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Abstract

Interval Gaussian algorithm is a popular method for enclosing solutions of linear interval equations. In this note we show that both versions of the method (preconditioned and unpreconditioned one) may yield large overestimations even in case $n = 4$.

Keywords
Linear interval system, interval Gaussian algorithm, overestimation

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1 Introduction

As is well known [1], interval Gaussian algorithm for enclosing solutions of a system of linear interval equations

$$A^I x = b^I \tag{1.1}$$

consists in solving the system (1.1) by Gaussian algorithm performed in interval arithmetic. The method often gives tighter enclosures if the system (1.1) is first preconditioned by the midpoint inverse [2]. In this note we show that even in the case $n = 4$ there exist examples with arbitrarily small data widths, arbitrarily large absolute overestimations, and with relative overestimations arbitrarily close to 2 in the unpreconditioned case and arbitrarily close to $\frac{1}{2}$ in the preconditioned one.

2 The example

For $\varepsilon > 0, \alpha > 0$ and $\beta > 1$, consider a linear interval system of the form

$$\begin{pmatrix} \frac{\varepsilon^2}{\alpha} & [-\varepsilon,\varepsilon] & [-\varepsilon,\varepsilon] & [-\varepsilon,\varepsilon] \\ 0 & \beta & 1 & 1 \\ 0 & 1 & \beta & 1 \\ 0 & 1 & 1 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ [-\varepsilon,\varepsilon] \\ [-\varepsilon,\varepsilon] \\ [-\varepsilon,\varepsilon] \end{pmatrix}. \tag{2.1}$$

The radii of all interval coefficients are $\leq \varepsilon$ and the interval matrix, if written in the form $[A_c - \Delta, A_c + \Delta]$, satisfies

$$\varrho(|A_c^{-1}|\Delta) = 0,$$

hence it is strongly regular [2]. Systems with matrices satisfying $\varrho(|A_c^{-1}|\Delta) < 1$ are usually considered "tractable". Yet it turns out that a system of type (2.1) is not well suited for solving by the interval Gaussian algorithm.

3 The result

First we derive some explicit formulae.

Theorem 1 *For each $\varepsilon > 0$, $\alpha > 0$ and $\beta > 1$ the system (2.1) satisfies:*

(i) the exact upper bound on x_1 is

$$\overline{x}_1 = \frac{(3\beta+5)\alpha}{(\beta-1)(\beta+2)}, \tag{3.1}$$

(ii) the upper bound on x_1 computed by preconditioned interval Gaussian algorithm (with or without partial pivoting) is

$$\overline{\overline{x}}_1 = \frac{(3\beta+9)\alpha}{(\beta-1)(\beta+2)}, \tag{3.2}$$

(iii) the upper bound on x_1 computed by unpreconditioned interval Gaussian algorithm (with or without partial pivoting) is

$$\overline{\overline{\overline{x}}}_1 = \frac{(3\beta^2+3\beta+2)\alpha}{\beta^2(\beta-1)}. \tag{3.3}$$

Proof. (i) Denote

$$A = \begin{pmatrix} \beta & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \beta \end{pmatrix},$$

then A is nonsingular for $\beta > 1$ and satisfies

$$A^{-1} = \frac{1}{(\beta-1)(\beta+2)} \begin{pmatrix} \beta+1 & -1 & -1 \\ -1 & \beta+1 & -1 \\ -1 & -1 & \beta+1 \end{pmatrix}. \tag{3.4}$$

¿From the form of (2.1) it can be easily seen that for the exact bound on x_1 we have (with $e = (1,1,1)^T$)

$$\begin{aligned} \overline{x}_1 &= \frac{\alpha}{\varepsilon^2}\max\{\varepsilon e^T|x'|;\, -\varepsilon e \le Ax' \le \varepsilon e\} \\ &= \alpha\max\{\|A^{-1}x''\|_1;\, -e \le x'' \le e\}, \end{aligned}$$

which in view of convexity of the norm implies

$$\overline{x}_1 = \alpha\max\{\|A^{-1}x''\|_1;\, |x''| = e\}. \tag{3.5}$$

Since A^{-1} is diagonally dominant for $\beta > 1$, for each x'' satisfying $|x''| = e$ (i.e., a ± 1-vector) there holds

$$\|A^{-1}x''\|_1 = (x'')^T A^{-1} x'' = \frac{3(\beta+2) - (e^T x'')^2}{(\beta-1)(\beta+2)} \le \frac{3\beta+5}{(\beta-1)(\beta+2)} \tag{3.6}$$

and the bound is achieved e.g. for $x'' = (1,1,-1)^T$, hence (3.5) and (3.6) imply (3.1).

