Deductive Systems of Fuzzy Logic (A Tutorial)

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Abstract

Fuzzy logic is analyzed from the point of view of formal logic; the underlying calculi and their properties are surveyed.

Keywords

uncertainty, medical informatics, health care management, fuzzy logic, possibility theory, probability theory, expert systems

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1 Introduction

This is a written material for a tutorial to be held at the international conference IFSA97 at Prague. The major part of the paper is a varied and extended version of [12] [2], written by Hájek; Section 5 was written by Godo. Full elaboration is to be found in the prepared book [13]. The present introduction is devoted to some informal discussion on the subject of fuzzy logic. (1) *Fuzzy logic is popular.* The number of papers dealing, in some sense, with fuzzy logic and its applications is immense, and the success in applications is evident, in particular in fuzzy control. From numerous books we mention at least [50], [11], [24]. As stated in the introduction to [24], in 1991 there were about 1400 papers dealing with fuzzy systems. Naturally, in this immense literature quality varies; a mathematician (logician) browsing in it is sometimes bothered by papers that are mathematically poor (and he/she may easily overlook those few that one mathematically excellent). This should not lead to a quick refusal of the domain. Let us quote Zadeh, the inventor of fuzzy sets ([24], Preface): "Although some of the earlier controversies regarding the applicability of fuzzy logic have abated, there are still influential voices which are critical and/or skeptical. Some take the position that anything that can be done with fuzzy logic can be done equally well without it. Some are trying to prove that fuzzy logic is wrong. And some are bothered by what they perceive to be exaggerated expectations. That may well be the case but, as Jules Verne had noted at the turn of the century, scientific progress is driven by exaggerated expectations."

To get insight into the domain let us first ask three questions: what is logic, what is fuzziness and what meaning(s) has the term "fuzzy logic".

(2) *Logic studies the notion(s) of consequence.* It deals with propositions (sentences), sets of propositions and the relation of consequence among them. The task of formal logic is to represent all this by means of well-defined logical calculi admitting exact investigation. Various calculi differ in their definitions of sentences and notion(s) of consequence (propositional logics, predicate logics, modal propositional/predicate logics, many-valued propositional/predicate logics etc.). Often a logical calculus has *two* notions of consequence: syntactical (based on a notion of proof) and semantical (based on a notion of truth); then the natural questions of soundness (does provability imply truth?) and completeness (does truth imply provability?) pose themselves.

(3) *Fuzziness is impreciseness (vagueness); a fuzzy proposition may be true in some degree.* The word "crisp" is used as meaning "non-fuzzy". Standard examples of fuzzy propositions use *linquistic variable* [47] as e.g. *age* with possible values *young, medium, old* or similar. The sentence "The patient is young" is true in some degree - the less is the age of the patient (measured e.g. in years), the more true is the sentence. *Truth of a fuzzy proposition is a matter of degree.*

We recommend to everybody interested in fuzzy logic to sharply distinguish fuzziness from uncertainty as degree of belief (e.g. probability). Compare the last proposition with the proposition "The patient will survive next week". This may well be considered as a crisp proposition which is either (absolutely) true or (absolutely) false;

[2] Thanks are due to the Springer-Verlag for the permission to republish the paper. Partial support of COST Action 15 is recognized.

but we do not know which is the case. We may have some *probability* (chance, degree of belief) that the sentence is true; but probability is *not* degree of truth. In more details, the probability of a crisp statement φ should not be understood as the truth-value of φ but may be well understood as the truth-value of the fuzzy statement saying "φ is probable". (This is elaborated in [17].)

(4) *The term "fuzzy logic" has two different meanings - wide and narrow.* This is a very useful distinction, made by Zadeh; we again quote from [24], preface: "In a narrow sense, fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic. However, the agenda of FLn is quite different from that of traditional multivalued logics. In particular, such key concepts in FLn as the concept of a linguistic variable, canonical form, fuzzy if-then rule, fuzzy quantification and defuzzification, predicate modification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. This is the reason why FLn has a much wider range of applications than traditional systems.

In its wide sense, fuzzy logic, FLw, is fuzzily synonymous with fuzzy set theory, FST, which is the theory of classes with unsharp boundaries. FST is much broader than FLn and includes the latter as one of its branches."

Let me add two comments: first, in the wide sense, everything dealing with fuzziness may be (and seems to be) called "fuzzy logic". Second, even if I agree with Zadeh's distinction between many-valued logic and fuzzy logic in the narrow sense, I consider (and hope Zadeh would agree) formal calculi of many-valued logic (including non-"traditional", of course) to be the kernel or base of fuzzy logic in narrow sense and the task to explain things Zadeh mentions by means of these calculi to be a very promising task (not yet finished).

(5) *This paper about the fuzzy logic in the narrow sense.* Our main aim is to survey strictly logical properties of the most important many-valued logics whose set of truth values is the unit interval [0,1]. In Section 2, we survey propositional calculi; in Section 3 predicate calculi. We present a basic fuzzy logic and three stronger logics: Łukasiewicz, Gödel, and product logic, as well as a graded form (extension) of Łukasiewicz logic invented by Pavelka. Section 4 is devoted to a very general notion of a fuzzy logic. In Section 5 we discuss the notions of equality and similarity in fuzzy logic.

The reader is asumed to have at least a partial experience with the classical (two-valued) propositional calculus; some knowledge of predicate calculus is very helpful.

We hope that the reader will agree, after having read the paper, that fuzzy logic is *not* (better: need not be) simple-minded poor man's logic but a powerful and interesting logical calculus.

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2 Propositional logics.

2.1 Classical logic.

We quickly review notions and facts assumed to be known to the reader. There are two *truth* values:1 and 0 (1 stands for *truth*, 0 for *falsity*). The *language* of the classical propositional calculus consists of a list of *propositional variables* $p, q, \ldots$ and connectives $\rightarrow$ (implication and $\neg$ (negation); each propositional variable is a *formula*; if φ as a formula then $\neg\varphi$ is a formula; if φ and ψ are formulas then $\varphi \rightarrow \psi$ is a formula. A *truth evaluation* is a mapping e assigning to each propositional variable p a truth value $e(p)$ (0 or 1). Each truth evaluation extends uniquely to an evaluation of all formulas, using the truth functions of connectives.

It is customary to represent this by *truth tables*:

	$\neg$
1	0
0	1

$\rightarrow$	1	0
1	1	0
0	1	1

Thus e.g. if $e(p) = 1$ and $e(q) = 0$ then $e(\neg q) = 1,\ e(p \rightarrow \neg q) = 1,\ e(\neg q \rightarrow p) = 1,\ e(\neg(\neg q \rightarrow p)) = 0,\ e((p \rightarrow \neg q) \rightarrow \neg(\neg q \rightarrow p)) = 0.$

Other connectives may be used as abbrevations: if φ and ψ are formulas then

$$\varphi \,\&\, \psi \text{ is an abbrevation for } \neg(\varphi \rightarrow \neg\psi),$$

$$\varphi \vee \psi \text{ is an abbrevation for } \neg\varphi \rightarrow \psi,$$

$$\varphi \equiv \psi \text{ is an abbrevation for } (\varphi \rightarrow \neg\psi) \rightarrow \neg(\neg\psi \rightarrow \varphi).$$

If we compute the corresponding truth tables we get

&	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

$\equiv$	1	0
1	1	0
0	0	1

A formula φ is a *tautology* if $e(\varphi) = 1$ for each truth evaluation e (i.e. φ is identically true).

The following formulas are *axioms* of the predicate calculus:

$$p \rightarrow (q \rightarrow p) \tag{1}$$

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \tag{2}$$

$$(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) \tag{3}$$

One easily verifies that for each φ, ψ, the formulas (1), (2), (3) are tautologies. *Modus ponens* is the following deduction rule: from φ and $\varphi \rightarrow \psi$ derive ψ.

A *proof* in propositional calculus is an arbitrary sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that, for each $i = 1, \ldots, n$, either φ_i is an axiom or φ_i follows from some preceding

formulas φ_j, φ_k $(j,k < i)$ by modus ponens. A formula is *provable* in propositional calculus if it is the last element of a proof.

It is easy to show that if φ and $\varphi \to \psi$ are tautologies then φ is also a tautology; we say that modus pones preserves tautologicity. Therefore each provable formula is a tautology (soundness); conversely, we have *completeness*: each tautology is provable. (The *completeness theorem* says that the calculus is sound and complete, i.e. provable formulas are exactly all tautologies.)

A *theory* is a set T of formulas called *special axioms* of T. A *proof from* T is defined as above but with the additional possibility that φ_i may be an element of T. We write $T \vdash \varphi$ if φ is provable from T (is the least element of a proof from T). The *deduction theorem* says that for each theory T and formulas φ, ψ, $T \vdash \varphi \to \psi$ iff $(T \cup \{\varphi\}) \vdash \psi$.

An evaluation is a *model* of a theory T if it assigns 1 to each special axiom of T (makes all the axioms true). *Strong completeness theorem* says that $T \vdash \varphi$ iff φ is true in each model of T. For details on propositional logic set any textbook of Mathematical logic, e.g. [25].

2.2 More values and truth-functionality.

We have good reasons to generalize the two-valued logic to logics having more truth values and undoubtedly this can be done in many ways. To grasp degree of truth let us decide that our set of truth values will be linearly ordered, with 1 as maximum and 0 as minimum. The most obvious choice is the unit interval [0,1] of reals and *this will be our choice* throughout.

