diagonal matrices* $(\mathbf{D}_1, \ldots, \mathbf{D}_k)$ *with* $diag(\mathbf{D}_1 + \ldots + \mathbf{D}_k) > 0$,

$$\det(\mathbf{M}_1\mathbf{D}_1 + \ldots + \mathbf{M}_k\mathbf{D}_k) \neq 0.$$

3. $\mathbf{M}_k$ *is invertible and* $(\mathbf{I}, \mathbf{M}_k^{-1} \cdot \mathbf{M}_1, \ \ldots, \ \mathbf{M}_k^{-1} \cdot \mathbf{M}_k)$ *has the W-property.*

2 Product properties

The preceding lemmas allow us to present the following result about product of arbitrary matrices:

Theorem 4 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be nonsingular matrices of order* N. *Then*

1. $\mathbf{A} \cdot \mathbf{B} \in P$ *if and only if* $(\mathbf{A}^T, (\mathbf{B}^{-1})^T)$ *has the W-property.*

2. $\mathbf{B} \cdot \mathbf{A} \in P$ *if and only if* $(\mathbf{A}, \mathbf{B}^{-1})$ *has the W-property.*

Proof: This proof is based on Lemma 1 and on idea of proof in [Roh90]. Let us denote a subset $\mathbf{H}$ of R^N space by

$$\mathbf{H} = \left\{ \mathbf{x} \in R^N | (\exists i \in \{1, \ldots, N\})(\mathbf{x}_i \in [0,1] \quad \text{and} \quad \mathbf{x}_j \in \{0,1\} \text{ for } i \neq j) \right\}$$

$\mathbf{H}$ is union of all edges of hypercube). Now, since the determinant of the matrix is linear function with regard to its rows, the second part of statement 1) it implies, that the property $\det((\mathbf{I} - \mathbf{Y})\mathbf{A} + \mathbf{Y}\mathbf{B}^{-1}) \cdot \det((\mathbf{I} - \mathbf{Z})\mathbf{A} + \mathbf{Z}\mathbf{B}^{-1}) > 0$ is satisfied for all nonnegative diagonal matrices $\mathbf{Y}$ and $\mathbf{Z}$ with $\mathbf{Y}_{ii} \leq 1, \mathbf{Z}_{ii} \leq 1, i \in \{1, \ldots, N\}$.
In fact, if we move from one vertex of hypercube $\mathbf{H}$ to some adjacent vertex along the edge, the determinant will be changed linearly. Hence the same sign of determinants in all vertices it implies that there does not exist a point on the edge, where the determinant has a value which is zero or which has the opposite sign.

Now we prove 1) in several steps. First we show that all leading principal minors $\gamma_j, j \in \{1, \ldots, N\}$, of $\mathbf{A} \cdot \mathbf{B}$ are positive. Put $\mathbf{R} = \mathbf{A} - \mathbf{B}^{-1}$ so that $\mathbf{A} \cdot \mathbf{B} = \mathbf{I} + \mathbf{R} \cdot \mathbf{B}$, and denote by $\mathbf{R}_j$, $j \in \{1, \ldots, N\}$, the matrix whose first j rows are identical with those of $\mathbf{R}$ and the remaining ones are zero. Then

$$\gamma_j = \det(\mathbf{I} + \mathbf{R}_j \cdot \mathbf{B})$$

holds for all $j \in \{1, \ldots, N\}$. We shall prove by induction that $\gamma_j > 0, j \in \{1, \ldots, N\}$:
Case $j = 1$: Since $\gamma_1 = \det(\mathbf{I} + \mathbf{R}_1 \cdot \mathbf{B}) = 1 + (\mathbf{R}_1 \cdot \mathbf{B})_{11}$, Lemma 1 implies $\gamma_1 > 0$, for otherwise the matrix $\mathbf{B}^{-1} + \alpha\mathbf{R}_1$ would be singular for some $\alpha \in (0, 1]$ which contradicts linearity of determinant.
Case $j > 1$: Assume that $\gamma_{j-1} > 0$ and consider the matrix

$$(\mathbf{I} + \mathbf{R}_j \cdot \mathbf{B}) \cdot (\mathbf{I} + \mathbf{R}_{j-1} \cdot \mathbf{B})^{-1} = \mathbf{I} + (\mathbf{R}_j - \mathbf{R}_{j-1}) \cdot \left(\mathbf{B}^{-1} + \mathbf{R}_{j-1}\right)^{-1}.$$

Taking determinant of both sides we obtain

$$\frac{\gamma_j}{\gamma_{j-1}} = 1 + \left((\mathbf{R}_j - \mathbf{R}_{j-1}) \cdot \left(\mathbf{B}^{-1} + \mathbf{R}_{j-1}\right)^{-1}\right)_{jj}.$$

If the right-hand side is nonpositive then according to Lemma 1 $\mathbf{B}^{-1} + \mathbf{R}_{j-1} + \alpha(\mathbf{R}_j - \mathbf{R}_{j-1})$ would be singular for some $\alpha \in (0, 1]$ which is a contradiction with linearity of determinant. Hence $\gamma_j > 0$ holds, which concludes the inductive proof.

