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Abstract

Well-formed formulas of the classical first-order predicate language without functions are evaluated in such a way that truthvalues are subsets of the set of all positive integers. Such an evaluation is projected in two different ways into the unit interval of real numbers so that two real-valued evaluations are obtained. The set of tautologies is proved to be identical, in all the three cases, with the set of classical first-order predicate tautologies, but the induced evaluations meet the properties of probability and possibility measures with respect to non-standard supremum and infimum operations induced in the unit interval of real numbers.

Keywords

probabilistic logic, first–order predicate calculus, Boolean algebra, nonstandard models, Boolean–valued models, probability measure, possibility measure

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1 Introduction

Even if there are the probability theory, and the mathematical statistics based on this theory, which have been playing, since the 18th century, the role of the dominant mathematical tool for uncertainty quantification and processing, the attempts to build alternative mathematical tools for these sakes, paradigmatically more close to formalized logical deductive calculi, are also numerous, important and interesting. The resulting mathematical models are usually subsumed under the common general notion "non-classical logics" and can be divided, roughly speaking, into three groups.

(i) *Modal logics* follow the pattern which emphasizes rather the qualitative than the quantitative aspects of the notions like possibility or necessity. This goal is reached by enriching the language of an appropriate logical calculus by new symbols for the functors like "it is possible that..." or "it is necessary that...", and by choosing a collection of axioms for the original as well as for the new, modal functors. Such a choice leads, as a rule, to a compromise between the intuitions and the common language feelings and connotations behind the modal functors, and the methodological (meta-logical) demands which must be obeyed when creating a deductive formalized system. The qualitative character of modal logics is also demonstrated by the fact that when defining semantical models of these logics based on the space of possible worlds (Kripke semantics), what matters is the fact whether the subsets of possible worlds, corresponding to some formulas, are empty or finite, or whether their complements possess these qualitative properties. However, if these subsets are beyond the scope of these extremal or almost extremal cases, their relative or absolute sizes (extends), measured by some quantitative numerical measure, do not play any important role.

(ii) *Fuzzy logics* are oriented toward quantification and processing of the notions like vagueness, impreciseness or fuzzines. They are based on the idea that formulas of the formalized language in question may be interpreted as taking not only the two qualitative values "true" and "false", but also some values "between these two ones". From the formal point of view this goal is reached in such a way that the classical qualitative truthvalues are identified with the extremal points 1 (true) and 0 (false) of the unit interval of real numbers, and the formulas are supposed to be able to take also truthvalues identified with (some or all) real numbers from the inside of the unit interval, i.e. from $(0,1)$. There are numerous variants of such formalized calculi based on different systems of functors and quantifiers, and on different ways of interpretations of these functors and quantifiers as functions from the truthvalues of the composing more elementary formula(s) to the truthvalue of the resulting composed formula. In every case, two aspects are emphasized by fuzzy logics: (1) the extensional character of all functiors and quantifies, i.e., as just mentioned, truthvalues of composed formulas are functions of truthvalues of their components, and (2) the notions of vagueness, impreciseness or fuzziness, to the description and processing of which fuzzy logics are applied, are supposed to be qualitatively different from the notions of uncertainty and randomness described and processed by probabilistic and statistical tools, and they are also supposed to be of extensional character, or at least to be allowed to be processed by formal tools preserving the extensional character of functors without the risk of arriving at some contradiction.

(iii) *Probabilistic logics* copy fuzzy logics as far as the truthvalues ranging over the unit interval of real numbers are concerned. However, probabilistic logics insist on the possibility to understand these values as probabilities, even if this demand implies the non-extensionality of the used functors (contrary to fuzzy logics when the possibility to interprete truthvalues as probabilities is abandoned in every case when it conflicts the demand of extensionality of all functors and quantifiers). Hence, probabilistic logics can be seen as alternative apparatus, if related to probability theory and mathematical statistics, for uncertainty and randomness quantification and processing based rather on the paradigmatical and methodological grounds of deductive formalized systems than on the grounds of measure theory, real functions and integral calculus, as it is the case of probability theory and mathematical statistics.

In what follows, we shall continue in our effort from [6] and we shall equip the classical syntax of the first-order predicate calculus without functions by a boolean-valued semantic. This semantic will induce two real-valued semantics taking their values in the unit interval of real numbers. One of them will be extensional, the other one will be intensional (i. e., non-extensional), but both of them will copy (each of them in a different way) the flexibility of probabilistic measures when various kinds and degrees of stochastical (statistical) dependence among propositions taken as random events are considered.

2 The syntax of the first-order predicate calculus without functions

Let us consider the formalized language $\mathcal{L}_{\rm PC}$ of the first-order predicate calculus (FOPC) without functions (function symbols) defined recursively as follows (cf. [1] or [9] for more detail).

(i) R_1, $R_2, \ldots, R_n$ are *relation symbols*; to each R_i a positive integer r_i called the *arity* of (the relation symbol) R_i is ascribed. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ denote the set of all relation symbols of the language $\mathcal{L}_{\rm PC}$.

(ii) x_1, $x_2, \ldots$ is an infinite sequence of *variables* or *indeterminates* (variable or indeterminate symbols), for the sake of convenience we shall denote variables also by the lower-case letters y, z, u, v and w, perhaps with indices. Let $\mathit{Var} = \{x_1, x_2, \ldots\}$ denote the set of all variables.

(iii) c_1, $c_2, \ldots$ is an infinite sequence of *constants* (constant symbols). The set of all constants is denoted by *Const*. It is also possible to take constants as relation symbols of zero arity.

(iv) For each $i \leq n = \mathrm{card}(\mathcal{R})$, and for each $j \leq r_i$, let $t_j \in \mathit{Var}$ or $t_j \in \mathit{Const}$. Then the finite sequence $R_i(t_1, t_2, \ldots, t_{r_i})$ of symbols is called an *elementary well-formed formula* (w. f. f.) of the language $\mathcal{L}_{\rm PC}$.

(v) Elementary w. f. f.'s of $\mathcal{L}_{\rm PC}$ are w. f. f.'s of $\mathcal{L}_{\rm PC}$. If A, B are w. f. f.'s, then $(A \to B)$ and $(\neg A)$ are w. f. f.'s of $\mathcal{L}_{\rm PC}$. $(A \to B)$ is called *implication* and $\to$ is called the *implication functor* or *operator*, $(\neg A)$ is called *negation* and $\neg$ is called the *negation functor* or *operator*.

(vi) If A is a w. f. f. of $\mathcal{L}_{\mathrm{PC}}$ and $x \in \mathit{Var}$, then the sequence $((\forall\, x)\, A)$ of symbols is also a w. f. f. of $\mathcal{L}_{\mathrm{PC}}$. The symbol $\forall$ is called the *universal quantifier*.

(vii) New binary operators $\wedge$ (*conjunction*), $\vee$ (*disjunction*) and $\leftrightarrow$ (*equivalence*) and new *existential quantifier* $\exists$ are introduced in order to abbreviate the notation for some w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$. Namely,

$$(A \vee B) \;=_{\mathrm{df}}\; ((\neg A) \to B), \tag{2.1}$$

$$(A \wedge B) \;=_{\mathrm{df}}\; (\neg((\neg A) \vee (\neg B))), \tag{2.2}$$

$$(A \leftrightarrow B) \;=_{\mathrm{df}}\; ((A \to B) \wedge (B \to A)), \tag{2.3}$$

$$((\exists\, x)\, A) \;=_{\mathrm{df}}\; (\neg((\forall\, x)(\neg A))). \tag{2.4}$$

As a rule, the outermost pairs of brackets will be omitted, if no misunderstanding menaces.

(viii) There are no other w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ beyond these generated by a finite number of applications of the rules (i) – (vii).

Hence, we consider only the most simple terms, namely variables and constants, but we do not take into consideration functional terms of the kind $f(t_1, t_2, \ldots, t_k)$, where f is a k-ary functional symbols and t_1, $t_2, \ldots, t_k$ are terms of the predicate language in question. This simplification does not reduce substantially the expressive powers of the defined first-order predicate language, as functions can be defined by relations satisfying some further conditions. We also do not consider different sorts of variables.

Let us introduce, in the usual way, the notion of *theorem* (*provable* or *deducible formula*) of the first-order predicate formula without functions. Let A be a w. f. f. of $\mathcal{L}_{\mathrm{PC}}$ obtained in such a way that w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ are substituted into a propositional tautology under the condition that all occurrences of the same propositional variables are replaced by the same w. f. f. of $\mathcal{L}_{\mathrm{PC}}$. Then A is called an *axiom* of FOPC. Let A be a w. f. f. of $\mathcal{L}_{\mathrm{PC}}$, let $x \in \mathit{Var}$, and let $t \in \mathit{Var} \cup \mathit{Const}$ be such that, if $t \in \mathit{Var}$ and if replacing all the occurrences of x in A by t, then in the resulting w. f. f. $A(t)$ no occurrence of t will be bound by a quantifier at a position where x occurred freely in the original formula A. Then the formulas

$$((\forall\, x)\, A) \to A(t), \qquad A(t) \to ((\exists\, x)\, A) \tag{2.5}$$

are also called axioms of FOPC.

All axioms of FOPC are *theorems* (*provable* or *deducible formulas*) of FOPC. If A and $A \to B$ are theorems of FOPC, then B is also a theorem of FOPC. If A, B are w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ and x is a variable such that x does not occur freely in A, then the following implications hold: if $A \to B$ is a theorem, then also $A \to ((\forall\, x)\, B)$ is a theorem, and if $B \to A$ is a theorem, then also $((\exists\, x)\, B) \to A$ is a theorem of FOPC. In other terms, the schemata

$$\frac{A,\ A \to B}{B}, \qquad \frac{A \to B,\ x \text{ not free in } A}{A \to ((\forall\, x)\, B)}, \qquad \frac{B \to A,\ x \text{ not free in } A}{(\exists\, x)\, B) \to A} \tag{2.6}$$

are deduction rules of FOPC. By $Ded_{\rm PC} = Ded(\mathcal{L}_{\rm PC}) \subset \mathcal{L}_{\rm PC}$ we denote the set of all theorems of FOPC with the language $\mathcal{L}_{\rm PC}$. The notion of proof can be defined in the common way.

3 Classical and boolean semantics for the first–order predicate language $\mathcal{L}_{\rm PC}$

A *structure* $\mathcal{M}$ (fitted) for the language $\mathcal{L}_{\rm PC} = \mathcal{L}_{\rm PC}(R_1, \dots, R_n,\ Var,\ Const)$ defined above is an $n+2$-tuple $\langle M,\ \rho_1, \rho_2, \dots, \rho_n,\ CS\rangle$, where M is a nonempty set called the *support* of the structure $\mathcal{M}$. For each $i \leq n$, $\rho_i \subseteq M^{r_i}$ is an r_i-ary relation (total) on M, and $CS : Const \rightarrow M$ is a total mapping ascribing to each constant (symbol) $c \in Const$ an element $CS(c)$ of M. In the case of necessity we shall write $M_{\mathcal{M}}$, $\rho_{i,\mathcal{M}}$ and $CS_{\mathcal{M}}$ in order to express explicitly the structure to which the corresponding objects relate or belong.

An *evaluation of the variables* of the language $\mathcal{L}_{\rm PC}$ in the structure $\mathcal{M} = \langle M,\ \rho_1, \rho_2, \dots, \rho_n,\ CS\rangle$ is a total mapping $e^* : Var \rightarrow M$, ascribing to each variable x an element $e^*(x)$ of M. Each such evaluation of variables defines, together with the structure $\mathcal{M}$, uniquely a mapping $e : \mathcal{L}_{\rm PC} \rightarrow \{0,1\}$ (we shall write also $e_{\mathcal{M}}$, if it is useful to express explicitly the role of the structure $\mathcal{M}$), called *the evaluation of the (w. f. f.'s of the) language* $\mathcal{L}_{\rm PC}$ *induced by* e^* and defined recurrently as follows.

(i) For elementary w. f. f.'s, $e(R_i(t_1, t_2, \dots, t_{r_i})) = 1$, if $\langle e(t_1),\ e(t_2), \dots, e(t_{r_i})\rangle \in \rho_i$. where $e(t_j) = e^*(t_j)$, if $t_j \in Var$, and $e(t_j) = CS(t_j)$, if $t_j \in Const$, $e(R_i(t_1, t_2, \dots, t_{r_i})) = 0$ otherwise.

(ii) For w. f. f.'s $A, B \in \mathcal{L}_{\rm PC}$, $e(\neg A) = 1 - e(A)$ and $e(A \rightarrow B) = \max(1 - e(A),\ e(B))$.

(iii) For a w. f. f. $A \in \mathcal{L}_{\rm PC}$, $e((\forall\, x)\, A) = \inf\{e_{x|m}(A) : m \in M\}$. Here $e_{x|m}$ is defined by the mapping $e^*_{x|m}$ taking Var into M in such a way that $e^*_{x|m}(z) = e^*(z)$ for all $z \in Var$, $z \neq x$, and $e^*_{x|m}(x) = m$. The value $e_{x|m}(A)$ is uniquely defined by $e^*_{x|m}$ in the same way as $e(A)$ is defined by e^*, as the induction step supposes that $e_{x|m}(A)$ is defined for all proper subformulas of $(\forall\, x)\, A$, hence, also for A.