Needless to say, other choices are possible; notably one often works will a finite set of truth values. But this will not be studied here.

As we have seen, the classical propositional logic is *truth-functional*, i.e. the truth value of a composed formula can be computed from the truth values of its components, thanks to the truth functions of connectives.

Should our many-valued logics be also truth-functional? I.e. should the truth-value of a formula be determined by the truth values of its atoms via truth-functions of connectives, now assigning, e.g. for implication $\to$, to each pair of the values $x, y \in [0,1]$ a truth value $(x \Rightarrow y) \in [0,1]$? Most systems of fuzzy logic are truth-functional (e.g. one may take $x \cap y = min(x,y)$ for the truth-function of conjunction). Our main attention will be paid to truth-functional systems; but note that the systems of Section 4 will not be.

The reader should observe that there is nothing wrong on truth-functionality: the truth degree of a component formula is just *defined* by the truth functions. But one has to be careful: on then *cannot* interpret truth degrees of formulas e.g. as their probabilities, since probability is *not* truth-functional, as everybody knows. It is also not counter-intuitive that if you interpret conjunction by minimum and negation as $(-)x = 1-x$ (also a very popular choice) then $e(\varphi \wedge \neg\varphi)$ may be (and often is) positive. Our favourite example is of φ saying "I am old". It is rather true, but still the truth degree is (I hope) less than 1. Thus "I am not old" has a (small) positive value. And the conjunction "I am old and I am not old" (or: "I am old - yes and no") has a small but positive truth degree. This is impossible for crisp propositions, needless to say.

2.3 Where are truth-functions of connectives from?

Obviously, the truth-functions should behave classically on extremal truth values 0,1 and should satisfy some natural monotonicities (the truth function of conjunction (disjunction) should be non-decreasing in both arguments; the truth function of implication should be non-decreasing in the second argument but non-increasing in the first, i.e. the less true is the antecedent φ and the more is true the succedent ψ the more is true the implication $\varphi \to \psi$. $(-)$ should be non-increasing.) This leads to the notion of a t-norm: (cf. [41]) this is an operation $* : [0,1]^2 \to [0,1]$ which is commutative and associative, non-decreasing in both arguments and having 1 as unit element and 0 as zero element, i.c.

$$x * y = y * x$$

$$(x * y) * z = x * (y * z)$$

$$x \leq x' \text{ and } y \leq y' \text{ implies } x * y \leq x' * y'$$

$$1 * x = x,\ 0 * x = 0.$$

We shall only work with *continuous* t-norms as good candidates for truth functions of a conjunction. Each t-norm t determines uniquely its corresponding implication $\Rightarrow$ (not necessarily continuous) satisfying, for all $x, y, z \in [0,1]$

$$z \leq x \Rightarrow y \text{ iff } x * z \leq y.$$

For each such system we define an *evaluation* to be a mapping e assigning to each atom p its truth degree $e(p)$, $0 \leq e(p) \leq 1$; a *1-tautology* is a formula whose value is 1 for each evaluation.

We present three outstanding examples:

(1) *Lukasiewicz logic* [23] with the conjunction

$x * y = max(x + y - 1, 0)$ and the corresponding implication

$x \Rightarrow y = 1$ for $x \leq y$ and $x \Rightarrow y = 1 - x + y$ otherwise;

(2) *Gödel logic* [9] will the conjunction

$x * y = min(x, y)$ and the corresponding implication

$x \Rightarrow y = 1$ iff $x \leq y$ and $x \Rightarrow y = y$ otherwise;

(3) *Product logic* will the conjunction $x * y = x.y$ (product) and $x \Rightarrow y = 1$ iff $x \leq y$, $x \Rightarrow y = y/x$ otherwise.

Negation $(-)$ is defined as follows: $(-)x = x \Rightarrow 0$

One can show (see e.g. [31]) that each t-norm is composed in a certain way from these three examples. Thus our question reads: what is the logic of these examples?

We show that min and max are definable from $*$ and $\Rightarrow$.

For each continuous t-norm $*$, the following identities are true in $L(*)$:

(i) $\min(x, y) = x * (x \Rightarrow y)$,

(ii) $\max(x, y) = \min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x)$.

In the next subsection we shall present a *basic fuzzy logic* BL. Formulas provable in BL are 1-tautologies for each continuous t-norm. We shall formulate a completeness theorem formulated with the help of *coordinate lattices*. Then in three following sections we shall develop logics of the three main t-norms defined above.

2.4 The basic many-valued logic

Fix a continuous t-norm $*$: you fix a propositional calculus (whose set of truth values is $[0,1]$): This means is the truth function of the (strong) conjunction $\&$, the residuum $\Rightarrow$ of $*$ becomes the truth function of the implication. Further connectives are defined as follows:

$$\begin{aligned} \varphi \wedge \psi &\text{ is } \varphi\&(\varphi \to \psi), \\ \varphi \vee \psi &\text{ is } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg\varphi &\text{ is } \varphi \to \bar{0}, \\ \varphi \equiv \psi &\text{ is } (\varphi \to \psi)\&(\psi \to \varphi). \end{aligned}$$

An *evaluation of propositional variables* is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0,1]$.

This extends to each formula via truth-functions as follows:

$$\begin{aligned} &e(\bar{0}) = 0, \\ &e(\varphi \to \psi) = (e(\varphi) \Rightarrow e(\psi)), \\ &e(\varphi\&\psi) = (e(\varphi) * e(\psi)). \end{aligned}$$

A formula φ is a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e.
The following formulas are axioms of the basic logic:
(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi\&\psi) \to \varphi$
(A3) $(\varphi\&\psi) \to (\psi\&\varphi)$
(A4) $(\varphi\&(\psi\&\chi)) \to ((\varphi\&\psi)\&\chi)$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)$
(A5b) $((\varphi\&\psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\bar{0} \to \varphi$
The *deduction rule* of BL is modus ponens. Given this, the notions of a *proof* and of a *provable formula* in BL are defined in the obvious way

All axioms of BL are 1-tautologies in each $PC(*)$. If φ and $\varphi \to \psi$ are 1-tautologies of $PC(*)$ then ψ is also a 1-tautology of $PC(*)$. Consequently, each formula provable in BL is a 1-tautology of each $PC(*)$. Let us present a list of some formulas provable in BL.

BL proves the following properties of implication:

(1) $\varphi \to (\psi \to \varphi)$
(2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(3) $\varphi \to \varphi$
BL proves the following properties of strong conjunction:
(4) $(\varphi\&(\varphi \to \psi)) \to \psi$
(5) $\varphi \to (\psi \to (\varphi\&\psi))$
(6) $(\varphi \to \psi) \to ((\varphi\&\chi) \to (\psi\&\chi))$
(7) $((\varphi_1 \to \psi_1)\&(\varphi_2 \to \psi_2)) \to ((\varphi_1\&\varphi_2) \to (\psi_1\&\psi_2))$
(8) $(\varphi\&\psi)\&\chi \to \varphi\&(\psi\&\chi)$.
BL proves the following properties of min-conjunction:
(9) $(\varphi \wedge \psi) \to \varphi, (\varphi \wedge \psi) \to \psi, (\varphi\&\psi) \to (\varphi \wedge \psi)$
(10) $(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$
(11) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(12) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
BL proves the following properties of max-disjunction:
(13) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$, $(\varphi \vee \psi) \to (\psi \vee \varphi)$,
(14) $(\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$
(15) $(\varphi \to \psi) \vee (\psi \to \varphi)$
(16) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to (\varphi \vee \psi) \to \chi$.
BL proves the following properties of negation.
(17) $\varphi \to (\neg\varphi \to \psi)$, in particular, $\varphi \to \neg\neg\varphi$
(18) $(\varphi \to (\psi\&\neg\psi)) \to \neg\varphi$
$\bar{1}$ stands for $\bar{0} \to \bar{0}$.
BL proves the following:
(19) $\bar{1}$,
(20) $\varphi \to (\bar{1}\&\varphi)$.
BL proves the following additional properties of $\wedge, \vee$:
(21) $(\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)$
$((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi))$
(associativity of $\wedge$),
(22) analogous associativity for $\vee$,
(23) $\varphi \to \varphi \wedge (\varphi \vee \psi)$
$(\varphi \vee (\varphi \wedge \psi)) \to \varphi$
BL proves
(24) $\varphi \equiv \varphi$, $(\varphi \equiv \psi) \to (\psi \equiv \varphi)$,
$((\varphi \equiv \psi)\&(\psi \equiv \chi)) \to (\varphi \equiv \chi)$,
(25) $(\varphi \equiv \psi) \to (\varphi \to \psi)$,
$(\varphi \equiv \psi) \to (\psi \to \varphi)$
(26) $(\varphi \equiv \psi) \to ((\varphi\&\chi) \equiv (\psi\&\chi))$,
(27) $(\varphi \equiv \psi) \to ((\varphi \to \chi) \equiv (\psi \to \chi))$,
(28) $(\varphi \equiv \psi) \to ((\chi \to \varphi) \equiv (\chi \to \psi))$,
(29) $(\varphi \equiv \psi) \to ((\varphi \to \psi) \wedge (\psi \to \varphi))$.
BL proves the following distributive laws:

(30) $\varphi\&(\psi \vee \chi) \equiv (\varphi\&\psi) \vee (\varphi\&\chi)$
$\varphi\&(\psi \wedge \chi) \equiv (\varphi\&\psi) \wedge (\varphi\&\chi)$

(31) $(\varphi \wedge (\psi \vee \chi)) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
$(\varphi \vee (\psi \wedge \chi)) \equiv ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

BL proves:

(32) $(\varphi \vee \psi)\&(\varphi \vee \psi) \rightarrow ((\varphi\&\varphi) \vee (\psi\&\psi))$
$(\varphi \wedge \psi)\&(\varphi \wedge \psi) \rightarrow ((\varphi\&\varphi) \wedge (\psi\&\psi))$

(33) $(\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n$, for each n,
where α^n is $\alpha\& \ldots \alpha, n$ times.