(ii) Preconditioning (2.1) with the midpoint inverse

$$A_c^{-1} = \begin{pmatrix} \frac{\alpha}{\varepsilon^2} & 0^T \\ 0 & A^{-1} \end{pmatrix}$$

yields the system

$$\begin{pmatrix} 1 & [-\frac{\alpha}{\varepsilon},\frac{\alpha}{\varepsilon}] & [-\frac{\alpha}{\varepsilon},\frac{\alpha}{\varepsilon}] & [-\frac{\alpha}{\varepsilon},\frac{\alpha}{\varepsilon}] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{\varepsilon(\beta+3)}{(\beta-1)(\beta+2)} \begin{pmatrix} 0 \\ [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}, \tag{3.7}$$

and interval Gaussian algorithm applied to it (which consists of the backward step only) gives the upper bound (3.2).

(iii) The forward step of the unpreconditioned interval Gaussian algorithm (which is the same with or without partial pivoting due to $\beta > 1$) results in the system

$$\begin{pmatrix} \frac{\varepsilon^2}{\alpha} & [-\varepsilon,\varepsilon] & [-\varepsilon,\varepsilon] & [-\varepsilon,\varepsilon] \\ 0 & \beta & 1 & 1 \\ 0 & 0 & \frac{\beta^2-1}{\beta} & \frac{\beta-1}{\beta} \\ 0 & 0 & 0 & \frac{(\beta-1)(\beta+2)}{\beta+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ [-\varepsilon,\varepsilon] \\ [-\frac{(\beta+1)\varepsilon}{\beta}, \frac{(\beta+1)\varepsilon}{\beta}] \\ [-\frac{(\beta+2)\varepsilon}{\beta}, \frac{(\beta+2)\varepsilon}{\beta}] \end{pmatrix},$$

and the backward step gives (3.3). ■

Notice that the values of $\overline{x}_1$, $\overline{\overline{x}}_1$, $\overline{\overline{\overline{x}}}_1$ do not depend on ε. As a result, we obtain:

Theorem 2 *For arbitrary data width $\varepsilon > 0$, for the system (2.1) we have:*

(i) for each $K > 0$ and $r \in (0, \frac{1}{2})$ there exist $\alpha > 0$ and $\beta > 1$ such that

$$\overline{\overline{x}}_1 - \overline{x}_1 > K \tag{3.8}$$

and

$$\frac{\overline{\overline{x}}_1 - \overline{x}_1}{\overline{x}_1} > r, \tag{3.9}$$

(ii) for each $K > 0$ and $r \in (0, 2)$ there exist $\alpha > 0$ and $\beta > 1$ such that

$$\overline{\overline{\overline{x}}}_1 - \overline{x}_1 > K$$

and

$$\frac{\overline{\overline{\overline{x}}}_1 - \overline{x}_1}{\overline{x}_1} > r.$$

Proof. (i) According to (3.1) and (3.2),

$$\lim_{\beta \to 1_+} \frac{\overline{\overline{x}}_1 - \overline{x}_1}{\overline{x}_1} = \lim_{\beta \to 1_+} \frac{4}{3\beta + 5} = \frac{1}{2},$$

hence for each $r \in (0, \frac{1}{2})$ there exists a $\beta > 1$ such that (3.9) holds, and a choice of

$$\alpha > \frac{1}{4}(\beta - 1)(\beta + 2)K$$

assures (3.8) to hold.

The proof of (ii) is quite analogous since

$$\lim_{\beta \to 1_+} \frac{\overline{\overline{\overline{x}}}_1 - \overline{x}_1}{\overline{x}_1} = \lim_{\beta \to 1_+} \frac{4\beta^2 + 8\beta + 4}{3\beta^3 + 5\beta^2} = 2$$

and the derivative of the rational function is negative at $\beta = 1$. ■

4 Concluding remark

We have shown that overestimations (3.8) and (3.9) are possible for systems of the form (2.1). But since $\overline{\overline{x}}_1$ is the *exact* bound on the solution of the preconditioned system (3.7) (as it can be easily seen from its form), the result of Theorem 2, (i) also holds true for any method based on solving a preconditioned linear interval system, as e.g. Rump's method in [4] or the classical bounds derived via Neumann series, cf. [3].

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