An easy reasoning yields that $e(A) \in \{0,1\}$ is uniquely defined for each w. f. f. A of $\mathcal{L}_{\rm PC}$. Moreover, for the abbreviations using the functors $\wedge$, $\vee$, $\equiv$ and the existential quantifier $\exists$, the following indentities obviously hold:

$$
\begin{aligned}
e(A \wedge B) &= \min\{e(A),\ e(B)\}, && (3.1)\\
e(A \vee B) &= \max\{e(A),\ e(B)\}, && (3.2)\\
e(A \leftrightarrow B) &= 1 \quad \text{iff} \quad e(A) = e(B), && (3.3)\\
e((\exists\, x)\, A) &= \sup\{e_{x|m}(A) : m \in M\}. && (3.4)
\end{aligned}
$$

A w. f. f. A of $\mathcal{L}_{\rm PC}$ is called a *sentence*, if it does not contain any free occurrence of any variable, let $Sent_{PC} \subset \mathcal{L}_{\rm PC}$ denote the set of all sentences. A sentence A of $\mathcal{L}_{\rm PC}$ is called a *tautology* of the FOPC, if for each structure $\mathcal{M} = \langle M,\ \rho_1, \rho_2, \dots, \rho_n,\ CS\rangle$

fitted for $\mathcal{L}_{\mathrm{PC}}$ and for *each* mapping $e^* : Var \to M$ the identity $e_{\mathcal{M}}(A) = 1$ holds. Let us denote by $Taut_{\mathrm{PC}}$ the set of all tautologies of FOPC. The following statement is a well-known fundamental property of FOPC.

Proposition 1. (Completeness theorem for the first-order predicate calculus)

$$Ded_{\mathrm{PC}} \cap Sent_{\mathrm{PC}} = Taut_{\mathrm{PC}}. \qquad \square$$

Verbally, a sentence is deducible from the axioms of FOPC by (a finite number of applications of) the deduction rules of FOPC iff it is valid (i. e., its truthvalue is 1) in all structures fitted for the language $\mathcal{L}_{\mathrm{PC}}$ and under all evaluations of variables into the support of the structure in question.

Leaving aside any informal introductory comments and motivations, let us consider an alternative, many-valued but at the same time non-numerical and, in particular, boolean-valued semantic for the language $\mathcal{L}_{\mathrm{PC}}$. Let $\mathcal{N}^+=\{1,2,\ldots\}$ be the set of all positive integers, let $\mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$ be the power-set of all subsets of $\mathcal{N}^+$. A *boolean-valued* or, more correctly, *$\mathcal{P}_0$-valued structure fitted for $\mathcal{L}_{\mathrm{PC}}$* is an $n+2$-tuple $\mathcal{M}^b = \langle M, \rho_1^b, \rho_2^b, \ldots, \rho_n^b, CS\rangle$, where

(i) M is a nonempty set called the *support* of the structure $\mathcal{M}^b$,

(ii) for each $i \leq n$, $\rho_i^b : M^{r_i} \to \mathcal{P}_0$ is a mapping ascribing to each r_i-tuple $\langle m_1, m_2, \ldots, m_{r_i}\rangle \in M^{r_i}$ a subset $\rho_i^b(m_1, m_2, \ldots, m_{r_i})$ of $\mathcal{N}^+$. The interpretation behind can read that this subset quantifies, in a non-numerical way, the degree in which the elements $m_1, m_2, \ldots, m_{r_i}$ of M, in the given order, satisfy the relation ρ_i^b. In other terms, perhaps more close to some readers, ρ_i^b is a fuzzy subset of the Cartesian product M^{r_i} with membership degrees taking their values in the lattice $\mathcal{P}_0$ (every Boolean algebra, including that of all subsets of a set with respect to the usual set-theoretic operations, is a lattice, cf. [2] or [8] for more detail about lattice-valued fuzzy sets). In the classical case, only the two extremal values are considered, either $\rho(m_1, \ldots, m_{r_i}) = 1$, hence, the r_i-tuple $\langle m_1, \ldots, m_{r_i}\rangle$ fulfils the relation ρ_i, or $\rho(m_1, \ldots, m_{r_i}) = 0$, if the r_i-tuple in question does not satisfy ρ_i. We shall see, in what follows, that these two cases will be covered by the particular cases of ρ_i^b, when $\rho_i^b(m_1, m_2, \ldots, m_{r_i}) = \mathcal{N}^+$ and when $\rho_i^b(m_1, m_2, \ldots, m_{r_i}) = \emptyset$ (the empty subset of $\mathcal{N}^+$).

(iii) $CS : Const \to M$ is a mapping ascribing to each constant (symbol) $c \in Const$ an element $CS(c)$ of M, so, constant are interpreted, for the sake of simplicity, as in the classical case above. As a matter of further investigation we could consider a more general case with CS replaced by $CS^b : Const \times M \to \mathcal{P}_0$, where $CS^b(c, m) \subset \mathcal{N}^+$ quantifies the degree in which the constant symbol c denotes the element m of the support M.

We shall write $M_{\mathcal{M}^b}$, $\rho_{i,\mathcal{M}^b}^b$ and $CS_{\mathcal{M}^b}$, if it is convenient or necessary to denote explicitly the structure in question.

An *evaluation of the variables* of the language $\mathcal{L}_{\mathrm{PC}}$ in a $\mathcal{P}_0$-valued structure $\mathcal{M}^b = \langle M, \rho_1^b, \ldots, \rho_n^b, CS\rangle$ fitted for $\mathcal{L}_{\mathrm{PC}}$ is a total mapping $e^* : Var \to M$ ascribing to each variable x an element $e^*(x)$ of M, like as in the classical case above. Each such evaluation of variables defines, together with the structure $\mathcal{M}^b$, uniquely a mapping $e : \mathcal{L}_{\rm PC} \to \mathcal{P}_0$ (denoted also by $e_{\mathcal{M}^b}$, if the dependence on $\mathcal{M}^b$ is to be explicitly stated), called the *boolean* or $\mathcal{P}_0$*-evaluation of the (w. f. f.'s of the) language* $\mathcal{L}_{\rm PC}$ *induced by* e^* and defined recurrently as follows.

(i) For elementary w. f. f.'s,

$$e\left(R_i(t_1, t_2, \ldots, t_{r_i})\right) = \rho_i^b\left(e^{**}(t_1),\, e^{**}(t_2), \ldots, e^{**}(t_{r_i})\right) \subset \mathcal{N}^+,$$

where $e^{**}(t_i) = e^*(t_i)$, if $t_i \in \mathit{Var}$, and $e^{**}(t_i) = CS(t_i)$, if $t_i \in \mathit{Const}$.

(ii) For w. f. f.'s $A,\, B \in \mathcal{L}_{\rm PC}$, $e(\neg A) = \mathcal{N}^+ - e(A)$ and $e(A \to B) = (\mathcal{N}^+ - e(A)) \cup e(B)$.

(iii) For a w. f. f. $A \in \mathcal{L}_{\rm PC}$ and a variable x,

$$e(\forall\, x)\, A) = \textstyle\bigcap_{m \in M} e_{x|m}(A) \subset \mathcal{N}^+, \tag{3.5}$$

where $e_{x|m}$ is defined by the mapping $e^*_{x|m}$ taking *Var* into M in such a way that $e^*_{x|m}(z) = e^*(z)$ for all $z \in \mathit{Var}$, $z \neq x$, and $e^*_{x|m}(x) = m$. Obviously, the value $e((\forall\, x)\, A)$ is uniquely defined.

An easy reasoning yields that the value $e(A) \subset \mathcal{N}^+$ is uniquely defined for each w. f. f. $A \in \mathcal{L}_{\rm PC}$. Moreover, for the abbreviations using the functors $\wedge$, $\vee$, $\leftrightarrow$ and the existential quantifier $\exists$ the following identities can be easily proved:

$$\begin{aligned} e(A \wedge B) &= e(A) \cap e(B), & (3.6)\\ e(A \vee B) &= e(A) \cup e(B), & (3.7) \end{aligned}$$

$e(A \leftrightarrow B) = \mathcal{N}^+ - (e(A) \div e(B))$, where $\div$ denotes the symmetric difference of sets, so that $e(A \leftrightarrow B) = \mathcal{N}^+$ iff $e(A) = e(B)$, i. e., iff $A \leftrightarrow B$ is a tautology, as will be proved below, and $e(A \leftrightarrow B) = \emptyset$ iff $e(A) = \mathcal{N}^+ - e(B) = e(\neg B)$, i. e. iff $A \leftrightarrow (\neg B)$ is a tautolgy,

$$e((\exists\, x)\, A) = \textstyle\bigcup_{m \in M} e_{x|m}(A). \tag{3.8}$$

A sentence A of $\mathcal{L}_{\rm PC}$ is called a $\mathcal{P}_0$*-tautology* of the FOPC, if for *each* $\mathcal{P}_0$-valued structure $\mathcal{M}^b = \langle M,\, \rho_1^b, \ldots, \rho_n^b,\, CS\rangle$ fitted for $\mathcal{L}_{\rm PC}$ and for *each* mapping $e^* : \mathit{Var} \to M$ the identity $e_{\mathcal{M}^n}(A) = \mathcal{N}^+$ holds. Let us denote by $\mathit{Taut}^b_{\rm PC}$ the set of all $\mathcal{P}_0$-tautologies of FOPC. Our aim will be to prove, in the next chapter, whether an analogy of Proposition 1 holds for $\mathit{Taut}^b_{\rm PC}$.

An intuitive interpretation of $\mathcal{P}_0$-valued semantics for $\mathcal{L}_{\rm PC}$ can be obtained as follows. Let $\mathcal{M}_i = \langle M,\, \rho_1^i, \rho_2^i, \ldots, \rho_n^i,\, CS\rangle$, $i \in \mathcal{N}^+$, be an infinite sequence of classical crisp structures fitted for $\mathcal{L}_{\rm PC}$ over the same support set M and with the same evaluation CS of constant symbols of $\mathcal{L}_{\rm PC}$. Given $e^* = \mathit{Var} \to M$, let $e_i^0 = e^0_{\mathcal{M}_i} : \mathcal{L}_{\rm PC} \to \{0, 1\}$ be the classical two-valued evaluation of w. f. f.'s of $\mathcal{L}_{\rm PC}$ induced by e^* and by the structure $\mathcal{M}_i$ according to the recurrent rules described above. Define, now, a mapping $e : \mathcal{L}_{\rm PC} \to \mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$ in such a way that $e(A) = \{i \in \mathcal{N}^+ : e_i^0(A) = 1\}$ for each $A \in \mathcal{L}_{\mathrm{PC}}$. An easy reasoning proves that

$$e(\neg A) = \{i \in \mathcal{N}^+ : e_i^0(\neg A) = 1\} = \mathcal{N}^+ - \{i \in \mathcal{N}^+ : e_i^0(A) = 1\} = \mathcal{N}^+ - e(A),$$

$$\begin{aligned} e(A \to B) &= \{i \in \mathcal{N}^+ : e_i^0(A \to B) = 1\} = \{i \in \mathcal{N}^+ : e_i^0(A) = 0\} \cup \{i \in \mathcal{N}^+ : e_i^0(B) = 1\} = \\ &= \left(\mathcal{N}^+ - \{i \in \mathcal{N}^+ : e_i^0(A) = 1\}\right) \cup \{i \in \mathcal{N}^+ : e_i^0(B) = 1\} = (\mathcal{N}^+ - e(A)) \cup e(B), \end{aligned}$$

$$e((\forall\, x)\, A) = \{i \in \mathcal{N}^+ : e_i^0((\forall\, x)\, A) = 1\} = \textstyle\bigcap_{m \in M} \{i \in \mathcal{N}^+ : e_{i,x|m}^0(A) = 1\} = \bigcap_{m \in M} e_{x|m}(A).$$

Hence, the mapping $e : \mathcal{L}_{\mathrm{PC}} \to \mathcal{P}_0$ is identical with that one defined by $e(R_j(t_1, \dots, t_{r_j})) = \{i \in \mathcal{N}^+ : e_i^0(R_j(t_1, \dots, t_{r_j})) = 1\}$ for each elementary formula $R_j(t_1, \dots, t_{r_j})$ and extended to $\mathcal{L}_{\mathrm{PC}}$ according to the recurrent rules for $\mathcal{P}_0$-evaluations defined above.