A *theory* over BL is a set of formulas. A *proof* in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas whose each member is either an axiom of BL or a member of T (special axiom) or follows from some preceding members of the sequence using the deduction rule modus ponens.

$T \vdash \varphi$ means that φ is *provable* in T, i.e. is the last member of of a proof in T. The deduction theorem for BL reads as follows:

Deduction theorem: Let T be a theory and let φ, ψ be formulas.

$T \cup \{\varphi\} \vdash \psi$ iff there is an n such that $T \vdash \varphi^n \rightarrow \psi$ (where φ^n is $\varphi\& \ldots \&\varphi, n$ factors). Now we shall introduce some algebras corresponding to BL similarly as Boolean algebras correspond to classical logic.

A *regular residuated lattice* is an algebra

$$(L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative, $1 * x = x$ for all x.

(iii) the following conditions hold:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z.
(2) $x \cap y = x * (x \Rightarrow y)$
(3) $x \cup y = ((x \Rightarrow y) \Rightarrow y)) \cap ((y \Rightarrow x) \Rightarrow x)$
(4) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

An **L**-evaluation of propositional variables is any mapping e assigning to each propositional variable p an element $e(p)$ of **L**. This extends in the obvious may to an evaluation of all formulas using the operations on **L** as truth functions.

The logic BL is sound with respect to **L**-tautologies: if φ is provable in BL then φ is an **L**-tautology for each regular linearly ordered residuated lattice. More generally, if T is a theory over BL and T proves φ then, for each regular linearly ordered residuated lattice **L** and each **L**-evaluation e of propositional variables assigning the value 1 to all the axioms of T, $e(\varphi) = 1$.

Classes of provably equivalent formulas (w.r.t. a theory T) form a regular residuated lattice.

Completeness theorem. For each formula φ the following there things are equivalent:

(i) φ is provable in BL,

(ii) for each linearly ordered regular residuated lattice **L**, φ is an **L**-tau-tology;

(iii) for each regular residuated lattice **L**, φ is an **L**- tautology.

We shall generalize this completeness theorem as follows:

(1) An *axiom schema* given by a formula
$\Phi(p_1, \ldots, p_n)$ is the set of all formulas
$\Phi(\varphi_1, \ldots, \varphi_n)$ resulting by the substitution of φ_i for $p_i (i = 1, \ldots, n)$ in $\Phi(p_1, \ldots, p_n)$.

(2) A logical calculus $\mathcal{C}$ is a *schematic extension* of BL if it results from BL by adding some (finitely or infinitely many) axiom schemata to its axioms. (The deduction rule remains to be modus ponens.)

(3) Let $\mathcal{C}$ be a schematic extension of BL and let **L** be a lattice. **L** is a *$\mathcal{C}$-lattice* if all axioms of $\mathcal{C}$ are **L**-tautologies.

Completeness. Let $\mathcal{C}$ be a schematic extension of BL and let φ be a formula. The following are equivalent:

(i) $\mathcal{C}$ proves φ,

(ii) φ is an **L**-tautology for each linearly ordered $\mathcal{C}$-lattice **L**,

(iii) φ is an **L**-tautology for each $\mathcal{C}$-lattice **L**.

Remark. Results of the present section are new; but rely very heavily on related results of Höhle [21].

2.5 Łukasiewicz logic

This logic results by extending BL by the following axiom (L4):

$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \tag{L4}$$

; Similarly as classical logic, Łukasiewicz logic Ł may be alternatively developed from implication $\to$ and negation $\neg$ (or just from $\to$ and 0); the truth function of negation is $(-)x = x \Rightarrow 0 = 1 - x$. We can define two different conjunctions and disjunctions:

$\varphi \,\&\, \psi$ is $\neg(\varphi \to \neg\psi)$, $\quad x * y = max(x + y - 1, 0)$

$\varphi \underline{\vee} \psi$ is $\neg(\neg\varphi \,\&\, \neg\psi)$, $\quad x \underline{\cup} y = min(x + y, 1)$

$\varphi \vee \psi$ is $(\varphi \to \psi) \to \psi$, $\quad x \cup y = max(x.y)$

$\varphi \wedge \psi$ is $\neg(\neg\varphi \vee \neg\psi)$, $\quad x \cap y = min(x, y)$

The following are the original axioms of Łukasiewicz logic:

$$\varphi \to (\psi \to \varphi) \tag{L1}$$

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{L2}$$

$$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi) \tag{L3}$$

$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \tag{L4}$$

The only deduction rule is modus ponens; the definition of a proof is as in classical logic (relative to our set of axioms).

As mentioned above, this set of axioms is equivalent to BL + (L4)

Completeness of this set of axioms was conjectured by Łukasiewicz in Thirties, but first proved by Rose and Rosser [37]; a good proof can be found in [10]. The relevant algebras are particular regular residuated lattices called MV-algebras.

Needless to say, details are non-trivial and laborious but the structure is the same in all our three logics.

Completeness. A formula φ is provable in Łukasiewicz logic Ł iff it is a 1-tautology of Łukasiewicz logic.

Remark. Observe the difference from the completeness theorem for BL: here we do and work work with all linearly ordered regular residuated lattices but with just one: the real interval [0,1] with the truth functions of Łukasiewicz logic.

2.6 Gödel logic

Kurt Gödel (born 1906 in Brno, now Czech Republic), probably the most important mathematical logician, published in 1932 an extremely short paper [9] concerning intuitionistic logic (a subsystem of classical logic with a different meaning of connectives; e.g. $\varphi \vee \neg\varphi$ is not provable). Gödel's aim was to show that there is no finitely valued logic for which axioms of intuitionistic logic would be complete. For this purpose he created a semantics of (possibly infinite-valued) propositional calculus which is now called Gödel logic G. (Needless to say, this was more than three decades before fuzzy sets have been defined).

Gödel logic has the following connectives: $\to, \wedge, \vee, \neg$ (implication, conjunction, disjunction, negation; negation may be replaced by 0). The semantics is as follows (cf. Sect. 2.3):

$x \Rightarrow y = 1$ if $x \leq y$, $x \Rightarrow y = y$ otherwise,

$x \cap y = \min(x, y)$,

$x \cup y = \max(x, y)$,

$(-)x = 1$ for $x = 0$, $(-)x = 0$ for $x > 0$.

The axioms are as follows (G1 - G11 are axioms of intuitionistic logic, G12 is an axiom of "linearity"):

(G1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(G2) $\varphi \to (\varphi \vee \psi)$

(G3) $\psi \to (\varphi \vee \psi)$

(G4) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)))$
(G5) $(\varphi \wedge \psi) \to \varphi$
(G6) $(\varphi \wedge \psi) \to \psi$
(G7) $(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi)))$
(G8) $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$
(G9) $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(G10) $(\varphi \wedge \neg\varphi) \to \psi$
(G11) $(\varphi \to (\varphi \wedge \neg\varphi)) \to \neg\varphi$
(G12) $(\varphi \to \psi) \vee (\psi \to \varphi)$

It is an easy checking to show that all these are 1-tautologies. The deduction rule is modus ponens; this defines the notion of a proof.

One can show that G is equivalently axiomatized by BL plus $\varphi \to (\varphi\&\varphi)$ – idempotence of &. It follows easily that $\varphi\&\psi$ is equivalent to $\varphi \wedge \psi$ so that & is redundant.

Completeness theorem: Each 1-tautology is provable. Again here the proof is rather non-trivial with a different class of algebras, called Heyting algebras or pseudo-boolean algebras. We have no room for details; [10] is recommended for a readable elaborated proof originally given by Dummett [7].

Deduction theorem is valid for G: $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash (\varphi \to \psi)$. Note that G is *the only* many-valued logic having the deduction theorem, more precisely: if a logic contains a conjunction given by a t-norm and the corresponding implication $\to$, is completely axiomatized and satisfies the deduction theorem then the t-norm is minimum and hence $\to$ is Gödel implication.

Gödels logic satisfies the following form of *strong completeness*: Say that a theory *semantically entails* φ if for each evaluation e there is a conjunction α of finitely many axioms of T such that $e(\alpha) \leq e(\varphi)$. (Observe that in classical logic this is equivalent to saying that φ is true in each model of T.)

Strong completeness: For each theory T and formula φ, $T \vdash \varphi$ iff T semantically entails φ.

Note that the easy part of this equivalent (soudness) implies that if $T \vdash \varphi$ and $e(\alpha) \geq r$ for each axiom α of T then $e(\alpha) \geq r$. The difficult part can be obtained by combining the (normal) completeness of G with the techniques of Takeuti and Titani [43].