Second we shall prove that each principal minor of $\mathbf{A} \cdot \mathbf{B}$ is positive. Consider the principal minor formed from the rows and columns with indices in $\phi \subset \{1, \ldots, N\}$. Let $\mathbf{P}$ be any permutation matrix such that the above minor is equal to the leading principal minor of

$$\mathbf{P}^T(\mathbf{A} \cdot \mathbf{B})\mathbf{P} = \left(\mathbf{P}^T\mathbf{A}\mathbf{P}\right) \cdot \left(\mathbf{P}^T\mathbf{B}^{-1}\mathbf{P}\right)^{-1}.$$

Since the matrices $\mathbf{P}^T\mathbf{A}\mathbf{P}$ and $\mathbf{P}^T\mathbf{B}^{-1}\mathbf{P}$ satisfy the assumption of 1), all leading principal minors of $\left(\mathbf{P}^T\mathbf{A}\mathbf{P}\right) \cdot \left(\mathbf{P}^T\mathbf{B}^{-1}\mathbf{P}\right)^{-1}$ are positive due to the preceding part of the proof.

To prove part 2), consider that $\mathbf{B} \cdot \mathbf{A} \in P \Leftrightarrow \mathbf{A}^T \cdot \mathbf{B}^T \in P$. According to part 1), it is equivalent to the fact that $(\mathbf{A}, \mathbf{B}^{-1})$ has the W-property. This completes the proof.

■

As follows from this theorem, the question if a product of two square matrices is in the class P can be solved by checking signs of determinants of 2^N matrices.

Example:
Let $\mathbf{A} = \left(\begin{smallmatrix} 2 & 1 \\ 4 & 6 \end{smallmatrix}\right), \mathbf{B}^{-1} = \frac{1}{8}\left(\begin{smallmatrix} 6 & 4 \\ 1 & 2 \end{smallmatrix}\right)$. Then (note that both $\mathbf{A}$ and $\mathbf{B}$ are P-matrices):

1. $\mathbf{A} \cdot \mathbf{B} = \left(\begin{smallmatrix} 3 & -2 \\ 2 & 20 \end{smallmatrix}\right) \in P$ because $\left|\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right| > 0$ and $\left|\begin{smallmatrix} 6 & 4 \\ 4 & 6 \end{smallmatrix}\right| > 0$
2. $\mathbf{B} \cdot \mathbf{A} = \left(\begin{smallmatrix} -12 & -22 \\ 22 & 35 \end{smallmatrix}\right) \notin P$ because $\left|\begin{smallmatrix} 2 & 4 \\ 4 & 2 \end{smallmatrix}\right| < 0$.

The preceding theorem allows us to formulate an equivalent condition to the fact that for given square matrices $\mathbf{A}$ and $\mathbf{B}$ all matrices in the form $\mathbf{A} \cdot \mathbf{D} \cdot \mathbf{B}^{-1}$ are P-matrices for arbitrary diagonal positive matrix $\mathbf{D}$.

Notation:
Let Δ be the set of square diagonal matrices of the order N with positive diagonal and let Ω be the set of all diagonal matrices $\mathbf{D}$ with $\mathbf{D}_{ii} \in \{0, 1\}$.

Theorem 5 *Let $\mathbf{A}$, $\mathbf{B}$ be nonsingular matrices of order N. Then the following statements are equivalent:*

1. $\mathbf{A}\mathbf{D}\mathbf{B}^{-1}$ *is a P-matrix for each positive diagonal matrix* $\mathbf{D}$.
2. $\mathbf{A}^{-1}\mathbf{D}\mathbf{B}$ *is a P-matrix for each positive diagonal matrix* $\mathbf{D}$.

3. $\det((\mathbf{I}-\mathbf{Y})\mathbf{A}(\mathbf{I}-\mathbf{Z})+\mathbf{YBZ})\cdot\det\mathbf{A}\geq 0$ *holds for all* $\mathbf{Y},\mathbf{Z}\in\Omega$ *for which* $\sum_{i=1}^{N}\mathbf{Y}_{ii}=\sum_{i=1}^{N}\mathbf{Z}_{ii}$.