4 Completeness theorem for the first–order predicate calculus without functions and with boolean–valued semantics

Lemma 1. If $A \in \mathcal{L}_{\mathrm{PC}}$ is a substitution of w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ into a propositional tautology, then $A \in \mathit{Taut}_{\mathrm{PC}}^b$. $\square$

Proof. Let P be a tautology of the propositional calculus, let the set $\{p_1, p_2, \dots, p_k\}$ of propositional variables contain all the propositional variables occurring in P, let A be a w. f. f. of $\mathcal{L}_{\mathrm{PC}}$ resulting when replacing all occurrences of p_1 by A_1, of p_2 of A_2,..., and of p_k by A_k, where $A_1, A_2, \dots, A_k$ are w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$. Being a propositional tautology, P is also a $\mathcal{P}_0$-valued tautology. so, for each mapping $e^0 : \{p_1, p_2, \dots, p_k\} \to \mathcal{P}_0$, and for the same rules as above extending e^0 from propositional variables as elementary formulas to all propositional formulas by induction on propositional functors $\neg$ and $\to$, we obtain that $e^0(P) = \mathcal{N}^+$ (cf. [6] for more detail concerning the boolean semantic for propositional calculus). Hence, if $e^* : \mathit{Var} \to M$ is given and $e(A_1), \dots, e(A_k)$ are defined, then also $e(A) = \mathcal{N}^+$, as $e(A) = e^0(P)$ for $e^0 : \{p_1, \dots, p_k\} \to \mathcal{N}^+$ defined in such a way that $e^0(p_i) = e(A_i)$ for each $i \leq k$. Consequently, for each $e^* : \mathit{Var} \to M$, $e(A) = \mathcal{N}^+$ for each w. f. f. $A \in \mathcal{L}_{\mathrm{PC}}$ obtained by substitution of w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ into a propositional tautology.

$\square$

Lemma 2. Let A be a w. f. f. of $\mathcal{L}_{\mathrm{PC}}$, x a variable, and t a variable or a constant such that the w. f. f.'s $((\forall\, x)\, A) \to A(t)$ and $A(t) \to ((\exists\, x)\, A)$ are axioms of FOPC. Then both these w. f. f.'s are also $\mathcal{P}_0$-tautologies of the FOPC. $\square$

Proof. Let $((\forall\, x)\, A) \to A(t)$ be an axiom of the FOPC, let $t \in \mathit{Const}$, let $e^* : \mathit{Var} \to M$ be an evaluation of variables. By induction on the syntactical depth of

A (the number of applications of syntactical rules when building w.f.f. A from elementary formulas) we obtain that $e(A(t)) = e_{x|CS(t)}(A)$, where $e^*_{x|CS(t)}(y) = e^*(y)$, if $y \in Var$, $y \neq x$, and $e^*_{x|CS(t)}(x) = CS(t)$. But,

$$\begin{aligned} e(((\forall\, x)\, A) \rightarrow A(t)) &= (\mathcal{N}^+ - e((\forall\, x)\, A)) \cup e(A(t)) = \\ &= \left(\mathcal{N}^+ - \bigcap_{m \in M} e_{x|m}(A)\right) \cup e_{x|CS(t)}(A) \supset \\ &\supset (\mathcal{N}^+ - e_{x|CS(t)}(A)) \cup e_{x|CS(t)}(A) = \mathcal{N}^+. \end{aligned} \tag{4.1}$$

If $t \in Var$, then $e(A)$ can be written in the form $e_{x|e^*(t)}(A)$ and the same computation as above yields that $e((\forall\, x)\, A) \rightarrow A(t) = \mathcal{N}^+$ holds as well.

If $A(t) \rightarrow ((\exists\, x)\, A)$ is an axiom, then

$$\begin{aligned} e(A(t) \rightarrow ((\exists\, x)\, A)) &= (\mathcal{N}^+ - e(A(t))) \cup e((\exists\, x)\, A) = \\ &= (\mathcal{N}^+ - e(A(t))) \cup \bigcup_{m \in M} e_{x|m}(A) = \\ &= (\mathcal{N}^+ - e_{x|K}(A)) \cup \bigcup_{m \in M} e_{x|m}(A) = \mathcal{N}^+, \end{aligned} \tag{4.2}$$

where $K = CS(t)$, if $t \in Const$, and $K = e^*(t)$, if $t \in Var$, in both the cases $K \in M$, so that $e_{x|K}(A) \subset \bigcup_{m \in M} e_{x|m}(A)$ holds. The lemma is proved. □

Lemma 3. (i) Let A, B be w.f.f.'s of $\mathcal{L}_{\mathrm{PC}}$, let $e^* : Var \rightarrow M$ be such that $e(A) = e(A \rightarrow B) = \mathcal{N}^+$. Then $e(B) = \mathcal{N}^+$.

(ii) Let A, B be w.f.f.'s of $\mathcal{L}_{\mathrm{PC}}$, let x be a variable not occurring freely in A, let $e^* : Var \rightarrow M$ be such that $e(A \rightarrow B) = \mathcal{N}^+$. Then $e(A \rightarrow ((\forall\, x)\, B)) = \mathcal{N}^+$.

(iii) Let A, B and x be as in (ii), let $e^* : Var \rightarrow M$ be such that $e(B \rightarrow A) = \mathcal{N}^+$. Then $e(((\exists\, x)\, B) \rightarrow A) = \mathcal{N}^+$. □

Proof. (i) If $e(A) = e(A \rightarrow B) = \mathcal{N}^+$, then $e(A \rightarrow B) = (\mathcal{N}^+ - e(A)) \cup e(B) = e(B) = \mathcal{N}^+$ immediately follows.

(ii) Let A, B and x be as supposed, let $e(A \rightarrow B) = \mathcal{N}^+$. Then

$$e(A \rightarrow ((\forall\, x)\, B)) = (\mathcal{N}^+ - e(A)) \cup \bigcap_{m \in M} e_{x|m}(B) \tag{4.3}$$

by definition. An induction on the syntactical depth of the w.f.f. B proves that if x is not free in B, then $e_{x|m}(B) = e(B)$ for each $m \in M$, so that, by assumption,

$$e(A \rightarrow ((\forall\, x)\, B)) = (\mathcal{N}^+ - e(A)) \cup e(B) = e(A \rightarrow B) = \mathcal{N}^+. \tag{4.4}$$

(iii) Dually, for the same A, B and x, if $e(B \rightarrow A) = \mathcal{N}^+$, then

$$\begin{aligned} &e(((\exists\, x)\, B) \rightarrow A = (\mathcal{N}^+ - e((\exists\, x)\, B)) \cup e(A) = \left(\mathcal{N}^+ - \bigcup_{m \in M} e_{x|m}(B)\right) \cup \\ &e(A) = (\mathcal{N}^+ - e(B)) \cup e(A) = e(B \rightarrow A) = \mathcal{N}^+. \end{aligned} \tag{4.5}$$

The lemma is proved. □

Lemma 4. $Ded_{\rm PC} \cap Sent_{\rm PC} \subset Taut^b_{\rm PC}$. □

Proof. The assertion immediately follows from Lemmas 1, 2, and 3, and from the recurrent definition of the set $Ded_{\rm PC}$ of deducible (provable) formulas (theorems) of the FOPC.

□

A $\mathcal{P}_0$-valued structure $\mathcal{M}^b = \langle M, \rho^b_1, \ldots, \rho^b_n, CS\rangle$ fitted for $\mathcal{L}_{\rm PC}$ is called *crisp*, if for each $i \leq n$ and each r_i-tuple $\langle m_1, m_2, \ldots, m_{r_i}\rangle \in M^{r_i}$, $\rho^b_i(m_1, m_2, \ldots, m_{r_i}\} \in \{\emptyset, \mathcal{N}^+\}$. Hence, defining a classical crisp r_i-ary relation ρ_i in M^{r_i} in such a way that $\langle m_1, m_2, \ldots, m_{r_i}\rangle \in \rho_i$ (or, under a slightly different notation, $\rho_i(m_1, m_2, \ldots, m_{r_i}) = 1$) iff $\rho^b_i(m_1, m_2, \ldots, m_{r_i}) = \mathcal{N}^+$, hence, $\rho_i(m_1, m_2, \ldots, m_{r_i}) = 0$ iff $\rho^b_i(m_1, m_2, \ldots, m_{r_i}) = \emptyset$, we obtain the classical crisp structure $\mathcal{M} = \langle M, \rho_1, \rho_2, \ldots, \rho_n, CS\rangle$ fitted for $\mathcal{L}_{\rm PC}$ and induced by $\mathcal{M}^b$. An easy induction on the syntactical depths of w. f. f.'s of $\mathcal{L}_{\rm PC}$ yields that, for each w. f. f. $A \in \mathcal{L}_{\rm PC}$ and each $e^* : Var \to M$, $e_{\mathcal{M}^b}(A) = \mathcal{N}^+$ iff $e_{\mathcal{M}}(A) = 1$ and $e_{\mathcal{M}^b}(A) = \emptyset$ iff $e_{\mathcal{M}}(A) = 0$.

Theorem 1. (Completeness theorem for the first-order predicate calculus without functions and with respect to the $\mathcal{P}_0$-semantic)

$$Ded_{\rm PC} \cap Sent_{\rm PC} = Taut^b_{\rm PC}. \tag{4.6}$$

□

Proof. Lemma 4 yields that $Ded_{\rm PC} \cap Sent_{\rm PC} \subset Taut^b_{\rm PC}$. If $A \in Taut^b_{\rm PC}$, then by definition $e_{\mathcal{M}^b}(A) = \mathcal{N}^+$ for all $\mathcal{M}^b = \langle M, \rho^b_1, \ldots, \rho^b_n, CS\rangle$ fitted for $\mathcal{L}_{\rm PC}$ and for all $e^* : Var \to M$, in particular, $e_{\mathcal{M}^b}(A) = \mathcal{N}^+$ for all *crisp* structures $\mathcal{M}^b$ and evaluations e^* of variables as before. So, $e_{\mathcal{M}}(A) = 1$ for all classical crisp structures $\mathcal{M} = \langle M, \rho_1, \rho_2, \ldots, \rho_n, CS\rangle$ fitted for $\mathcal{L}_{\rm PC}$ and for all $e^* : Var \to M$, as each such structure is induced by some (in particular, crisp) $\mathcal{P}_0$-valued structure fitted for $\mathcal{L}_{\rm PC}$. Consequently, $A \in Ded_{\rm PC} \cap Sent_{\rm PC}$ according to the completeness theorem for the first-order predicate calculus without functions and with the classical (two-valued) semantics (cf. Proposition 1), and the equality in Theorem 1 is proved. □

5 A nonstandard boolean–like numerical semantic for the first–order predicate calculus without functions

As the reader perhaps noticed, when building the $\mathcal{P}_0$-valued semantic for $\mathcal{L}_{\rm PC}$, we used only the obvious and trivial fact that the power-set $\mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$ of all sets of positive integers defines, together with the usual set-theoretic operations, a complete Boolean algebra. In other words, everything done till now could be also done when replacing $\mathcal{N}^+ = \{1, 2, \ldots\}$ by any nonempty set (infinite, if we want to take profit of the possibility to ascribe different truthvalues to an infinite number of w. f. f.'s), and we have not taken profit, till now, from the specific properties of the set $\mathcal{N}^+$, its elements, and structures with which it can be endowed. We shall do so in this chapter, when the subsets of $\mathcal{N}^+$, i. e., the truthvalues of the $\mathcal{P}_0$-valued semantic, will be projected, in two different ways, into the unit interval of real numbers.

Let $\mathcal{B} = \{0,1\}^\infty$ be the set of all infinite binary sequences, let $\mathcal{C}$ be the well-known Cantor subset of the unit interval of real numbers, let us recall that $\mathcal{C}$ is the set of all numbers $x \in \langle 0,1\rangle$ for which there exists a ternary decomposition (decomposition to the base 3), not containing any occurrence of the numeral 1. Let $\chi : \mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+) \rightarrow \mathcal{B}$ be defined in such a way that $\chi(A) = \langle \chi(A)\,(1),\, \chi(A)\,(2), \ldots\rangle$, and $\chi(A)\,(i) = 1$ iff $i \in A$. Hence, $\chi(A)$ is the *characteristic function (sequence)* or *identifier* of the set A of positive integers. Let $\varphi : \mathcal{P}_0 \rightarrow \mathcal{C}$ be defined in such a way that

$$\varphi(A) = \sum_{i=1}^{\infty} 2\chi(A)\,(i)\,3^{-i}, \tag{5.1}$$

we shall also take φ as $\varphi : \mathcal{B} \rightarrow \mathcal{C}$ and write $\varphi(\chi(A))$ instead of $\varphi(A)$. Both the mappings χ and φ are obviously one-to-one mappings between $\mathcal{P}_0$ and $\mathcal{B}$ ($\mathcal{P}_0$ and $\mathcal{C}$, resp.). Set also

$$w(A) = \lim_{n\rightarrow\infty} (1/n) \sum_{i=1}^{n} \chi(A)\,(i), \tag{5.2}$$

if this limit value is defined, $w(A)$ being undefined otherwise.