2.7 Product logic.

The logic based on the product t-norm has been considerably less investigated them the two preceding ones (see [1]). The paper [18] investigates product logic and proves completeness theorem using a class of algebras called *product algebras*. There are several open problems related to this (rather interesting and unjustly overlooked) logic.

We write $\odot$ instead of &.

The *axioms* of Π are those of BL plus

(Π1) $\neg\neg\chi \to ((\varphi \odot \chi \to \psi \odot \chi) \to (\varphi \to \psi))$,

(Π2) $\varphi \wedge \neg\varphi \to \bar{0}$.

The axioms are 1-tautologies over the algebra $[0,1]_P$ of the truth functions.

Π proves the following formulas:

(1) $\neg(\varphi \odot \psi) \to \neg(\varphi \wedge \psi)$
(2) $(\varphi \to \neg\varphi) \to \neg\varphi$
(3) $\neg\varphi \vee \neg\neg\varphi$

The axiom ($\Pi 2$) can be equivalently replaced by each of the following formulas:

$$\neg(\varphi \odot \varphi) \to \neg\varphi,\ (\varphi \to \neg\varphi) \to \neg\varphi,\ \neg\varphi \vee \neg\neg\varphi.$$

Following the general approach we define a Π-*algebra* (or *product algebra*) to be a regular residuated lattice satisfying

$$\neg\neg z \leq ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y)),$$
$$x \cap -x = 0.$$

Using this notion one proves the following

Completeness theorem.

(1) A formula φ is provable in the product logic Π iff it is a 1-tautology of the product logic.

(2) Let T be a finite theory over Π, φ a formula. T proves φ over the product logic iff it is true in each model of T (in the sense of Π).

2.8 Rational Pavelka logic

Till now we have been interested almost exclusively only in axiomatizing 1-tautologies, i.e. proving formulas that are absolutely true. But in fuzzy logic we are interested in deriving consequences from assumptions that are only partially true, true in some degree. (We met a result of this type at the end of 2.5 - for Gödel logic.) Logics of partial truth were studied, in a very general manner, as early as in the seventies by the Czech mathematician Jan Pavelka [32] and since then have been substantially simplified; We refer to [14] but here we describe a still simpler version. It is very different from the original Pavelka's version and looks as an "innocent" extension of *L*ukasiewicz's Ł; but the main completeness result of Pavelka still holds.

The idea is as follows: assume that $e(\varphi) = r$; then for each φ, $e(\psi) \geq r$ iff $e(\varphi \to \psi) = 1$. Thus if φ is a formula whose value is r in *all* evaluations then the axiom $\varphi \to \psi$ would just postulate that ψ is at least r-true.

Thus we extend the language of Ł by adding truth constant $\overline{r}$ for some $r \in [0,1]$ as new atomic formulas, postulating that $e(\overline{r}) = r$ for each evaluation (we already have had $\overline{0}$ and $\overline{1}$). Our choice will be to add truth constants $\overline{r}$ for each *rational* $r \in [0,1]$ (thus we have truth constants for a countable dense recursirely representable set of reals from [0,1], this is all we need).

Thus for example if φ, ψ are formulas then $(\varphi \to \overline{0.7})\ \&\ (\overline{0.4} \to \neg\psi)$ is a formula. We have some obvious tautologies like $\neg\overline{0.7} \equiv \overline{0.3}$ and $\overline{0.7} \to \overline{0.5} \equiv \overline{0.8}$; in general, for each rational $r, s \in [0,1]$ we have

(P1) $\neg\overline{r} \equiv \overline{(-)r}$,

(P2) $(\overline{r} \to \overline{s}) \equiv \overline{r \Rightarrow s}$

We add these schemas as new logical axioms; the resulting logic (with the language extended by truth constants and axioms extended by (P1), (P2)) will by called RPL (rational propositional logic or rational Pavelka logic). The only deduction rule is modus ponens.

If φ is a formula and $r \in [0,1]$ is rational then (φ, r) denotes just the formula $(\overline{r} \to \varphi)$ (saying that φ is at least r-true). We have same derived deduction rules.

Lemma. Let T be a theory in RPL (a set of special axioms); for each formula α, $T \vdash \alpha$ means that α is provable in T.

(1) If $T \vdash (\varphi, r)$ and $T \vdash (\varphi \to \psi, s)$, then $T \vdash (\psi, r * s)$.

(2) If $T \vdash (\varphi, r)$ then $T \vdash (\overline{s} \to \varphi, s \Rightarrow r)$.

Definition. Let T be a theory in RPL. (1) The *truth degree* of φ in T is $\| \varphi \|_T = inf\{e(\varphi) \mid e \text{ is a model of } T\}$.

(2) *The provability degree* of φ in T is

$$| \varphi |_T = sup\{r \mid T \vdash (\varphi, r)\}.$$

Thus $\| \varphi \|_T$ is the infimum of values of φ in models of T; $| \varphi |_T$ is the supremum of rationals r such that $T \vdash \overline{r} \to \varphi$.

Completeness theorem for RPL: Let T be a theory in RPL; then, for each formula φ, $\| \varphi \|_T = | \varphi |_T$.

This is a very pleasing and elegant result (invented originally by Pavelka); the proof is moderately difficult (much easier than the proof of completeness of Ł, but using the fact that we have the Rose-Rosser's complete axiom system for Ł).

Remarks. (1) A *fuzzy theory* is a fuzzy set of formulas, i.e. a mapping T associating to each formula φ the degree $T(\varphi)$ of being an axiom. An evaluation e is a *model* of T of for each φ, $e(\varphi) \geq, T(\varphi)$, i.e. each formula is at least as much true, as the theory demands. It is natural to assume that each $T(\varphi)$ is a rational number. The notion of a fuzzy theory is central in Pavelka's approach but we see that it is superfluouns; if you define $T' = \{(\varphi, T(\varphi)) \mid \varphi \text{ formula}\}$ (thus for each φ, if $T(\varphi) = r$ we put $(\overline{r} \to \varphi)$ into T') then T' is a (crisp) theory having the same models as T.

(2) The set of all formulas is a recursive set and the syntax is recursive; thus we may call a theory T *recursive* if T is a recursive set of formulas. Note that $| \varphi |_T$ may be irrational; on the other hand, if $r > 0$ is rational then we can construct a recursive theory T such that the set of all φ such that $| \varphi |_T \geq r$ is "badly" non-recursive (for experts: it may be Π_2-complete; see [14] for details).

(3) We can similarly other logics, e.g. Gödel logic or product logic but unfortunately we cannot hope for Pavelka style completeness (as Pavelka himself tells us) since the truth function of implication is not contuous in these logics. To see this take the theory $T = \{p \to (\overline{\frac{1}{n}}) \mid n \text{ natural }\}$; then

$\| p \to 0 \|_T = 1$ for each of Ł, G, P;

$|p \to 0|_T = 1$ for Ł but $|p \to 0 |_T = 0$ for both G and P (verify).

Note that RPL satisfies the same generalized deduction theorem as Ł (and of course does not satisfy the classical deduction theorem).

3 Predicate calculi

3.1 The classical predicate calculus

In the present section we assume the reader to have some basic knowledge of the classical predicate calculus. In this subsection we survey the basic notions and facts, for comparison with their many-valued generalizations. We shall restrict ourselves to calculi without function symbols. Details may be found e.g. in [25].

A *language* consists of *predicates* $P, Q, \ldots$, *object constants* $c, d, \ldots$, *object variables* $x, y, \ldots$. Each predicate is assigned a positive natural number as its *arity*. If P is an n-ary predicate and $t_1, \ldots, t_n$ are variables and/or constants then $P(t_1, \ldots, t_n)$ is an atomic *formula*. Non-atomic formulas are from atomic ones using *connectives* $\rightarrow, \neg$ and the *universal quantifier* $\forall$: if φ, ψ are formulas and x is an object variable then $\varphi \rightarrow \psi$, $\neg\varphi$, $(\forall x)\varphi$ are *formulas*. The variable x is bound in $(\forall x)\varphi$; other variables are free/bound in φ iff they are free/bound in $(\forall x)\varphi$. A variable is free/bound in $\neg\varphi$ iff of is such in φ; it is free/bound in $\varphi \rightarrow \psi$ iff it is such in φ or in ψ. A formula is *closed* of it has no free variable.

Other connectives are introduced as abbreviations as in propositional quantifier; the *existential quantifies* $\exists$ is defined thus: $(\exists x)\varphi$ abbreviates $\neg(\forall x)\neg\varphi$.

An *interpretaion* of a language L is given by the following:

- a non-empty *domain* M,
- for each n-ary predicate P, an n-ary *relation* $r_P \subseteq M^n$ (set of n-tuples of elements of M)
- for each constant c, an element $m_c \in M$.