Proof:
Without loss of generality we can assume that $\det\mathbf{A}>0$. Then the property $(\forall\mathbf{D}\in\Delta)(\mathbf{ADB}^{-1}\in P)$ is equivalent to (see statement 1) of Theorem 4)

$$(\forall\mathbf{D}\in\Delta)(\forall\mathbf{Y}\in\Omega)(\det((\mathbf{I}-\mathbf{Y})\mathbf{AD}+\mathbf{YB})>0)\ .$$

But by Lemma 2 it is equivalent to

$$(\forall\mathbf{Z}\in\Omega)(\forall\mathbf{Y}\in\Omega)(\det((\mathbf{I}-\mathbf{Y})\mathbf{A}(\mathbf{I}-\mathbf{Z})+\mathbf{YBZ}\geq 0) \tag{2.1}$$

and by the same lemma it is equivalent to the fact that

$$(\forall\mathbf{D}\in\Delta)(\forall\mathbf{Z}\in\Omega)(\det(\mathbf{A}(\mathbf{I}-\mathbf{Z})+\mathbf{DBZ})\neq 0).$$

But at the same time $\det\mathbf{A}>0$ which follows

$$(\forall\mathbf{D}\in\Delta)(\forall\mathbf{Z}\in\Omega)(\det(\mathbf{A}(\mathbf{I}-\mathbf{Z})+\mathbf{DBZ})>0).$$

Using statement 2) of Theorem 4 we finally obtain

$$(\forall\mathbf{D}\in\Delta)(\mathbf{A}^{-1}\mathbf{DB}\in P)\ .$$

Hence 1) $\Rightarrow$ 2). Implication 2) $\Rightarrow$ 1) follows from reversing above arguments. Therefore both 1) and 2) are equivalent to (2.1). But each matrix from (2.1) can be by some convenient (not necessary simultaneous) permutation of its rows and columns expressed in the form

$$\begin{pmatrix}\mathbf{A}^* & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^*\end{pmatrix}$$

where matrices $\mathbf{A}^*$ and $\mathbf{B}^*$ are submatrices of $\mathbf{A}$ and $\mathbf{B}$ respectively. Hence the determinant of this matrix can be nonzero only if both matrices $\mathbf{A}^*$ and $\mathbf{B}^*$ are square matrices and this can happen if and only if $\sum_{i=1}^{N}\mathbf{Y}_{ii}=\sum_{i=1}^{N}\mathbf{Z}_{ii}$.

■

It follows that each statement of the preceding theorem implies both statements of Theorem 4. The converse implication is not true. We can demonstrate it by the counter-example.

Example:
Let $\mathbf{A}=\left(\begin{smallmatrix}2 & 1\\ -1 & 2\end{smallmatrix}\right),\mathbf{B}=\left(\begin{smallmatrix}2 & -1\\ 1 & 2\end{smallmatrix}\right)$. Then $\mathbf{B}^{-1}=\frac{1}{5}\mathbf{A}$ and therefore $\mathbf{A}\cdot\mathbf{B}^{-1}=\frac{1}{5}\mathbf{A}^2=(\mathbf{A}^{-1}\cdot\mathbf{B})^{-1}=\frac{1}{5}\left(\begin{smallmatrix}3 & 4\\ -4 & 3\end{smallmatrix}\right)\in P$ which implies that both statements of Theorem 4 are satisfied. Now let $\left(\begin{smallmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{smallmatrix}\right)=\mathbf{A}\cdot\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\cdot\mathbf{B}^{-1}$. Then $\gamma_1=\frac{1}{5}(4\alpha-\beta)$ and $\gamma_4=\frac{1}{5}(4\beta-\alpha)$ and the statements of Theorem 5 can not be satisfied, which corresponds to the fact that for $\mathbf{Z}=\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $\mathbf{Y}=\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$ we have $\det((\mathbf{I}-\mathbf{Y})\mathbf{A}(\mathbf{I}-\mathbf{Z})+\mathbf{YBZ})=\left|\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right|<0$.

Note that there exist $\binom{2n}{n}$ matrices of the preceding form in general case.

3 Conclusion

Results concerning the connection between properties of product of square matrices and the class P are presented. It is shown that W-property of two matrices is equivalent to the fact that product one of them with inverse of second of them belongs to the set of P-matrices.

In spite of the fact that W-property is applicable in exponential time in a general and also that a decision problem if a matrix $\mathbf{M}$ posses a nonpositive principal minor is NP-Complete [Cox94], techniques used to prove results presented here could be useful in investigation of the above problems restricted to more special matrices.

Bibliography

[Cox94] Gregory E. Coxson. The P-matrix problem is co-NP-complete. *Mathematical Programing*, 64:173–178, 1994.

[FP66] Miroslav Fiedler and Vlastimil Pták. Some generalizations of positive definiteness and monotonicity. *Numerische Mathematik*, 9:163–172, oct 1966.

[JT95] C. R. Johnson and M. J. Tsatsomeros. Convex sets of nonsingular and P-matrices. *Linear and Multilinear Algebra*, 38:233–239, 1995.

[Roh90] Jiří Rohn. Nonsingularity and P-matrices. *Aplikace Matematiky*, 35(3):215–219, 1990.

[SG95] R. Sznajder and M. S. Gowda. Generalization of P_0-and P-properties; extended vertical and horizontal linear complementarity problems. *Linear Algebra and its Applications*, 223:695–715, jul 1995.

[Wil71] A. N. Wilson Jr. A useful generalization of the P_0-matrix concept. *Numerische Mathematik*, 17:62–70, 1971.