As $\varphi : \mathcal{P}_0 \rightarrow \mathcal{C}$ is one-to-one, the inverse mapping $\varphi^{-1} : \mathcal{C} \rightarrow \mathcal{P}_0$ is uniquely defined, so that the following three operations over the Cantor set $\mathcal{C}$ are well-defined for each $x,\, y \in \mathcal{C}$:

$$\begin{aligned} 1\dot{-}x &= \varphi(\mathcal{N}^+ - \varphi^{-1}(x)), \\ x \vee y &= \varphi(\varphi^{-1}(x) \cup \varphi^{-1}(y)), \\ x \wedge y &= \varphi(\varphi^{-1}(x) \cap \varphi^{-1}(y)). \end{aligned} \tag{5.3}$$

An easy calculation yields that $1\dot{-}x$ agrees with $1 - x$ for the usual substraction but, in general, $x \vee y \neq \max\{x,y\}$ and $x \wedge y \neq \min\{x,y\}$ for the usual operations max and min in $\langle 0,1\rangle$. In more detail, if $x,\, y \in \mathcal{C}$ are such that their corresponding (and obviously uniquely defined) ternary decompositions not containing 1 are $\langle x_1, x_2, \ldots\rangle \in \{0,2\}^\infty$ and $\langle y_1, y_2, \ldots\rangle \in \{0,2\}^\infty$, then $x \vee y$ is defined by the ternary decomposition $\langle z_1, z_2, \ldots\rangle$ such that $z_i = \max\{x_i, y_i\}$ for each $i \in \mathcal{N}^+$, consequently, $x \vee y = \sum_{i=1}^{\infty}(\max\{x_i,y_i\})\,3^{-i}$, similarly, $x \wedge y = \sum_{i=1}^{\infty}(\min\{x_i,y_i\})\,3^{-i}$. E. g., if $x = 1/3 = 0,0222\ldots$ and $y = 2/3 = 0,2000\ldots$, then $x \vee y = 1$ and $x \wedge y = 0$ (their alternative decompositions $x = 0,1000\ldots$ and $y = 0,1222\ldots$ do not meet the condition not to contain any occurrence of 1).

Both the operations $\vee$ and $\wedge$ can be easily extended to any nonempty set $D \subset \mathcal{C}$ of real numbers from the Cantor set. Either we can define

$$\begin{aligned} \bigvee D &= \bigvee_{x\in D} x = \varphi\left(\bigcup_{x\in D} \varphi^{-1}(x)\right), \\ \bigwedge D &= \bigwedge_{x\in D} x = \varphi\left(\bigcap_{x\in D} \varphi^{-1}(x)\right), \end{aligned} \tag{5.4}$$

copying (5.3), so that $x \vee y = \bigvee\{x, y\}$ and $x \wedge y = \bigwedge\{x, y\}$ hold for each $x, y \in \mathcal{C}$. Or, what can be easily proved to be the same, if $\langle x_1, x_2, \ldots\rangle \in \{0, 2\}^\infty$ is the only admissible ternary decomposition of $x \in \mathcal{C}$, then

$$\begin{aligned} \bigvee D &= \sum_{i=1}^{\infty} (\sup\{x_i : x \in D\})\, 3^{-i}, \\ \bigwedge D &= \sum_{i=1}^{\infty} (\inf\{x_i : x \in D\})\, 3^{-i}, \end{aligned} \tag{5.5}$$

where sup and inf are the usual supremum and infimum operations in the space of integers (reducing to max and min in our case when the space of values of x_i is finite).

Nonstandard c-valued evaluations of the w. f. f.'s of the first-order predicate calculus without functions can be defined in two ways which can be easily proved to be equivalent. Either, let $e^* : \mathit{Var} \to M$ be an evaluation of the variables of the language $\mathcal{L}_{\mathrm{PC}}$, let $\mathcal{M}^b$ be a $\mathcal{P}_0$-valued structure fitted for $\mathcal{L}_{\mathrm{PC}}$, and let $e : \mathcal{L}_{\mathrm{PC}} \to \mathcal{P}_0$ be defined by e^* and $\mathcal{M}^b$ as above. Then the mapping $e_c : \mathcal{L}_{\mathrm{PC}} \to \mathcal{C}$ defined by $e_c(A) = \varphi(e(A))$ for each w. f. f. $A \in \mathcal{L}_{\mathrm{PC}}$ is called a (nonstandard) c-valued evaluation of $\mathcal{L}_{\mathrm{PC}}$. Equivalently, given $e^* : \mathit{Var} \to M$, let

$$\begin{aligned} & e_c(R_i(t_1, t_2, \ldots, t_{r_i})) = \varphi(e(R_i(t_1, t_2, \ldots, t_{r_i}))) = \\ = \ & \varphi(\rho_i^b(e^{**}(t_1),\, e^{**}(t_2), \ldots, e^{**}(t_{r_i}))) = \\ = \ & \sum_{j=1}^{\infty} \left(\chi(\rho_i^b(e^{**}(t_1),\, e^{**}(t_2), \ldots, e^{**}(t_{r_i})))\,(j)\right)\, 3^{-j} \in \mathcal{C}, \end{aligned} \tag{5.6}$$

where $e^{**}(t_k) = e^*(t_k)$, if $t_k \in \mathit{Var}$ and $e^{**}(t_k) = CS(t_k)$, if $t_k \in \mathit{Const}$, and let

$$\begin{aligned} e_c(\neg A) &= 1 - e_c(A),\ e_c(A \to B) = (1 - e_c(A)) \vee e_c(B), \\ e_c((\forall\, x)\, A) &= \bigwedge\{e_{x|m,c}(A) = m \in M\}, \end{aligned} \tag{5.7}$$

where $e_{x|m,c}$ is defined, for all subformulas of the w. f. f. $((\forall\, x)\, A)$ including A itself, by $e^*_{x|m}$ in the same way as $e(A)$ by e^*.

A sentence A of $\mathcal{L}_{\mathrm{PC}}$ is called a *c-tautology* of the FOPC, if for *each* $\mathcal{P}_0$-valued structure $\mathcal{M}^b = \langle M,\, \rho_1^b, \ldots, \rho_n^b,\, CS\rangle$ fitted for $\mathcal{L}_{\mathrm{PC}}$ and for *each* mapping $e^* : \mathit{Var} \to M$ the identity $e_c(A) = e_{c,\mathcal{M}^b}(A) = \varphi(e_{\mathcal{M}^b}(A)) = 1$ holds. Let us denote by $\mathit{Taut}^c_{\mathrm{PC}}$ the set of all c-tautologies of FOPC.

Theorem 2. (Completeness theorem for the first-order predicate calculus without functions and with nonstandard c-valued semantics)

$$\mathit{Sent}_{\mathrm{PC}} \cap \mathit{Ded}_{\mathrm{PC}} = \mathit{Taut}^c_{\mathrm{PC}}. \tag{5.8}$$

□

Proof. Due to Theorem 1, the only we have to prove is the equality $\mathit{Taut}^b_{\mathrm{PC}} = \mathit{Taut}^c_{\mathrm{PC}}$. Let $A \in \mathit{Taut}^b_{\mathrm{PC}}$, let $\mathcal{M}^b$ be any $\mathcal{P}_0$-valued structure fitted for $\mathcal{L}_{\mathrm{PC}}$, let e^* take

Var into M arbitrarily. Then $e_{\mathcal{M}^b}(A) = \mathcal{N}^+$ due to the definition of $Taut^b_{\mathrm{PC}}$, hence, $e_{c,\mathcal{M}^b}(A) = \varphi(e_{\mathcal{M}^b}(A)) = \varphi(\mathcal{N}^+) = 1$, so that $A \in Taut^c_{\mathrm{PC}}$. Vice versa, if $A \in Taut^c_{\mathrm{PC}}$, then for any $\mathcal{M}^b$ as above and any $e^* : Var \to M$, $e_{c,\mathcal{M}^b}(A) = \varphi(e_{\mathcal{M}^b}(A)) = 1$, so that $e_{\mathcal{M}^b}(A) = \mathcal{N}^+$, as $\varphi^{-1}(1) = \mathcal{N}^+$. Hence, $A \in Taut^b_{\mathrm{PC}}$ and the theorem is proved. □

Theorem 2 is nothing else than a rather trivial consequence of the fact that the projection φ of the power-set $\mathcal{P}_0$ of all subsets of $\mathcal{N}^+$ into the unit interval of real numbers has been defined rather with the aim to encode, in a one-to-one way, subsets of $\mathcal{N}^+$ by real numbers than to quantify somewhat their respective sizes. We have paid for such an encoding projection by the fact that the binary relation $\leq_*$ on $\mathcal{C}$, defined for each $x, y \in \mathcal{C}$ by $x \leq_* y$ if $x \vee y = y$ (or, what can be proved to be the same, iff $x \wedge y = x$), i. e., the relation with respect to which $\vee$ and $\wedge$ fulfil the properties of supremum and infimum operations, is just a *partial* ordering on $\mathcal{C}$, copying the partial ordering of $\mathcal{P}_0$ by the relation of set-theoretic inclusion. As can be easily proved, $x \leq_* y$ holds iff $x_i \leq y_i$ holds for each $i \in \mathcal{N}^+$, where $\langle x_1, x_2, \ldots\rangle$ and $\langle y_1, y_2, \ldots\rangle$ are the corresponding ternary decompositions from $\{0,1\}^\infty$. It follows immediately, that for each $x, y \in \mathcal{C}$, $x \leq_* y$ implies $x \leq y$ for the usual (linear) ordering $\leq$ in $\langle 0,1\rangle$ but not, in general, vice versa. E. g., neither $1/3 \leq_* 2/3$ nor $2/3 \leq_* 1/3$ hold. Nevertheless, the following statement can be proved.

Lemma 5. (i) For each w. f. f.'s A, $B \in \mathcal{L}_{\mathrm{PC}}$, for each $\mathcal{P}_0$-valued structure $\mathcal{M}^b$ fitted for $\mathcal{L}_{\mathrm{PC}}$, and for each $e^* : Var \to M$, $e(A \to B) = \mathcal{N}^+$ iff $e(A) \subset e(B)$.

(ii) For each A, B, $\mathcal{M}^b$ and e^* as in (i), $e_{c,\mathcal{M}^b}(A \to B) = 1$ iff $e_{c,\mathcal{M}^b}(A) \leq_* e_{c,\mathcal{M}^b}(B)$. Hence, if $e_{c,\mathcal{M}^b}(A \to B) = 1$, then $e_{c,\mathcal{M}^b}(A) \leq e_{c,\mathcal{M}^b}(B)$. □

Proof. Let A, B, $\mathcal{M}^b$ and e^* be as in (i). Then $e(A \to B) = (\mathcal{N}^+ - e(A)) \cup e(B) = \mathcal{N}^+$ yields that $e(A) \subset e(B)$, at the same time, $e(A) \subset e(B)$ yields that $e(A \to B) = \mathcal{N}^+$. For the same A, B, $\mathcal{M}^b$ and e^*, $e_{c,\mathcal{M}^b}(A \to B) = 1$ holds iff $e(A \to B) = \mathcal{N}^+$, hence, iff $e(A) \subset e(B)$. However, $e(A) \subset e(B)$ holds iff $\varphi(e(A)) = e_{c,\mathcal{M}^b}(A) \leq_* \varphi(e(B)) = e_{c,\mathcal{M}^b}(B)$ holds. The assertion is proved. □

6 A nonstandard intensional numerical semantic for the first–order predicate calculus without functions

Given a $\mathcal{P}_0$-valued evaluation $e : \mathcal{L}_{\mathrm{PC}} \to \mathcal{P}_0$ generated by an evaluation $e^* : Var \to M$ of variables and by a $\mathcal{P}_0$-valued structure $\mathcal{M}^b = \langle M, \rho^b_1, \ldots, \rho^b_n, CS\rangle$ fitted for $\mathcal{L}_{\mathrm{PC}}$, there exists still another way how to project the values of e, i. e., subsets of $\mathcal{N}^+$, into $\langle 0,1\rangle$ than the mapping φ. Namely, (*nonstandard intensional*) *w-evaluation* of w. f. f.'s of the language $\mathcal{L}_{\mathrm{PC}}$ is a mapping $e_w : \mathcal{L}_{\mathrm{PC}} \to \langle 0,1\rangle$ such that there exists a $\mathcal{P}_0$-valued evaluation $e : \mathcal{L}_{\mathrm{PC}} \to \mathcal{P}_0$ induced by $e^* : Var \to M$ and by $\mathcal{M}^b$, such that, for each $A \in \mathcal{L}_{\mathrm{PC}}$, the value $w(\chi(e(A))) = \lim_{n\to\infty} \sum_{i=1}^{n} \chi(e(A))(i)$ exists and $e_w(A) =$

$w(\chi(e(A)))$. Nonstandard intensional w-evaluations differ substantially from the $\mathcal{P}_0$-valued and $\boldsymbol{c}$-valued ones, as they cannot be defined in the recurrent way beginning from evaluations of elementary w. f. f.'s and extended recurrently (inductively) to all w. f. f.'s using rules for propositional functors and quantifiers. In other words, $\mathcal{P}_0$-valued and $\boldsymbol{c}$-valued evaluations are *extensional*, but w-evaluations are not, hence, they are *intensional*. E. g., if $e(A) = \{1, 3, 5, 7, 9, \ldots\}$ for a w. f. f. $A \in \mathcal{L}_{\mathrm{PC}}$, then $e(\neg A) = \{2, 4, 6, \ldots\}$, hence, $w(e(A)) = w(e(\neg a)) = 1/2$, but $w(e(A \vee (\neg A))) = w(\mathcal{N}^+) = 1$, and $w(e(A \vee A)) = w(e(A)) = 1/2$, so that $w(e(A \vee B))$ is not defined, in general, by the values $w(e(A))$ and $w(e(B))$ (in order to simplify our notation we write $w(e(A))$ instead of $w(\chi(e(A))))$.