The interpretation is *witnessed* if each element $m \in M$ is the meaning of a constant c, $m = m_c$. (This can be achieved by extending the language by some additional constants.) For each closed formula φ and each interpretation

$$\mathbf{M} = \langle M, (r_P)_{P \text{ predicate}}, (m_c)_{c \text{ constant}} \rangle,$$

The *truth value* of φ in $\mathbf{M}$ is defined as follows:

- If $P(c, \ldots, d)$ is a closed atomic formula then $\| P(c, \ldots, d) \|_{\mathbf{M}} = 1$ iff $\langle m_c, \ldots, m_d \rangle \in r_P$ (the tuple of meanings of $c, \ldots, d$ is in the relation r_p which is the meaning of P); otherwise $\| P(c, \ldots, d) \|_{\mathbf{M}} = 0$;
- $\| \varphi \rightarrow \psi \|_{\mathbf{M}} \Rightarrow \| \varphi \|_{\mathbf{M}} \Rightarrow \| \psi \|_{\mathbf{M}}$, $\| \neg\varphi \|_{\mathbf{M}} = (-) \| \varphi \|_{\mathbf{M}}$;
- $\| (\forall x)\varphi \|_{\mathbf{M}} = \min_c \| \varphi(c) \|_{\mathbf{M}}$, where $\varphi(c)$ results from φ by substituting the constant c for (free occurences of) x.

We write $M \models \varphi$ for $\| \varphi \|_{\mathbf{M}} = 1$ and read: φ is true in $\mathbf{M}$. If φ is not closed then $\mathbf{M} \models \varphi$ means that $M \models (\forall x_1) \ldots (\forall x_n)\varphi$, where $x_1, \ldots x_n$ are the variables free in φ.

A *theory* is a set of formulas (special axioms). **M** is a *model* of a T if each $\varphi \in T$ is true in **M**.

Logical axioms: axioms of classical propositional calculus plus

(A1) $\qquad (\forall x)\varphi \to \varphi(t)$

where t is either a constant or an object variable free for x in φ (this is a simple condition preventing "clash of free and bound variables") - the *substitution* axiom,

(A2) $\qquad (\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$

where ν is a formula in which x is not free.

Deduction rules: Modus ponens and *generalization*: from φ derive $(\forall x)\varphi$.

A *proof* in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas (not necessarily closed) such that each φ_i either is a logical axiom or belongs to T (is a special axiom) or results from some previous formulas(s) using one of the deduction rules. A formula φ is *provable* in T (notation: $T \vdash \varphi$) if φ is the last member of a proof in T.

Gödel's completeness theorem: $T \vdash \varphi$ iff φ is true in each model of T. In particular, φ is a tautology (true in all interpretations) iff $\vdash \varphi$ (φ is provable using only logical axioms).

3.2 The basic many-valued predicate logic

A *predicate language* consists of *predicates* $P, Q, \ldots$, each together with its *arity* and *object constants.* $c, d, \ldots$. *Logical symbols* are *object variables* $x, y, \ldots$, *connectives* $\&, \to$, *truth constants* $\bar{0}, \bar{1}$ and *quantifiers* $\forall, \exists$. Other connectives ($\wedge$, $\vee$, $\neg$, $\equiv$) are defined as in propositional calculus. *Terms* are object variables and object constants.

Atomic formulas have the form $P(t_1, \ldots, t_n)$ where P is a predicate of arity n and $t_1, \ldots, t_n$ are terms. If φ, ψ are formulas and x is an object variable then $\varphi \to \psi$, $\varphi\&\psi$, $(\forall x)\psi$, $(\exists x)\varphi$, $\bar{0}$, $\bar{1}$ are formulas; each formula results from atomic formulas by iterated use of this rule.

Let $\mathcal{J}$ be a predicate language and let **L** be a regular residuated lattice. An **L**-*structure* $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ *for* $\mathcal{J}$, $M \neq \emptyset$, for each n-ary predicate P a **L**-fuzzy n-ary relation $r_P : M^n \to \mathbf{L}$ on M and for each object constant c, m_c is an element of M.

An **M**-*evaluation* of *object variables* is a mapping v assigning to each object variable x an element $v(x) \in M$. *Values* of terms and formulas are defined as follows: $\|x\|_{M,v} = v(x)$; $\|c\|_{M,v} = m_c$.

$\|P(t_1, \ldots, t_n)\|^L_{M,v} = r_p(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v})$;
$\|\varphi \to \psi\|^L_{M,v} = \|\varphi\|^L_{M,v} \Rightarrow \|\psi\|^L_{M,v}$;
$\|\varphi\&\psi\|^L_{M,v} = \|\varphi\|^L_{M,v} * \|\psi\|^L_{M,v}$;
$\|\bar{0}\|_{M,v}$; $\qquad \|\bar{1}\|_{M,v}$;
$\|(\forall x)\varphi\|^L_{M,v} = \inf\{\|\varphi\|^L_{M,v'} | v \equiv_x v'\}$;
$\|(\exists x)\varphi\|^L_{M,v} = \sup\{\|\varphi\|^L_{M,v'} | v \equiv_x v'\}$;
provided the infimum/supremum exists in the sense of **L**;

The structure $\mathbf{M}$ is $\mathbf{L}$-*safe* if all the needed infima and suprema exist, i.e. $\|\varphi\|^L_{M,v}$ is defined for all φ, v.

$\|\varphi\|_M = \inf\{\|\varphi\|_{M,v} | \, v \ \ \mathbf{M} - \text{evaluation}\}$.

A formula φ of a language $\mathcal{J}$ is an $\mathbf{L}$-*tautology* if $\|\varphi\|_M = 1_{\mathbf{L}}$ for each safe $\mathbf{L}$-structure $\mathbf{M}$.

The following are *logical axioms on quantifiers:*
($\forall 1$) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
($\exists 1$) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
($\forall 2$) $(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
($\exists 2$) $(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
($\forall 3$) $(\forall x)(\nu \vee \varphi) \to (\nu \vee (\forall x)\varphi)$ (x not free in ν)

The predicate calculus $\mathcal{C}\forall$ (over a given predicate language $\mathcal{J}$) has the following axioms:

- all formulas resulting from the axioms of $\mathcal{C}$ by substituting arbitrary formulas of $\mathcal{J}$ for propositional variables, and
- the axioms $(\forall 1), (\forall 2), (\exists 1), (\exists 2), (\forall 3)$ for quantifiers

  and deduction rules
- modus ponens (from φ, $\varphi \to \psi$ infer ψ) and
- generalization (from φ infer $(\forall x)\varphi$).

In particular, we are interested in BL$\forall$ and three stronger logics: Ł$\forall$ (Lukasiewicz), G$\forall$ (Gödel), $\Pi\forall$ (product). Also note in passing that if $\mathcal{C}$ is the classical propositional calculus (as described above) then in $\mathcal{C}\forall$ the axioms $(\forall 3), (\exists 1), (\exists 2)$ are redundant (provable from the rest); $(\forall 1), (\forall 2)$ are the usual axioms of the classical predicate logic.

The axioms ($\forall 1$)–($\forall 3$), ($\exists 1$)–($\exists 2$) are $\mathbf{L}$-tautologies for each regular residuated lattice $\mathbf{L}$.

(Soundness of provability.) Let $\mathcal{C}$ be a schematic extension of BL, let T be a theory in the language of T over $\mathcal{C}\forall$, let φ be a formula of T. If $T \vdash \varphi$ (φ is provable in T) then $\|\varphi\|^L_M = 1$ for each $\mathcal{C}$-lattice $\mathbf{L}$ and each $\mathbf{L}$-model $\mathbf{M}$ of T.

Let φ be an arbitrary formula, ν a formula not containing x freely. Then BL$\forall$ proves the following:

(1) $(\forall x)(\nu \to \varphi) \equiv (\nu \to (\forall x)\varphi)$
(2) $(\forall x)(\varphi \to \nu) \equiv ((\exists x)\varphi \to \nu)$
(3) $(\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$
(4) $(\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$

The converse implications in (3), (4) are *not* provable in BL. We shall see later that neither of them is a tautology of G$\forall$; the converse of (3) is but the converse of (4) is not a tautology of $\Pi\forall$; and both converses are tautologies of Ł$\forall$.

For arbitrary formulas φ, ψ, BL$\forall$ proves the following:

(5) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
(6) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$
(7) $((\forall x)\varphi \& (\exists x)\psi) \to (\exists x)(\varphi \& \psi)$
For arbitrary φ and for ν not containing x freely, BL$\forall$ proves
(9) $(\exists x)(\varphi \& \nu) \equiv ((\exists x)\varphi \& \nu)$,
(10) $(\exists x)(\varphi \& \varphi) \equiv ((\exists x)\varphi \& (\exists x)\varphi)$.
BL$\forall$ proves the following:
(11) $(\exists x)\varphi \to \neg(\forall x)\neg\varphi$
(12) $\neg(\exists x)\varphi \to (\forall x)\neg\varphi$

Completeness. Let T be a theory over $\mathcal{C}\forall$.

(1) T is *consistent* if there is a formula φ unprovable in T.

(2) T is *complete* if for each pair φ, ψ of closed formulas, $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$.

(3) T is *Henkin* if for each closed formula of the form $(\forall x)\varphi(x)$ unprovable in T there is a constant c in the language of T such that $\varphi(c)$ is unprovable in T.

3.3 Łukasiewicz predicate logic

Ł$\forall$ proves

$$(\exists x\varphi) \equiv \neg(\forall x)\neg\varphi.$$
$$(\forall x)(\varphi \& \nu) \to ((\forall x)\varphi \& \nu).$$

Axioms $(\exists 1)(\exists 2)(\forall 3)$ are redundant (provable from the others).