Setting

$$\begin{aligned} Taut^w_{\mathrm{PC}} \;=\; & \{A \in \mathcal{L}_{\mathrm{PC}} \cap Sent_{\mathrm{PC}} : w(e_{\mathcal{M}^b}(A)) \text{ is defined and } w(e_{\mathcal{M}^b}(A)) = 1 \\ & \text{for each } \mathcal{P}_0\text{-valued structure } \mathcal{M}^b \text{ fitted for } \mathcal{L}_{\mathrm{PC}} \\ & \text{and each } e^* : Var \to M\}, \qquad (6.1) \end{aligned}$$

we obtain easily that $Taut^b_{\mathrm{PC}} \subset Taut^w_{\mathrm{PC}}$ holds, as if $e_{\mathcal{M}^b}(A) = \mathcal{N}^+$, then trivially $w(e_{\mathcal{M}^b}(A)) = 1$.

The inverse inclusion $Taut^w_{\mathrm{PC}} \subset Taut^b_{\mathrm{PC}}$ is not so evident, but it can be also proved. Let us assume, in order to arrive at a contradiction, that there exists a sentence $A \in \mathcal{L}_{\mathrm{PC}}$ such that $A \in Taut^w_{\mathrm{PC}} - Taut^b_{\mathrm{PC}}$. As $Taut^b_{\mathrm{PC}} = Taut_{\mathrm{PC}}$, A is not a classical tautology of the FOPC, consequently, there exist a classical crisp structure $\mathcal{M}_0 = \langle M, \rho_1, \ldots, \rho_n, CS\rangle$ fitted for $\mathcal{L}_{\mathrm{PC}}$ and a mapping $e^* : Var \to M$ such that $e_{\mathcal{M}_0}(A) = 0$. Consider the $\mathcal{P}_0$-valued structure $\mathcal{M}^b_0 = \langle M, \rho^b_1, \ldots, \rho^b_n, CS\rangle$ with the same support set M and evaluation CS of constant symbols as $\mathcal{M}_0$ and such that, for each $i \leq n$ and each r_i-tuple $\langle m_1, m_2, \ldots, m_{r_i}\rangle \in M^{r_i}$, $\rho^b_i(m_1, \ldots, m_{r_i}) = \mathcal{N}^+$, if $\rho_i(m_1, \ldots, m_{r_i}) = 1$ (if $\langle m_1, \ldots, m_{r_i}\rangle \in \rho_i$, in the alternative notation), and $\rho^k_i(m_1, \ldots, m_{r_i}) = \emptyset$ (the empty subset of $\mathcal{N}^+$), if $\rho_i(m_1, \ldots, m_{r_i}) = 0(\langle m_1, \ldots, m_{r_i}\rangle \in M^{r_i} - \rho_i)$. Let us compute the value $e_{\mathcal{M}^b_0}(A)$ of the w. f. f. A with respect to the evaluation induced by the same substitution $e^* : Var \to M$ for variables as in the case of the classical crisp structure $\mathcal{M}_0$, but now applied to the $\mathcal{P}_0$-valued structure $\mathcal{M}^b_0$, which is obviously fitted for $\mathcal{L}_{\mathrm{PC}}$.

For each elementary w. f. f. $R_i(t_1, \ldots, t_{r_i})$, $i \leq n$, $t_j \in Var \cup Const$, $j \leq r_i$, $e_{\mathcal{M}^b_0}(R_i(t_1, \ldots, t_{r_i})) = \rho^b_i(t^{**}_1, \ldots, t^{**}_{r_i})$ by definition, where for all $j \leq r_i$, $t^{**}_j = e^*(t_j)$, if $t_j \in Var$, and $t^{**}_j = CS(t_j)$, if $t_j \in Const$, in every case, $\langle t^{**}_1, \ldots, t^{**}_{r_i}\rangle \in M^{r_i}$. By definition, $\rho^b_i(t^{**}_1, \ldots, t^{**}_{r_i}) = \mathcal{N}^+$ iff $\rho_i(t^{**}_1, \ldots, t^{**}_{r_i}) = 1$, but this last equality holds, due to the definition of the classical evaluation $e_{\mathcal{M}_0}$, iff $e_{\mathcal{M}_0}(R_i(t_1, \ldots, t_{r_i})) = 1$. Hence, $e_{\mathcal{M}^b_0}(R_i(t_1, \ldots, t_{r_i})) = \mathcal{N}^+$ iff $e_{\mathcal{M}_0}(R(t_1, \ldots, t_{r_i})) = 1$, and $e_{\mathcal{M}^b_0}(R_i(t_1, \ldots, t_{r_i})) = \emptyset$ iff $e_{\mathcal{M}_0}(R(t_1, \ldots, t_{r_i})) = 0$. By induction on the syntactical depth of w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ and by the recurrent rules for functors and quantifiers enabling to extend $e_{\mathcal{M}^b_0}$ and $e_{\mathcal{M}_0}$ uniquely from elementary w. f. f.'s to all w. f. f.'s of $\mathcal{L}_{\mathrm{PC}}$ we obtain easily that for each w. f. f. $B \in \mathcal{L}_{\mathrm{PC}}$, $e_{\mathcal{M}^b_0} = \mathcal{N}^+$, if $e_{\mathcal{M}_0}(B) = 1$, and $e_{\mathcal{M}^b_0}(B) = \emptyset$, if $e_{\mathcal{M}_0}(B) = 0$. In particular, $e_{\mathcal{M}^b_0}(A) = \emptyset$ and $w(e_{\mathcal{M}^b_0}(A)) = 0$ for the hypothetical w. f. f. A fixed above. So, there exist a $\mathcal{P}_0$-valued structure $\mathcal{M}^b_0 = \langle M, \rho^b_1, \ldots, \rho^b_n, CS\rangle$ fitted for $\mathcal{L}_{\mathrm{PC}}$ and a mapping $e^* : Var \to M$ such that $e_{\mathcal{M}^b_0}(A) = \emptyset$ and $w(e_{\mathcal{M}^b_0}(A)) = 0$, so that A cannot be in $Taut^w_{\mathrm{PC}}$ as assumed. Hence, the inclusion $Taut^w_{\mathrm{PC}} \subset Taut^b_{\mathrm{PC}}$ and the equality

$Taut^w_{\mathrm{PC}} = Taut^b_{\mathrm{PC}}$ are proved.

As a matter of fact, we have proved still more than this equality. Our notion of w-tautology can be (seemingly) weakened by setting

$$\begin{aligned} Taut^{ww}_{\mathrm{PC}} \;=\; & \{A \in \mathcal{L}_{\mathrm{PC}} : w(e_{\mathcal{M}^b}(A)) \text{ is defined and} \\ & w(e_{\mathcal{M}^b}(A)) > 0 \text{ holds for each } \mathcal{P}_0 \text{ valued structure} \\ & \mathcal{M}^b \text{ fitted for } \mathcal{L}_{\mathrm{PC}} \text{ and for each } e^* : Var \to M\}. \end{aligned} \tag{6.2}$$

What we have proved can be explicitly stated as follows.

Theorem 3. (Completeness theorem for the first-order predicate calculus without functions and with respect to the classical, $\mathcal{P}_0$-valued, nonstandard c-valued, and nonstandard intensional w-semantics)

$$Taut^{ww}_{\mathrm{PC}} = Taut^w_{\mathrm{PC}} = Taut^c_{\mathrm{PC}} = Taut_{\mathrm{PC}} = Ded_{\mathrm{PC}} \cap Sent_{\mathrm{PC}}. \tag{6.3}$$

□

7 Nonstandard semantics as probabilistic measures

As it is well-known, probability measure is not extensional in the sense that there is no function $G : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ such that, for a probability space $\langle \Omega, \mathcal{A}, P\rangle$ and for all A, $B \in \mathcal{A}$ the equalities

$$P(A \cup B) = G(P(A),\, P(B)) \tag{7.1}$$
$$P(A \cap B) = P\left(\Omega - ((\Omega - A) \cup (\Omega - B))\right) = 1 - G(1 - P(A),\, 1 - P(B)) \tag{7.2}$$

would hold. Obviously, when $P(A) = 1/2$, then $P(\Omega - A) = 1/2$ as well, but $P(A \cup A) = 1/2 \neq 1 = P(A \cup (\Omega - A))$. This property of intensionality (non-extensionality) of probabilistic measures follows from the fact that relative frequences are not extensional and from the basic (platonistic) idea on which probability theory relies and according to which relative frequences are imperfect images (shadows on the wall of the Platon caverne) of ideal and perfect values of probability measures; in other words, probabilities are theoretically accessible as limit values of certain sequences of relative frequences. So, looking for an appropriate mathematical tool for uncertainty quantification and processing, we are at the very beginning of our considerations faced to the necessity to choose between the intensionality of probabilistic measures and the extensionality of some other models, comparing the relative advantages and disadvantages of both the approaches. However, the ideas and results explained above bring us to the conclusion that the ultimate character of this choice is closely related to the classical linear ordering of the unit interval of real numbers and to the resulting operations of supremum and infimum, and that using the nonstandard (partial) ordering and operations presented in the foregoing chapters we could combine the extensional and the probabilistic properties of the numerical uncertainty degrees in a much larger extend that in the case of the classical structures over $\langle 0,1\rangle$.

Theorem 4. Let A, B be sentences of the FOPC, let $\mathcal{M}^b = \langle M,\, \rho_1^b, \ldots, \rho_n^b,\, CS\rangle$ be a $\mathcal{P}_0$-valued structure fitted for $\mathcal{L}_{\text{PC}}$, let $e^* : Var \to M$ be an evaluation of variables of $\mathcal{L}_{\text{PC}}$, let $e = e_{\mathcal{M}} : \mathcal{L}_{\text{PC}} \to \mathcal{P}_0$ be induced by $\mathcal{M}^b$ and e^*, let $e_w(A)$, $e_w(B)$ and $e_w(A \wedge B)$ be defined (let us recall that $e_w(A) = w(\chi(e(A))) = \lim_{n\to\infty}(1/n)\sum_{i=1}^n \chi(e(A))\,(i))$, let $e_c(A) = \varphi(e(A)) = \sum_{i=1}^\infty 2\chi(e(A))\,(i)\,3^{-i}$. Then

$$e_c(A \vee B) \;=\; e_c(A) + e_c(B) - e_c(A \wedge B), \tag{7.3}$$

$$e_w(A \vee B) \;=\; e_w(A) + e_w(B) - e_w(A \wedge B). \tag{7.4}$$

In particular, if $\neg(A \wedge B) \in Taut_{\text{cl}}$, then

$$e_c(A \vee B) \;=\; e_c(A) + e_c(B), \tag{7.5}$$

$$e_w(A \vee B) \;=\; e_w(A) + e_w(B), \tag{7.6}$$

and

$$e_c(T) = e_w(T) = 1 \tag{7.7}$$

for each $T \in Taut_{\text{PC}}$. Hence, both e_c and e_w possess the properties of finitely additive probability measures. $\square$

Proof. Let $A, B \in Sent_{\text{PC}}$, let $e : \mathcal{L}_{\text{PC}} \to \mathcal{P}_0$ be induced by a $\mathcal{P}_0$-valued structure $\mathcal{M}^b$ fitted for $\mathcal{L}_{\text{PC}}$ and by a mapping $e^* : Var \to M$. Then

$$e_c(A) = \varphi(e(A)) = \sum_{i=1}^{\infty} 2(\chi(e(A))\,(i))\,3^{-i} = \sum_{i \in e(A)} 2 \cdot 3^{-i}, \tag{7.8}$$

as $\chi(e(A))\,(i) = 1$, if $i \in e(A)$, $\chi(e(A))\,(i) = 0$ otherwise. If A, B are such that $\neg(A \wedge B) \in Taut_{\text{PC}}$, then $\neg(A \wedge B) \in Taut^b_{\text{PC}}$, so that $\mathcal{N}^+ - e(A \wedge B) = \mathcal{N}^+$ and $e(A \wedge B) = \emptyset$ follows. But $e(A \wedge B) = e(A) \cap e(B)$, so that, for $A \vee B$,

$$\begin{aligned} e_c(A \vee B) \;&=\; \sum_{i \in e(A \vee B)} 2 \cdot 3^{-i} = \sum_{i \in e(A) \cup e(B)} 2 \cdot 3^{-i} = \\ &=\; \sum_{i \in e(A)} 2 \cdot 3^{-i} + \sum_{i \in e(B)} 2 \cdot 3^{-i} = e_c(A) + e_c(B), \end{aligned} \tag{7.9}$$

and (7.5) is proved. Moreover, if $A \leftrightarrow B \in Taut_{\text{PC}}$, then $A \to B \in Taut_{\text{PC}}$ and $B \to A \in Taut_{\text{PC}}$, so that, by Lemma 5, $e(A) \subset e(B)$ and $e(B) \subset e(A)$, hence, $e(A) = e(B)$. Considering a general case of sentences A, B of FOPC and applying the results just obtained to the formulas $(A \wedge \neg B) \vee (A \wedge B)$ and $(B \wedge \neg A) \vee (A \vee B)$, we obtain immediately that

$$\begin{aligned} &\neg((A \wedge \neg B) \wedge (A \wedge B)) \in Taut_{\text{PC}}, \\ &\neg((B \wedge \neg A) \wedge (A \wedge B)) \in Taut_{\text{PC}}, \\ &((A \wedge \neg B) \vee (A \wedge B)) \leftrightarrow A \in Taut_{\text{PC}}, \\ &((B \wedge \neg A) \vee (A \wedge B)) \leftrightarrow B \in Taut_{\text{PC}}, \\ &((A \wedge \neg B) \vee (B \wedge \neg A) \vee (A \wedge B)) \leftrightarrow A \vee B \in Taut_{\text{PC}}, \end{aligned} \tag{7.10}$$

so that

$$
\begin{aligned}
e_c(A \vee B) &= \sum_{i\in e(A\vee B)} 2\cdot 3^{-i} = \sum_{i\in e(A\wedge\neg B)} 2\cdot 3^{-i} + \\
&\quad + \sum_{i\in e(A\wedge B)} 2\cdot 3^{-i} + \sum_{i\in e(B\wedge\neg A)} 2\cdot 3^{-i} = \\
&= \left(\sum_{i\in e(A\wedge\neg B)} 2\cdot 3^{-i} + \sum_{i\in e(A\wedge B)} 2\cdot 3^{-i}\right) + \\
&\quad + \left(\sum_{i\in e(B\wedge\neg A)} 2\cdot 3^{-i} + \sum_{i\in e(A\wedge B)} 2\cdot 3^{-i}\right) - \sum_{i\in e(A\wedge B)} 2\cdot 3^{-i} + \sum_{i\in e(A\wedge B)} 2\cdot 3^{-i} = \\
&= \sum_{i\in e(A)} 2\cdot 3^{-i} + \sum_{i\in e(B)} 2\cdot 3^{-i} - \sum_{i\in e(A\wedge B)} 2\cdot 3^{-i} = \\
&= e_c(A) + e_c(B) - e_c(A\wedge B), \qquad (7.11)
\end{aligned}
$$

and (7.3) is proved.

The proof for e_w is similar. If $\neg(A\wedge B) \in \mathit{Taut}_{\mathrm{PC}}$, then $(e(A)\cap[n])\cap(e(B)\cap[n]) = \emptyset$ for each $n \in \mathcal{N}^+$, where $[n] = \{1, 2, \ldots, n\}$ is the initial segment of $\mathcal{N}^+$ of the length n. So, $\neg(A \wedge B) \in \mathit{Taut}_{\mathrm{PC}}$ implies that

$$
\begin{aligned}
e_w(A \vee B) &= \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} \chi(e(A\vee B))\,(i) = \\
&= \lim_{n\to\infty} (1/n)\mathrm{card}\{i \le n : i \in e(A \vee B)\} = \\
&= \lim_{n\to\infty} (1/n)\mathrm{card}(e(A \vee B) \cap [n]) = \\
&= \lim_{n\to\infty} (1/n)\,[\mathrm{card}(e(A) \cap [n]) + \mathrm{card}(e(B) \cap [n])] = \\
&= \lim_{n\to\infty} (1/n)\mathrm{card}(e(A) \cap [n]) + \lim_{n\to\infty} (1/n)\mathrm{card}(e(B) \cap [n]) = \\
&= e_w(A) + e_w(B), \qquad (7.12)
\end{aligned}
$$

supposing that $e_w(A)$ and $e_w(B)$ are defined, so that (7.6) holds. Considering the general case of sentences A, $B \in \mathit{Sent}_{\mathrm{PC}}$, supposing that the values $e_w(A)$, $e_w(B)$, and $e_w(A \wedge B)$ are defined, and applying (7.12) to the sentences $A \wedge \neg B$, $B \wedge \neg A$, and $A \wedge B$, we obtain in the same way as above, that

$$
\begin{aligned}
&\mathrm{card}(e(A \vee B) \cap [n]) = \mathrm{card}(e(A \wedge \neg B) \cap [n]) + \\
&+\mathrm{card}(e(B \wedge \neg A) \cap [n]) + \mathrm{card}(e(A \wedge B) \cap [n]) = \\
= \;&\mathrm{card}(e(A) \cap [n]) + \mathrm{card}(e(B) \cap [n]) - \mathrm{card}(e(A \wedge B) \cap [n]), \qquad (7.13)
\end{aligned}
$$

so that (7.4) immediately follows when all the limit values are defined. As $e(T) = \mathcal{N}^+$ for each $T \in \mathit{Taut}_{\mathrm{PC}}$, (7.7) immediately follows, so that the theorem is proved. □

As can be almost obviously seen, but as it is perhaps worth being stated explicitly, Theorem 4 can be generalized to the case of finite disjunctions; let us limit ourselves to the case of logically disjoint components. Let A_1, $A_2, \ldots, A_m$ be sentences of $\mathcal{L}_{\mathrm{PC}}$

such that $\neg(A_i \wedge A_j) \in Taut_{\mathrm{PC}}$ holds for each $i, j \leq m,\ i \neq j$, let $\bigvee_{i=1}^{m} A_i$ abbreviate $A_1 \vee A_2 \vee \cdots \vee A_m$. Then

$$e_c\left(\bigvee_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} e_c(A_i), \tag{7.14}$$

and supposing that $e_w(A_i)$ for each $i \leq m$ is defined, also $e_w\left(\bigvee_{i=1}^{m} A_i\right)$ is defined and

$$e_w\left(\bigvee_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} e_w(A_i). \tag{7.15}$$

Generalized forms of (7.3) and (7.4) can be also deduced. However, the situation differs principially when considering the *σ-additivity* (the countable additivity) of the evaluations e_c and e_w. As the language $\mathcal{L}_{\mathrm{PC}}$ does not allow to define disjunctions of infinitely many formulas, we have to formalize the next statement in a slightly modified way.

Theorem 5. Let $A_1, A_2, \ldots$ be an infinite sequence of formulas of $\mathcal{L}_{\mathrm{PC}}$, let $\mathcal{M}^b$ be a $\mathcal{P}_0$-valued structure fitted for $\mathcal{L}_{\mathrm{PC}}$, let $e^* : Var \to M$ be an evaluation of variables, let $e : \mathcal{L}_{\mathrm{PC}} \to \mathcal{P}_0$ be induced by $\mathcal{M}^b$ and e^*, let $\neg(A_i \wedge A_j) \in Taut_{\mathrm{PC}}$ hold for each $i, j \geq 1,\ i \neq j$. Then

$$\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right) = \sum_{i=1}^{\infty} \varphi(e(A_i)) = \sum_{i=1}^{\infty} e_c(A_i). \tag{7.16}$$

Hence, the difference is that the value $\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right)$ cannot be written as $e_c\left(\bigvee_{i=1}^{\infty} A_i\right)$ because of the fact that $\bigvee_{i=1}^{\infty} A_i$ is not a w.f.f. of $\mathcal{L}_{\mathrm{PC}}$. A particular case when $\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right)$ can be written as $e_c((\exists\, x)\, A)$ for an appropriate w.f.f. A of $\mathcal{L}_{\mathrm{PC}}$ will be discussed below.

Proof. Like as in the proof of Theorem 4 we obtain that $e(A_i) \cap e(A_j) = \emptyset$ for each $i, j \geq 1,\ i \neq j$, so that

$$\begin{aligned} \varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right) &= \sum_{j=1}^{\infty} 2\chi\left(\bigcup_{i=1}^{\infty} e(A_i)\right)(j)\, 3^{-j} = \\ = \sum_{j \in \bigcup_{i=1}^{\infty} e(A_i)} 2 \cdot 3^{-j} &= \sum_{i=1}^{\infty} \sum_{j \in e(A_i)} 2 \cdot 3^{-j} = \\ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2\chi(e(A_i))(j)\, 3^{-j} &= \sum_{i=1}^{\infty} e_c(A_i), \end{aligned} \tag{7.17}$$

and the assertion is proved. □

Corollary 1. Let A be a w.f.f. of $\mathcal{L}_{\mathrm{PC}}$ containing x as the only free variable, let $\neg(A(c_1) \wedge A(c_2)) \in Taut_{\mathrm{PC}}$ hold for each $c_1, c_2 \in Const,\ c_1 \neq c_2$, where $A(c_1)$ ($A(c_2)$, resp.) is the sentence of $\mathcal{L}_{\mathrm{PC}}$ resulting when all free occurrences of x in A are replaced by c_1 (by c_2, resp.), let $\mathcal{M}^b = \langle M, \rho_1^b, \ldots, \rho_n^b, CS\rangle$ be a $\mathcal{P}_0$-valued structure fitted for

$\mathcal{L}_{\mathrm{PC}}$ and such that M is countable and CS is a one-to-one mapping between M and *Const*, let $e^* : \mathit{Var} \to M$ be an evaluation of variables of $\mathcal{L}_{\mathrm{PC}}$, let $e : \mathcal{L}_{\mathrm{PC}} \to \mathcal{P}_0$ be induced by $\mathcal{M}^b$ and e^*. Then

$$e_c((\exists\, x)\, A) = \sum_{m \in M} e_{c,x|m}(A). \tag{7.18}$$

$\square$

Proof. If $\neg(A(c_1) \wedge A(c_2)) \in \mathit{Taut}_{\mathrm{PC}}$ for some $c_1,\, c_2 \in \mathit{Const},\ c_1 \neq c_2$, then $\neg(A(c_1) \wedge A(c_2)) \in \mathit{Taut}^b_{\mathrm{PC}}$ as well, so that, for each $\mathcal{P}_0$-valued structure $\mathcal{M}^b$ fitted for $\mathcal{L}_{\mathrm{PC}}$, and for all evaluations of variables by elements of the support set of $\mathcal{M}^b$, $e(\neg(A(c_1) \wedge A(c_2))) = \mathcal{N}^+$, so that the same holds for $\mathcal{M}^b$ and e^* satisfying the conditions of the assertion. Consequently, $e(A(c_1) \wedge A(c_2)) = e(A(c_1)) \cap e(A(c_2)) = \emptyset$. so that

$$\begin{aligned} e_c((\exists\, x)\, A) &= \varphi(e((\exists\, x)\, A)) = \varphi\left(\bigcup_{m \in M} e_{x|m(A)}\right) = \\ &= \varphi\left(\bigcup_{c \in \mathrm{Const}} e_{x|CS^{-1}(c)}(A)\right) = \sum_{j \in \bigcup_{c \in \mathrm{Const}} e_{x|CS^{-1}(c)}(A)} 2 \cdot 3^{-j}. \end{aligned} \tag{7.19}$$

If $c_1,\, c_2 \in \mathit{Const},\ c_1 \neq c_2$, then $e_{x|CS^{-1}(c_1)(A)} = e(A(c_1))\ e_{x|CS^{-1}(c_2)(A)} = \emptyset$. Consequently, we obtain that

$$\begin{aligned} e_c((\exists\, x)\, A) &= \sum_{c \in \mathit{Const}} \left(\sum_{j \in e_{x|CS^{-1}(c)}(A)} 2 \cdot 3^{-j} \right) = \\ &= \sum_{m \in M} \left(\sum_{j \in e_{x|m}(A)} 2 \cdot 3^{-j} \right) = \sum_{m \in M} e_{c,x|m}(A), \end{aligned} \tag{7.20}$$

and the corollary is proved. $\square$

8 Two auxiliary lemmas

The two assertions of rather technical nature presented in this chapter will be used in the next chapter, where we shall evaluate the range in which probabilistic and possibilistic measures can be defined by nonstandard semantics of first-order predicate calculus without functions.

Lemma 6. Let $p \in \langle 0, 1\rangle$, let the infinite binary sequence $\langle x_1, x_2, \ldots \rangle \in \{0,1\}^\infty$ be defined in this way:

(i) $x_1 = 1,\ x_2 = 0$.

(ii) Let $x_1,\, x_2, \ldots, x_n$ be already defined. If $n^{-1} \sum_{i=1}^n x_i \leq p$, then $x_{n+1} = 1$, if $n^{-1} \sum_{i=1}^n x_i > p$, then $x_{n+1} = 0$.