Lemma. Ł$\forall$ proves the following:
$$(\nu \to (\exists x)\varphi) \to (\exists x)(\nu \to \varphi),$$
$$((\forall x)\varphi \to \nu) \to (\exists)(\varphi \to \nu).$$

Theorem. There is no recursive axiomatic system complete with respect to Ł$\forall$-tautologies (over $[0,1]_{\mathrm{L}}$). Moreover, the last set is Π_2-complete.[34, 35, 40]

3.4 Rational Pavelka quantification logic

We extend Lukasiewicz predicate logic by propositional constants $\overline{r}$ for each rational $r \in [0,1]$; for each $\mathbf{M}$, $\|\overline{r}\|_{\mathbf{M}} = r$. The *axioms* of RPL$\forall$ are those of RPL plus (A1), (A2) from 3.1 plus

We introduce (φ, r) as abbreviation of $(\overline{r} \to \varphi)$ as above; given a theory T, we define the provability degree and truth degree as above:

$$|\varphi|_T = \sup\{r \mid T \vdash (r \to \varphi)\},$$

$$\| \varphi \|_T = \inf\{ \| \varphi \|_{\mathbf{M}} \mid M \text{ a model of } T \}.$$

(We should say that for a non-closed φ, $\| \varphi \|_{\mathbf{M}}$ is defined as $\| (\forall x_n) \dots (\forall x_n) \varphi \|_{\mathbf{M}}$ analogously as above; $\mathbf{M}$ is a *model* of T if $\| \varphi \|_{\mathbf{M}} = 1$ for each $\varphi \in T$.

We have the following Pavelka-style

Completeness theorem (see [16]). For each theory T and formula φ,

$$\| \varphi \|_T = | \varphi |_T,$$

i.e. the truth degree equals the provability degree. Let T be a recursive theory. For each positive $r \in [0,1]$, the set $Pr(T,r)$ of all φ such that $| \varphi |_T \geq r$ is Π_2; there is a recursive theory T such that $Pr(T,1)$ is Π_2-complete. (See again [16].

Thus RPL$\forall$ is an elegant fuzzy predicate calculus with truth degree equal to provability degree; on the other hand, it badly undecidable. For details see [16] and its predecessors, in particular, [28].

3.5 Gödel predicate logic

This logic is, in contradistinction to Łukasiewicz logic, recursively axiomatizable.

One extends the notion of a language to contain *propositional* variables, thus: atomic formulas are either of the from $P(s, \dots, t)$ when $s, \dots, t$ are object variables or object constants or just z where z is a propositional variable. An other formulas are built using connectives $\rightarrow, \neg, \wedge, \vee$ and quantifiers $\forall, \exists$. An *interpretation* has the form

$$\mathbf{M} = \langle M, (r_P)_{P \text{ predicate}}, (m_c)_{c \text{ constant}}, (t_z)_{z \text{ prop.var}} \rangle$$

where for each propositional variable $z, t_z, \in [0,1]$ (a truth-value).

Clearly, $\| z \|_{\mathbf{M}} = t_z$; $\| \varphi \|_M$ for other formulas is defined in the obvious way, using truth functions of Gödel logic,

$$(\| (\exists x) \varphi \|_{\mathbf{M}} = \sup_c \| \varphi(c) \|_{\mathbf{M}}).$$

Logical axioms are those of Gödel propositional logic (see 2.5) plus the axioms $(\forall 1), (\forall 2), (\forall 3), (\exists 1), (\exists 2)$ of BL$\forall$ (see above) *Deduction rules* are modus ponens and generalization. The logic is *sound* in the following sense: if $T \vdash \varphi$ then for each M there is a conjunction α of finitely many elements of T such that $\| \alpha \|_T \leq \| \varphi \|_T$. It follows that if all axioms of T are 1-true in M ($\| \alpha \|_{\mathbf{M}} = 1$) and $T \vdash \varphi$ then $\| \varphi \|_{\mathbf{M}} = 1$ too. Moreover, if M is such that $\| \alpha \|_{\mathbf{M}} \geq r$ for some r and all $\alpha \in T$ and if $T \vdash \varphi$ then $\| \varphi \|_{\mathbf{M}} \geq r$.

Completeness $T \vdash \varphi$ iff for each M there is a conjunction α of finitely many elements of T such that $\| \alpha \|_{\mathbf{M}} \leq \| \varphi \|_{\mathbf{M}}$. In particular, φ is a 1-tautology ($\| \varphi \|_{\mathbf{M}} = 1$ for all $\mathbf{M}$ iff $\vdash \varphi$).

Hence, in contradistinction to Łukasiewicz predicate logic (and Rational Quantification Logic), the set of all 1-tautologies of Gödel predicate logic is recursively enumerable.

Historical remark. Recursive axiomatizability of Gödel predicate logic was first shown by Takeuti and Titani [43] using an auxiliary deduction rule. M. Baaz showed that the rule is superfluous (still unpublished).

We have surveyed two main systems of fuzzy predicate calculus: Łukasiewicz's calculus (with its extension RPL$\forall$ à la Pavelka-Novák) and Gödel's calculus (à la Takeuti-Titani). The investigation of a predicate calculus based on the product conjunction remains to be a future task. We know that the set of 1-tautologies of $\Pi\forall$ is not recursively enumerable; moreover, it is Π_2–hard.

4 General fuzzy logics

In this section we describe a very general approach to the syntax and semantics of fuzzy logics, developed by Pavelka [32]. This approach does not assume any truth functionality.

4.1 Formulas and models

We have a set $Form$ of *formulas*. These may be formulas of some propositional logic, predicate logic, or quite abstract entities. *Semantics* is given by a set $\mathcal{S}$ whose element are called *models*. Each model is a mapping $M : Form \to [0,1]$; thus M assigns to each formula the degree in which it is true (in the model).

For example, $Form$ consists of formulas of Łukasiewicz propositional calculus and $\mathcal{S}$ consists of all $e : Form \to [0,1]$ obeying the truth functions of connectives, i.e. $e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$, $e(\neg\varphi) = (-)e(\varphi)$.

Any $T : Form \to [0,1]$ may be understood as a *fuzzy theory*; $T(\varphi)$ is the degree in which φ is an axiom. An $M \in \mathcal{S}$ is a *model of* T if, for each φ, $M(\varphi) \geq T(\varphi)$ (each formula is at least as true as the theory T demands).

For each fuzzy theory T and formula φ, let $\| \varphi \|_T = \inf\{M(\varphi) \mid M \text{ is a model of } T\}$ (the truth degree of φ for T).

4.2 Provability

We shall work with *graded formulas*, i.e. pairs (φ, x) where φ is a formula and $x \in [0,1]$. An n-ary *deduction rule* assigns to some n-tuples $(\varphi_1, x_1, \ldots (\varphi_n, x_n)$ of graded formulas a graded formula $(r'(\varphi_1, \ldots, \varphi_n),\ r''(x_1, \ldots, x_n))$ (r', r'' are appropriate functions).

The function r'' is assumed to preserve all (infinite) suprema, i.e. if $\sup_{n\in I}(x_n) = y$ then $\sup_{n\in I}(r''(\ldots, x_n, \ldots) = r''(\ldots, \sup_{n\in I} x_n, \ldots)$.

For example, recall the *fuzzy modus ponens* in Łukasiewicz logic:

$$\frac{(\varphi, x), (\varphi \to \psi, y)}{(\psi, x * y)}.$$

A theory T is *closed under* the rule (r', r'') if for each tuple $\varphi_1, \ldots, \varphi_n$ of formulas, $T(r'(\varphi_1, \ldots, \varphi_n)) \geq r''(T(\varphi_1), \ldots T(\varphi_n))$, i.e. if $T(\varphi_i) = x_i$ and $T(r'(\varphi_1, \ldots, \varphi_n)) = y$ then from $(\varphi_1, x_1), \ldots, (\varphi_n, x_n)$ the rule derives

$(r'(\varphi_1,\ldots,\varphi_n),\ r''(x_1,\ldots,x_n))$ and T demands $r'(\varphi_1,\ldots\varphi_n)$ to be at least y-true, $y \geq r''(x_1,\ldots,x_n)$).

A *deductive structure* is given by a fuzzy theory A (of logical axioms) and a set $\mathcal{R}$ of deduction rules. For each fuzzy theory T, there is a unique theory $T' \supseteq T$ such that $T \supseteq A$ and T is closed under each rule from $\mathcal{R}$. T' is denoted $Cn_{A,\mathcal{R}}(T)$.

A *graded proof* in T (given $A, \mathcal{R}$) is a set of graded formulas $(\varphi_1, x_1), \ldots, (\varphi_n, x_n)$ such that each (φ_i, x_i) either is a logical axiom $(A(\varphi_i) = x_i)$ or is an axiom of T $(T(\varphi_i) = x_i)$ or (φ_i, x_i) results by a rule $R \in \mathcal{R}$ from some previous graded formulas. The *provability* degree $|\varphi|_T$ is $\sup\{r \mid T \vdash (\varphi, x)\}$ (where $T \vdash (\varphi, x)$ obviously means that (φ, x) is the last member of a proof.

The condition of *sup* preservation guarrantees that for each φ
$|\varphi|_T = Cn_{A\mathcal{S}}(T)(\varphi)$.