Then for each $n = 2,\, 3, \ldots$ the inequality $|n^{-1} \sum_{i=1}^n x_i - p| \leq n^{-1}$ holds, so that $\lim_{n \to \infty} \sum_{i=1}^n x_i = p$. $\square$

Proof. The proof is by induction on n. If $n = 2$, then $n^{-1}\sum_{i=1}^{n} x_i = (1/2)(1+0) = 1/2$ and the inequality $|1/2 - p| \leq 1/2$ holds trivially for each $p \in \langle 0, 1\rangle$. Let the inequality

$$p - (1/n) \leq n^{-1}\sum_{i=1}^{n} x_i \leq p + (1/n) \tag{8.1}$$

hold for some $n \geq 2$. Let $x_{n+1} = 1$. Then $n^{-1}\sum_{i=1}^{n} x_i \leq p$ must hold due to the rule by which x_{n+1} is defined. Combining this inequality with (8.1) we obtain that

$$p - (1/n) \leq n^{-1}\sum_{i=1}^{n} x_i \leq p, \tag{8.2}$$

hence,

$$np - 1 \leq \sum_{i=1}^{n} x_i \leq np. \tag{8.3}$$

As $0 \leq p \leq 1$ holds, $np - 1 \leq \sum_{i=1}^{n} x_i$ implies that $np + p - 1 \leq \sum_{i=1}^{n} x_i + 1$, and $\sum_{i=1}^{n} x_i \leq np$ implies that $\sum_{i=1}^{n} x_i + 1 \leq np + p + 1$, so that

$$np + p - 1 \leq \sum_{i=1}^{n} x_i + 1 \leq np + p + 1, \tag{8.4}$$

what turns into

$$p(n+1) - 1 \leq \sum_{i=1}^{n+1} x_i \leq p(n+1) + 1 \tag{8.5}$$

and

$$p - (1/(n+1)) \leq (n+1)^{-1}\sum_{i=1}^{n+1} x_i \leq p + (1/(n+1)). \tag{8.6}$$

Hence, $\left|(n+1)^{-1}\sum_{i=1}^{n+1} x_i - p\right| \leq (n+1)^{-1}$ holds and the induction step is proved, if $x_{n+1} = 1$.

Let (8.1) hold, let $x_{n+1} = 0$. Then $n^{-1}\sum_{i=1}^{n} x_i > p$ holds due to the rule defining x_{n+1}, but at the same time $p + n^{-1} \geq n^{-1}\sum_{i=1}^{n} x_i$ holds by the induction assumption. Hence, the inequalities

$$p < n^{-1}\sum_{i=1}^{n} x_i \leq p + n^{-1} \tag{8.7}$$

and

$$np < \sum_{i=1}^{n} x_i \leq np + 1 \tag{8.8}$$

hold. If $np < \sum_{i=1}^{n} x_i$, then $np + p - 1 < \sum_{i=1}^{n} x_i + 0 = \sum_{i=1}^{n+1} x_i$, as $p - 1 \leq 0$ and $x_{n+1} = 0$ hold. If $\sum_{i=1}^{n} x_i \leq np + 1$, then $\sum_{i=1}^{n+1} x_i \leq np + p + 1$ also follows, as $p \geq 0$. So,

$$np + p - 1 \leq \sum_{i=1}^{n+1} x_i \leq np + p + 1, \tag{8.9}$$

so that (8.6) and $\left|(n+1)^{-1}\sum_{i=1}^{n+1} x_i - p\right| \leq (n+1)^{-1}$ holds also in this case and Lemma 6 is proved. $\square$

Lemma 7. Let $p_1, p_2, \ldots$ be a probability distribution on the set $\mathcal{N}^+$ of positive integers, i.e., $0 \leq p_i \leq 1$ for each $i \in N^+$ and $\sum_{i=1}^{\infty} p_i = 1$ hold. Then there exists a sequence $\{\alpha_{ij}\}_{i,j=1}^{\infty}$ such that $\alpha_{ij} \in \{0,1\}$ for each $i, j \in \mathcal{N}^+$, and

(i) if $p_i = 0$, then $\alpha_{ij} = 0$ for all $j \in \mathcal{N}^+$,

(ii) if $p_i = 1$, then $\alpha_{ij} = 1$ for all $j \in \mathcal{N}^+$,

(iii) if $0 < p_i < 1$, then $w(\{\alpha_{ij}\}_{i,j=1}^{\infty}) = \lim_{n\to\infty}(1/n)\sum_{j=1}^{n} \alpha_{i,j} = p_i$,

(iv) for each $j \in \mathcal{N}^+$ there exists just one $i \in \mathcal{N}^+$ such that $\alpha_{ij} = 1$. $\square$

Proof. If $p_{i_0} = 1$ for some $i_0 \in \mathcal{N}^+$, then $p_i = 0$ for all $i \neq i_0$, hence, the sequence $\alpha_{i_0 j} = 1$ for all $j \in \mathcal{N}^+$, $\alpha_{ij} = 0$ for all $i, j \in \mathcal{N}^+$, $i \neq i_0$, obviously satisfies the assertion. If $p_i = 0$ for $i \in A \subset \mathcal{N}^+$, set $\alpha_{ij} = 0$ for all $i \in A$, $j \in \mathcal{N}^+$ and consider the subsequence $p_1', p_2', \ldots$ of $\langle p_1, p_2, \ldots\rangle$ containing just the positive p_i's. Having obtained, in the way to be defined below, α_{ij} for the sequence $\langle p_1', p_2', \ldots\rangle$, we interpolate the identically zero rows for the indices corresponding to $p_i = 0$ and re-enumerate the first indices in α_{ij}. Hence, we may limit ourselves to the case when $0 < p_i < 1$ and, consequently, $0 < \sum_{j=1}^{i} p_j < 1$ hold for each $i \in \mathcal{N}^+$.

Let $q_1 = p_1$, let $q_i = p_i\left(1 - \sum_{j=1}^{i-1} p_j\right)^{-1}$ $\left(= p_i \sum_{j=i}^{\infty} p_j\right)$ for all $i \geq 2$, due to the condition $0 < p_i < 1$, $i \in \mathcal{N}^+$, all q_i's are well-defined. Let $\boldsymbol{x}^i = \langle x_1^i, x_2^i, \ldots\rangle \in \{0,1\}^{\infty}$ be the sequence defined in Lemma 6 for $p = q_i$, so that $w(\boldsymbol{x}^i) = \lim_{n\to\infty}(1/n)\sum_{j=1}^{n} x_j^i = q_j$. Let us remark that $0 < q_i < p_i < 1$ holds for each $i \leq 2$. Let us define binary sequences $\boldsymbol{y}^i = \langle y_1^i, y_2^i, \ldots\rangle \in \{0,1\}^{\infty}$ for each $i \in \mathcal{N}^+$ by the following induction. Let $\boldsymbol{y}^1 = \boldsymbol{x}^1$ and suppose that $\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^n$ have been already defined in such a way that $w(\boldsymbol{y}^i) = p_i$ for all $i \leq n$ and $s(\boldsymbol{y}^i) \cap s(\boldsymbol{y}^j) = \emptyset$ for all $i, j \leq n$, $i \neq j$, where $s(\boldsymbol{y}^i) = \{k \in \mathcal{N}^+ : y_k^i = 1\}$.

As $\sum_{i=1}^{n} p_i < 1$ and $p_{n+1} > 0$ hold, y^{n+1} can be defined in this way. Let $\boldsymbol{z}^n = \langle z_1^n, z_2^n, \ldots\rangle \in \{0,1\}^{\infty}$ be such that $z_i^n = 1$ iff there exists $j \leq n$ such that $y_i^j = 1$ (such j is just one, as $s(\boldsymbol{y}^k) \cap s(\boldsymbol{y}^j) = \emptyset$ for each $k, j \leq n$, $k \neq j$). In other terms, $z_i^n = \max\{y_i^j : 1 \leq j \leq n\}$. Consequently, $s(\boldsymbol{z}^n) = \{k \in \mathcal{N}^+ : z_k^n = 1\} = \bigcup_{j=1}^{n} s(\boldsymbol{y}^j)$, so that

$$
\begin{aligned}
w(\boldsymbol{z}^n) &= \lim_{k\to\infty} k^{-1} \sum_{i=1}^{k} z_i^n = \lim_{k\to\infty} k^{-1} \sum_{i\in s(\boldsymbol{z}^n)}^{i\leq k} z_i^n = \\
&= \lim_{k\to\infty} k^{-1} \sum_{i\in\bigcup_{j=1}^{n} s(\boldsymbol{y}^j)}^{i\leq k} z_i^n = \\
&= \lim_{k\to\infty} k^{-1} \sum_{j=1}^{n} \sum_{i\in s(\boldsymbol{y}^i)}^{i\leq k} z_i^n = \\
&= \sum_{j=1}^{n} \lim_{k\to\infty} k^{-1} \sum_{i\in s(\boldsymbol{y}^n)}^{i\leq k} y_i^j =
\end{aligned}
$$

$$= \sum_{j=1}^{n} \lim_{k\to\infty} k^{-1} \sum_{i=1}^{k} y_i^j = \sum_{j=1}^{n} w(\boldsymbol{y}^j) = \sum_{j=1}^{n} p_j < 1, \tag{8.10}$$

as $y_i^j = z_i^n = 1$ for each $i \in s(\boldsymbol{y}^j)$ and $y_i^j = 0$ for each $i \in \mathcal{N}^+ - s(\boldsymbol{y}^j)$. Hence, there are infinitely many occurrences of zeros in $\boldsymbol{z}^n$, as in the opposite case $w(\boldsymbol{z}^n)$ would be one, in other terms, the sequence $j_1 < j_2 < \ldots$ of positive integers such that $k = j_m$ for some m iff $z_k^n = 0$, is infinite. Set $\boldsymbol{y}^{n+1} = \langle y_1^{n+1}, y_2^{n+1}, \ldots \rangle$ in this way: $y_{j_m}^{n+1} = x_m^{n+1}$, $y_k^{n+1} = 0$ otherwise, i. e., if k does not occur in the sequence $\langle j_1, j_2, \ldots \rangle$ of integers. Obviously, if $z_k^n = 1$, then $y_k^{n+1} = 0$, so that $s(\boldsymbol{y}^{n+1}) \cap s(\boldsymbol{z}^n) = \emptyset$, consequently, $s(\boldsymbol{y}^{n+1}) \cap s(\boldsymbol{y}^j) = \emptyset$ for each $j \leq n$. As $x_1^{n+1} = 1$ for all $n \in \mathcal{N}^+$ due to the construction of $\boldsymbol{x}^{n+1}$ defined in Lemma 6, $y_{j_1}^{n+1} = 1$ as well, so that $\bigcup_{n=1}^{\infty} s(\boldsymbol{y}^n) = \mathcal{N}^+$. The only thing we have to prove is that $w(\boldsymbol{y}^{n+1}) = p_{n+1}$.

Take $m \in \mathcal{N}^+$ and set $K_m^n = \sum_{j=1}^{m} z_j^m$. Then

$$m^{-1} \sum_{i=1}^{m} y_i^{n+1} = m^{-1} \sum_{i=1}^{m-K_m^n} x_i^{n+1}, \tag{8.11}$$

as K_m^n is the number of indices in $\boldsymbol{y}^{n+1}$ corresponding to the occurrences of units in $\boldsymbol{z}^n$ for which $y_i^{n+1} = 0$ due to the construction of $\boldsymbol{y}^{n+1}$. Hence, just the units occurring in the initial segment of the length $m - K_m^n$ of $\boldsymbol{x}^{n+1}$ occur in the initial segment of the length m of the sequence $\boldsymbol{y}^{n+1}$. Inequality (2.1) applied to the initial segment of the length $m - K_m^n$ of the sequence $\boldsymbol{x}^{n+1}$ yields that

$$q_{n+1} - (m - K_m^n)^{-1} \leq (m - K_m^n)^{-1} \sum_{i=1}^{m-K_m^n} x_i^{n+1} \leq q_{n+1} + (m - K_m^n)^{-1}. \tag{8.12}$$

Hence, setting $p_{n+1} \left(1 - \sum_{i=1}^{n} p_i\right)^{-1}$ for q_{n+1}, multiplying all the members of (8.12) by $m - K_m^n$ and then dividing them by m, we obtain that

$$\begin{aligned} & p_{n+1} \left(1 - \sum_{i=1}^{n} p_i\right)^{-1} m^{-1}(m - K_m^n) - m^{-1} \leq m^{-1} \sum_{i=1}^{m-K_m^n} x_i^{n+1} = \\ = \;& m^{-1} \sum_{i=1}^{m} y_i^{n+1} \leq p_{n+1} \left(1 - \sum_{i=1}^{n} p_i\right)^{-1} m^{-1}(m - K_m^n) + m^{-1}. \end{aligned} \tag{8.13}$$

But,

$$\begin{aligned} \lim_{n\to\infty} m^{-1}(m - K_m^n) &= \lim_{m\to\infty} \left(1 - m^{-1} \sum_{j=1}^{m} z_j^n\right) = \\ &= 1 - w(\boldsymbol{z}^n) = 1 - \sum_{i=1}^{n} p_i \end{aligned} \tag{8.14}$$

by (8.10). Hence, (8.13) yields that

$$\begin{aligned} & \lim_{m\to\infty} p_{n+1} \left(1 - \sum_{i=1}^{n} p_i\right)^{-1} m^{-1} \left(m - \sum_{j=1}^{m} z_j^n\right) - m^{-1} = \\ & \lim_{m\to\infty} p_{n+1} \left(1 - \sum_{i=1}^{n} p_i\right)^{-1} m^{-1} \left(m - \sum_{j=1}^{m} z_j^n\right) + m^{-1} = p_{n+1}, \end{aligned} \tag{8.15}$$

so that $\lim_{m\to\infty} m^{-1}\sum_{i=1}^{m} y_i^{n+1} = p_{n+1} = w(\boldsymbol{y}^{n+1})$ also holds. Setting $\alpha_{ij} = y_j^i$ for each $i,\, j \in \mathcal{N}^+$ we can easily see that Lemma 7 is proved. $\square$

9 Definability of probabilistic and possibilistic measures by nonstandard semantics over probabilistic logics

Considering the results of Chapter 7 according to which nonstandard evaluations of w. f. f.'s of the propositional calculus (and also of FOPC) satisfy the conditions imposed to probabilistic measures, a natural question arises: how large is the class of probability distributions and measures which can be defined by such nonstandard evaluations? The following statement proves that every probability distribution over a finite or countable set can be obtained in this way.