The deductive structure $(A, \mathcal{R})$ is *sound* for the semantics $\mathcal{S}$ if for each theory T and each formula $\varphi, |\varphi|_T \leq \|\varphi\|_T$ ($|\varphi|_T$ being defined using $(A, \mathcal{R})$, $\|\varphi\|_T$ using $\mathcal{S}$). It is *complete* if $|\varphi|_T = \|\varphi\|_T$.

5 Equality and similarity

5.1 Classical predicate calculus with equality

Classical predicate calculus is often extended with to deal with the relation of equality. This is achieved by introducing in the language a new predicate E of arity 2 and adding to the axioms of classical propositional calculus (i.e. the three axioms of classical propositional calculus plus the two axioms on quantifiers $(\forall 1)$ and $(\forall 2)$) the following two axioms for equality:

(E1) $E(x, x)$

(E2) $E(x, y) \to (P(\ldots, x, \ldots) \to P(\ldots, y, \ldots))$

where P is any predicate of the (extended) language. Then any extension of classical predicate calculus including these axioms is called a PC system with equality. Let T be such an extension. Then it is easy to show that T proves the following formulas about the equality predicate:

$(\forall x)E(x, x)$

$(\forall x)(\forall y)(E(x, y) \to E(y, x))$

$(\forall x)(\forall y)(\forall z)(E(x, y) \to (E(y, z) \to E(x, z))$

Thus since each of these must be true in any model of S, the predicate E has to be interpreted by an equivalence relation (reflexive, symmetric and transitive), but not necessarily as an equality $=$. However it can be also proved that any consistent PC system with equality has a model where E is interpreted by $=$.

5.2 Many-valued predicate calculi with fuzzy equality

The fuzzy counterpart of classical equivalence relations is the following notion of fuzzy similarity relation, also known as fuzzy equality relations.

Let W be a set and let L be a linearly ordered residuated lattice. A binary L-fuzzy relation S on W (i.e. a mapping $S : W \times W \to L$) is a $*$-*similarity relation* if it satisfies the following properties ([45]):

1. reflexivity: $S(w, w) = 1$
2. symmetry: $S(w, w') = S(w', w)$
3. $*$-transitivity: $S(w, w') * S(w', w'') \leq S(w, w'')$

When $S(W \times W) = \{0, 1\}$, S is clearly an equivalence relation on W. For simplicity we shall assume that L is the interval [0,1] with the structure given by a t-norm $*$ and its residuum $\Rightarrow$. Let us discuss our three basic t-norms:

- $*$ = minimum: then S is a similarity relation in the sense of Zadeh [46]). Especially, each level-cut $S_\alpha = \{(w, w') \mid S(w, w') \leq \alpha\}$ is an equivalence relation, and $1 - S$ defines a pseudo-ultrametric;
- $*$ = product: this type of fuzzy relation goes back to Menger [26] and has been studied by Ovchinnikov [30];
- $*$ = Lukasiewicz conjunction, i.e. $a * b = max(0, a + b - 1)$. This type of fuzzy relation is studied by Ruspini [38], Bezdek and Harris [2]) who call it a likeness relation. Then $1 - S$ is a pseudo-metric.

A similarity in thus a notion dual to a distance. A $*$-similarity relation will be called *fuzzy equality* if in addition it verifies the following separating property:

$$S(w, w') = 1 \text{ iff } w = w'.$$

If S is a fuzzy equality, the 1-cut of S (that is, $\{(w, w') \mid S(w, w') = 1\}$) is just the equality on W.

Therefore, in order to define what a many-valued logical system with equality is, it seems natural to add the following axioms for fuzzy equality to our basic many-valued predicate logic $BL\forall$:

(E1) $E(x, x)$

(E2) $E(x, y) \to (P(\ldots, x, \ldots) \to P(\ldots, y, \ldots))$

where P is any predicate.

Then, analogously to classical predicate logic, in any model of any theory of containing (E1), (E2) the predicate E must be interpreted by a fuzzy relation which must be a $*$-similarity relation and moreover, the interpretations of the rest of predicates have to be *extensional* [22]. Indeed, the following formulas

$(\forall x)E(x, x)$

$(\forall x)(\forall y)(E(x, y) \to E(y, x))$

$(\forall x)(\forall y)(\forall z)(E(x, y) \to (E(y, z) \to E(x, z))$

directly corresponding to the reflexivity, symmetry and *-transitivity properties of the similarity relations, are also provable in any theory over BL$\forall$ containing (E1) and (E2).

Moreover, if S is the interpretation of E and μ_P is the interpretation of the predicate P (we consider P of arity 1), since (E2) is a 1-tautology, it follows that

$$S(a,b) \leq \mu_P(a) \Rightarrow \mu_P(b)$$

that is,

$$\mu_P(a) * S(a,b) \leq \mu_P(b),$$

which is the condition for the fuzzy set μ_P to be extensional [22], which in turn is a generalization of the classical condition

$$\text{if } a \in A \text{ and } a \equiv b \text{ then } b \in A$$

for a subset A to behave well with respect to an equivalence relation $\equiv$, or in other words, the condition for A to be a union of equivalence classes.

Finally, let us mention, again analogously to the classical case, that the interpretation of the equality predicate in models of theories with equality need not be a fuzzy equality in the above sense. However, for any consistent theory with equality there is a model where the equality predicate is interpreted as a fuzzy equality relation. The proof is as follows.

Let T be a consistent theory with equality over $\mathcal{C}\forall$, and let $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ a model for T. Let $S = r_E$ be the interpretation of E in the model M. It is clear that S must be a $*$-similarity relation. Define the equivalence relation on M as follows: $a \sim b$ iff $S(a,b) = 1$, and denote the equivalence class containing a by $[a]$. Now define a new structure $\mathbf{M}' = \langle M', (r'_P)_P, (m'_c)_c \rangle$, where $M' = M/\sim$, $r'_P([a]) = [r'_P(a)]$ and $m'_c = [m_c]$. It can be checked that $\mathbf{M}'$ is a model of T and $S' = r'_E$ is a fuzzy equality.

5.3 Similarity-based logical systems

One of the possible semantics of fuzzy sets is in terms of similarity, namely a grade of membership of an item in a fuzzy set can be viewed as the degree of resemblance between this item and prototypes of the fuzzy set. In such a framework, an interesting question is how to devise a logic of similarity able to account for the proximity between interpretations.

A variety of uncertain reasoning models has been captured in the modal framework by equipping the set of interpretations or possible worlds with a suitable uncertainty measure (see e.g. [15]). It is thus tempting to model similarity-based reasoning by equipping a set of possible worlds with a proximity or generalized metric structure.

Similarity relations and fuzzy sets can be closely related. Namely let $A \subseteq \Omega$ be a non-empty subset of Ω. Then a similarity relation S allows us to define the non-empty normalized fuzzy set A^* of elements *close* to A as follows:

$$\mu_{A^*}(w) = max_{w' \in A} S(w,w')$$

. Conversely, any non-empty fuzzy set F on Ω can be viewed as deriving from a $*$-similarity relation S and a subset A such that

$$A = \{w \mid \mu_F(w) = 1\} \ (\neq \emptyset) \ S(w,w') = min(\mu_F(w) \Rightarrow \mu_F(w'), \mu_F(w') \Rightarrow \mu_F(w))$$

This is due to Valverde's theorem of representation of similarity relations by fuzzy sets [45], based on residuation. This result gives a formal justification to the fact that a degree of membership $\mu_F(w)$ in a fuzzy set can be interpreted as a degree of similarity of w to *prototypes* of F, which form the set A.

Moreover it points out that if q is a proposition in a formal propositional language L, of which Ω is the (finite) set of interpretations, then the similarity induces a fuzzy proposition q^* whose (fuzzy) set of models is $[q^*] = [q]^*$, defined by means of the fuzzy relation S, where $[q]$ denotes the (classical) set of models of q (the set of interpretations where q is true). Intuitively $q*$ means *approximately* q, *not far from* q, where "approximately", "not far from" is mathematically expressed by the similarity relation S.

Analogously to what we have said in the introduction on fuzziness and probability, the similarity-based approach in the frame of truth-functional fuzzy logic has to distinguish between a crisp proposition q and its fuzzy counterpart *approximately* q, keeping strictly in mind that *approximately* $p\&q$, i.e. $(p\&q)^*$ is not equivalent to *approximately* p *and approximately* q, i.e. $(p^*\&q^*)$. Then one may be safely truth-functional.

But our aim in this section is to describe another approach that consists in considering for eac q the corresponding *approximately* q, i.e. in defining a graded satisfaction relation on the formulas of the original given propositional language as follows:

$$w \models_S^\alpha q \text{ iff } \mu_{[q*]}(w) \geq \alpha$$

That is, in the finite case, $w \models_S^\alpha q$ if there exists a model w' of q which is α-similar to w. In other words, w belongs to the α-cut of $[q*]$ that will be denoted by $[q*]_\alpha$. The degree of approximate satisfaction of q by w in the sense of S has been introduced by Ruspini [39], and shall be denoted

$$I_S(q \mid w) = \mu_{[q*]}(w) = max_{w' \models q} S(w, w').$$

Note that, identifying each interpretation w of Ω with the conjunction of literals made true by w , we have that $I_S(w' \mid w) = S(w, w')$. Thus, one may have $w \models_S^\alpha w'$ for $w' \neq w$. Indeed it means that w and w' are close enough to each other in the sense that $S(w, w') \geq \alpha$. Note that $w \models_S^\alpha w'$ is equivalent to $w' \models_S^\alpha w$ since S is symmetric.