Theorem 6. Let q_i, $i \in \mathcal{N}^+$, be a probability distribution on $\mathcal{N}^+$, i. e., $q_i \geq 0$ for each $i \in \mathcal{N}^+$ and $\sum_{i=1}^{\infty} q_i = 1$ hold. Let $\mathcal{L}_{\mathrm{PROP}}$ be the propositional language obtained when replacing the elementary formulas of the language $\mathcal{L}_{\mathrm{PC}}$ in a one-to-one way by propositional variables $p_1,\, p_2, \ldots$ Then there exists an evaluation $e : \{p_1, p_2, \ldots\} \to \mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$ such that, extending e to $e : \mathcal{L}_{\mathrm{PROP}} \to \mathcal{P}_0$ in the same way as in the case of $\mathcal{L}_{\mathrm{PC}}$, setting

$$e_w(\varphi) = w(e(\varphi)) = \lim_{n\to\infty} (1/n)\, \mathrm{card}[e(\varphi) \cap \{1, 2, \ldots, n\}] \tag{9.1}$$

for each w. f. f. φ of $\mathcal{L}_{\mathrm{PROP}}$ for which this limit value is defined, setting $Q(A) = \sum_{i\in A} q_i$ for each $A \subset \mathcal{N}^+$, and setting $A^0 = \bigvee_{i\in A} p_i \in \mathcal{L}_{\mathrm{PROP}}$ for each finite set $A \subset \mathcal{N}^+$, the equality $w_w(A^0) = Q(A)$ holds. $\square$

Proof. Let $\{\alpha_{ij}^q\}_{i,j=1}^{\infty}$ be a binary sequence satisfying the statement of Lemma 7 with respect to the probability distribution $\{q_1, q_2, \ldots\}$ on $\mathcal{N}^+$. Let $e(p_i) = \{j \in \mathcal{N}^+ : \alpha_{ij}^q = 1\}$. For each finite $A \subset \mathcal{N}^+$, $e\left(\bigvee_{i\in A} p_i\right) = \bigcup_{i\in A}\{j \in \mathcal{N}^+ : \alpha_{ij}^q = 1\}$ and the sets $e(p_i)$ are disjoint for different i's due to the properties of $\{\alpha_{ij}^q\}_{i,j=1}^{\infty}$. So,

$$\begin{aligned} e_w(A^0) &= e_w\left(\bigvee_{i\in A} p_i\right) = w\left(e\left(\bigvee_{i\in A} p_i\right)\right) = w\left(\bigcup_{i\in A} e(p_i)\right) = \\ &= \lim_{n\to\infty} (1/n)\, \mathrm{card}\left[\bigcup_{i\in A} e(p_i) \cap \{1, 2, \ldots, n\}\right] = \\ &= \lim_{n\to\infty} (1/n) \sum_{i\in A} \mathrm{card}[e(p_i) \cap \{1, 2, \ldots, n\}] = \\ &= \sum_{i\in A} \lim_{n\to\infty} (1/n)\, \mathrm{card}[e(p_i) \cap \{1, 2, \ldots, n\}] = \sum_{i\in A} q_i = Q(A). \end{aligned} \tag{9.2}$$

The assertion is proved. $\square$

Nonstandard evaluations defined over the propositional language $\mathcal{L}_{\text{PROP}}$, induced by $\mathcal{L}_{\text{PC}}$ in the way described in Theorem 6, stand also in close connection to *possibilistic measures*, a very topical alternative (with respect to probability measures) tool for uncertainty quantification and processing. Namely, the $\boldsymbol{c}$-evaluations e_c, induced by a boolean-valued evaluation $e : \{p_1, p_2, \ldots\} \to \mathcal{P}_0$ in the way defined above and taking their values in the Cantor subset $\mathcal{C}$ of the unit interval, can be proved to possess the properties of possibilistic measures.

Given a nonempty set Ω, the most simple definition reads that possibilistic (or: possibility) measure on Ω is a mapping Π which takes the power-set $\mathcal{P}(\Omega)$ of all subsets of Ω into $\langle 0, 1\rangle$ in such a way that $\Pi(\emptyset) = 0$, $\Pi(\Omega) = 1$, and

$$\Pi(A \cup B) = \max\{\pi(A),\, \pi(B)\} \tag{9.3}$$

for all A, $B \subset \Omega$. Modifications in at least the three following directions can be introduced.

(i) (9.3) is strengthened in such a way that

$$\Pi\left(\bigcup_{A\in\mathcal{A}} A\right) = \sup\{\pi(A) : A \in \mathcal{A}\} \tag{9.4}$$

holds also for at least some infinite classes $\mathcal{A}$ of subsets of Ω (for finite $\mathcal{A}$'s it follows easily from (9.3)). E. g. (9.4) is supposed to hold for all the classes $\mathcal{A}$ the cardinality of which does not exceed a fixed infinite cardinal number.

(ii) (9.3) is weakened in the sense that it holds not for all A, $B \subset \Omega$, but only for those belonging to a proper subset $\mathcal{S}$ of $\mathcal{P}(\Omega)$, usually endowed by some natural properties, e. g., closed with respect to finite or even to certain infinite unions. Both the modifications sub (i) and (ii) can be combined with each other, so that we can arrive, e. g., at the definition of σ-complete possibilistic measure: it is a mapping Π defined on a nonempty σ-field $\mathcal{S}$ of subsets of Ω (hence, $\mathcal{S}$ is closed with respect to the set-theoretic operations of complement and countable unions) and such that (9.4) holds for each countable system $\mathcal{A} \subset \mathcal{S}$ of subsets of Ω.

(iii) The supremum operation in (9.4), tacitly assumed to be that related to the usual linear ordering of the unit interval of real numbers, can be replaced by the supremum operations induced by another, even a partial, ordering of this interval. E. g., we may define a Cantor-valued σ-complete possibilistic measure Π defined on the power-set $\mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$ of all sets of positive integers in such a way that $\Pi(\emptyset) = 0$, $\Pi(\mathcal{N}^+) = 1$, $\Pi(A) \in \mathcal{C}$ (Cantor set) for each $A \subset \mathcal{N}^+$, and

$$\Pi\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigvee_{A\in\mathcal{A}} \Pi(A) \tag{9.5}$$

holds for each countable $\mathcal{A} \subset \mathcal{P}_0$. Here $\bigvee$ is the supremum operation defined by

$$\bigvee_{\alpha\in D\subset\mathcal{C}} x = \sum_{i=1}^{\infty} 2\max\{x_i : x \in D\}\, 3^{-i} \tag{9.6}$$

for each nonempty subset D of the Cantor set $\mathcal{C}$ (cf. above for details) and corresponding to the partial ordering relation $\leq_*$ according to which $x \leq_* y$

holds for $x,\, y \in \mathcal{C}$ iff $x_i \leq y_i$ holds for each $i \in \mathcal{N}^+$. Here $\langle x_1, x_2, \ldots\rangle$ and $\langle y_1, y_2, \ldots\rangle \in \{0,2\}^\infty$ are the unique decompositions of the real numbers x, y to the base 3 not containing any occurrence of the numeral 1.

Theorem 7. Let $e^* : \{p_1, p_2, \ldots\} \to \mathcal{P}_0$ be a $\mathcal{P}_0$-valued evaluation of the propositional variables of the language $\mathcal{L}_{\mathrm{PROP}}$ induced by the language $\mathcal{L}_{\mathrm{PC}}$ of the FOPC, let $e : \mathcal{L}_{\mathrm{PROP}} \to \mathcal{P}_0$ be the $\mathcal{P}_0$-valued evaluation of $\mathcal{L}_{\mathrm{PROP}}$ uniquely defined by e^* and by the induction rules for propositional connectives $\neg$ and $\to$, let e_c be the $\boldsymbol{c}$-valued evaluation of $\mathcal{L}_{\mathrm{PROP}}$ defined by e. Let

$$\mathcal{A} = \left\{A \subset \mathcal{N}^+ : A = e(\alpha),\, \alpha \in \mathcal{L}_{\mathrm{PROP}}\right\} \tag{9.7}$$

be the system of inverse images of w. f. f.'s of $\mathcal{L}_{\mathrm{PROP}}$ under the evaluation e, let $\mathcal{S} = \sigma(A) \subset \mathcal{P}_0$ be the minimal σ-field of subsets of $\mathcal{N}^+$ containing the system $\mathcal{A}$. Then e_c can be uniquely extended to a σ-complete possibilistic measure defined on $\mathcal{S}$, i. e.,

$$\hat{e}_c\left(\bigcup\nolimits_{A\in\mathcal{A}} A\right) = \bigvee\nolimits_{A\in\mathcal{A}} \hat{e}_c(A) \tag{9.8}$$

holds for each countable system $\mathcal{A}$ of subsets of $\mathcal{A}^+$ belonging to $\mathcal{S}$, here $\hat{e}_c(A) = e_c(\alpha)$, if $e(\alpha) = A$.

Proof. By the definition of $\mathcal{P}_0$-valued evaluations (item (ii)) and by (3.7), $e(\neg\alpha) = \mathcal{N}^+ - e(\alpha)$ and $e(\alpha \vee \beta) = e(\alpha) \cup e(\beta)$ for $\alpha,\, \beta \in \mathcal{L}_{\mathrm{PROP}}$, so that the system $\mathcal{A}$ of subsets of $\mathcal{N}^+$, defined by (9.7) is an algebra (field) of sets. For each w. f. f. $\alpha \in \mathcal{L}_{\mathrm{PROP}}$, $e(\alpha \wedge \neg\alpha) = e(\alpha) \cap (\mathcal{N}^+ - e(\alpha)) = \emptyset$ and $e(\alpha \vee \neg\alpha) = e(\alpha) \cup (\mathcal{N}^+ - e(\alpha)) = \mathcal{N}^+$, so that $\hat{e}_c(\emptyset) = 0$ and $\hat{e}_c(\mathcal{N}^+) = 1$. Set, for each $A \subset \mathcal{N}^+$, $\varphi(A) = \sum_{i\in A} 2 \cdot 3^{-i}$. then, for each $A_1,\, A_2, \ldots \cdot c\, N^+$,

$$\begin{aligned}
\varphi\left(\bigcup\nolimits_{i=1}^\infty A_i\right) &= \sum_{j\in\bigcup_{i=1}^\infty A_i} 2\cdot 3^{-j} = \\
&= \sum_{j=1}^{\infty} 2 \sup\left\{\chi_{A_i}(j) : i \in \mathcal{N}^+\right\} 3^{-j} = \bigvee\nolimits_{i=1}^\infty \varphi(A_i),
\end{aligned} \tag{9.9}$$

where $\chi_{A_i}(j) = 1$, if $j \in A_i$, $\chi_{A_i}(j) = 0$ otherwise. Hence, (9.9) holds in particular also for $A_1,\, A_2, \ldots \in \mathcal{S}$, as in this case $\bigcup_{i=1}^\infty A_i \in \mathcal{S}$ holds as well. For finite sequences $A_1,\, A_2, \ldots, A_n$ of subsets of $\mathcal{N}^+$ such that $A_i = e(\alpha_i)$, $\alpha_i \in \mathcal{L}_{\mathrm{PROP}}$, $i = 1, 2, \ldots, n$, the identity

$$\varphi\left(\bigcup\nolimits_{i=1}^\infty A_i\right) = \varphi\left(e(\alpha_1 \vee \cdots \vee \alpha_n)\right) = c_n(\alpha_1 \vee \cdots \vee \alpha_n)$$

is obvious, so that $\hat{e}_c$ extends e_c. As $\hat{e}_c$ is a σ-additive probability measure on the algebra $\mathcal{A}$ (cf. (7.16)), its extension to $\mathcal{S}$ is also a probabilistic measure on $\mathcal{S}$ and it is defined uniquely due to the theorem on the extension of (probabilistic) measure, cf. [3] or [7]. Hence, $\hat{e}_c$ is a σ-complete $\boldsymbol{c}$-valued possibilistic measure on $\mathcal{S}$ with respect to the nonstandard supremum operation $\bigvee$ on $[0,1]$. $\square$

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