The graded satisfaction relation can be extended over a graded entailment between (boolean) propositions in the obvious way:

$$p \models_S^\alpha q \text{ iff } w \models_S^\alpha q \text{ for each } w \text{ model of } p$$

In other words, $p \models_s^\alpha q$ holds if each model of p is similar, at least to the degree α, to some model of q. An equivalent definition is $p \models_S^\alpha q$ iff $I_S(q \mid p) \geq \alpha$, where $I_S(q \mid p) = inf_{w \models p} I_S(q \mid w)$ is the Ruspini's *implication measure* of q given p. The graded entailment has been characterized in terms of the following properties [4]:

Nested:	If $p \models^{\alpha} q$ then $p \models^{\beta} q$, for $\beta \leq \alpha$.
Extremals:	$p \models^{1} q$ iff $p \models q$; $p \models^{0} q$.
$*$-*Transitivity*:	If $p \models^{\alpha} q$ and $q \models^{\beta} r$ then $p \models^{\alpha * \beta} r$.
Left Or:	$p \vee q \models^{\alpha} r$ iff $p \models^{\alpha} r$ and $q \models^{\alpha} r$.
Right Or:	If r has a single model then
	$r \models^{\alpha} p \vee q$ iff $r \models^{\alpha} p$ or $r \models^{\alpha} q$.
Consistency preservation:	If $p \not\equiv \top$ then $p \models^{\alpha} \top$ only when $\alpha = 0$.
Continuity from below:	If $p \models^{\beta} q$ for each $\beta < \alpha$ then $p \models^{\alpha} q$.

One can understand this as a general fuzzy logic in the sense of Section 4. But one has to be aware of the fact that such a logic cannot be truth-functional. Namely given S, the truth-value evaluation $I_S(q \mid w)$ of q associated to the interpretation w is truth-functional neither for the negation nor for the conjunction since only the following inequalities hold in the general case:

$I_S(\neg q \mid w) \geq 1 - I_S(q \mid w)$

$I_S(p \wedge q \mid w) \leq min(I_S(p \mid w), I_S(q \mid w))$

However for disjunction we do have that $[p \vee q]^* = [p]^* \cup [q]^*$, hence $I_S(p \vee q \mid w) = max(I_S(p \mid w), I_S(q \mid w))$. This fact stresses the difference between similarity logic and many other logics underlying fuzzy sets like the truth-functional fuzzy logics described in Section 2. This lack of truth-functionality has also been noticed in the theory of rough sets (Pawlak, 1991). Rough sets are a theory of similarity based on equivalence relations that handles upper and lower approximations of sets. The lack of truth-functionality is thus not due to the fuzziness of similarity.

A natural logical setting for similarity-based reasoning is the one of modal logics which is tailored to account for relations on the set of interpretations. The similarity relation S can be considered as a family of nested accessibility relations R_α on the set of possible worlds Ω defined as $wR_\alpha w'$ iff $S(w, w') \geq \alpha$. Therefore, enlarging the logical language, we can define, for each α, a pair of dual modal operators $\Box_\alpha$ and $\Diamond_\alpha$ with the following standard semantics:

$w \models \Diamond_\alpha p$ iff there exists w' such that $wR_\alpha w'$ and $w' \models p$

$w \models \Box_\alpha p$ iff for every w' such that $wR_\alpha w'$ then it holds $w' \models p$

If the similarity relation is min-transitive, i.e.

$$S(w, w') \geq min(S(w, w"), S(w", w')),$$

then the accessibility relations R_α are equivalence relations, and therefore, for each α, $\Box_\alpha$ and $\Diamond_\alpha$ are a pair of dual S5 modal operators. These types of modal logics generalize rough set logics (Orlowska, 1984) and have been studied by Nakamura (1992). It is easy to check that the above defined graded satisfaction $\models_\alpha$ is directly related to the possibility operator $\Diamond_\alpha$ in the sense that if q is a non-modal proposition, then $w \models_\alpha q$ iff $w \models \Diamond_\alpha q$.

In the following we shall describe the multi-modal system axiomatizing the graded modal operators $\Box_\alpha$ and $\Diamond_\alpha$.

To define the language we fix a range $G \subset [0, 1]$ of possible similarity values. Further assumptions on G are that $\{0, 1\} \subseteq G$ and that, for the sake of simplicity, we shall assume that G is denumerable. Then, the multi-modal propositional language is built,

in the usual way, upon a denumerable set of propositional variables $p, q, \ldots$, connectives $\rightarrow$ (implication) and $\neg$ (negation), and (unary) modal operators $\Diamond^o_\alpha$ and $\Diamond^c_\alpha$, for each $\alpha \in G$. We shall use φ, $\psi, \ldots$ to denote arbitrary formulas. We shall also use the classical definitions of $\wedge$ and $\vee$ in terms of $\rightarrow$ and $\neg$, and furthermore $\Box^o_\alpha\varphi$ and $\Box^c_\alpha\varphi$ will stand for abbreviations of $\neg\Diamond^o_\alpha\neg\varphi$ and $\neg\Diamond^c_\alpha\neg\varphi$ respectively.

A *similarity Kripke model* is a struture $\mathcal{M} = \langle W, S, \| \ \| \rangle$ where:

1. W is a non empty set of possible worlds,
2. $S : W \times W \rightarrow G$ is a $*$-similarity fuzzy relation on W, for some t-norm $*$ on G,
3. $\| \ \|$ is a function that given an atomic formula p return the set $\|p\| \subseteq W$ where p is considered to be true.

The notion of a formula φ being true in a world w in a similarity Kripke model $\mathcal{M} = \langle W, S, \| \ \| \rangle$, written $(\mathcal{M}, w) \models \varphi$ is defined in the usual way, except for the modal formulas, which is defined as follows:

$(\mathcal{M}, w) \models \Diamond^c_\alpha\varphi$ iff $I^{\mathcal{M}}_S(\varphi \mid w) \geq \alpha$

$(\mathcal{M}, w) \models \Diamond^o_\alpha\varphi$ iff $I^{\mathcal{M}}_S(\varphi \mid w) > \alpha$

where the implication measure $I^{\mathcal{M}}_S$ is defined as follows:

$$I^{\mathcal{M}}_S(\varphi \mid w) = Inf_{(\mathcal{M},w)\models\varphi} S(w, w').$$

Notice that $\Diamond^o_\alpha$ is a normal modal operator in the sense that it has an associated accessibility relation R^o_α which provides it with the standard Kripke semantics:

$$(\mathcal{M}, w) \models \Diamond^o_\alpha\varphi \text{ iff } (\mathcal{M}, w') \models \varphi \text{ for some } w' \text{ such that } (w, w') \in R^o_\alpha,$$

where the accessibility relation R^o_α is defined as

$$(w, w') \in R^o_\alpha \text{ iff } S(w, w') > \alpha.$$

This is not the general case for the operators $\Diamond^c_\alpha$, i.e. they do not have, in general, a corresponding accessibility relation. However they do have it whenever the *sup* appearing in the expression of $I^{\mathcal{M}}_S(\varphi \mid w)$ is reached for any A and any w, i.e. when $I^{\mathcal{M}}_S(\varphi \mid w)$ becomes $max_{(\mathcal{M},w')\models\varphi} S(w, w')$. In particular, this is the case when either the range G is finite or the set of possible worlds W is finite.

Given a range G and a t-norm operation $*$ on G, the class of structures $\mathcal{C}^G_*$ is the set of similarity structures $\mathcal{M} = \langle W, S, \| \ \| \rangle$ where S is a $(G, *)$-similarity on W. The notation $\mathcal{FC}^G_*$ will denote the subclass of $\mathcal{C}^G_*$ consisting of similarity structures with a finite set of worlds W.

The basic similarity multi-modal logic $\mathbf{MS5}(\mathbf{G}, *)$ is the smallest set of sentences containing every instance of the following axiom schemes and closed under the last two inference rules:

PL: Propositional tautologies
K^c: $\Box^c_\alpha(\varphi \to \psi) \to (\Box^c_\alpha\varphi \to \Box^c_\alpha\psi)$, $\forall\alpha \in G$
K^o: $\Box^o_\alpha(\varphi \to \psi) \to (\Box^o_\alpha\varphi \to \Box^o_\alpha\psi)$, $\forall\alpha \in G$
T^c: $\Box^c_\alpha\varphi \to \varphi$, $\forall\alpha \in G$
B^o: $\varphi \to \Box^o_\alpha\Diamond^o_\alpha\varphi$, $\forall\alpha \in G$
4^c: $\Box^c_{\alpha*\beta}\varphi \to \Box^c_\beta\Box^c_\alpha\varphi$, $\forall\alpha \in G$
N^c: $\Box^c_\alpha\varphi \to \Box^c_\beta\varphi$, for $\beta \geq \alpha$,
EX^c: $\Diamond^c_0\varphi$,
EX^o: $\neg\Diamond^o_1\varphi$,
CO: $\Box^c_\alpha\varphi \to \Box^o_\alpha\varphi$, $\forall\alpha \in G$
OC: $\Box^o_\alpha\varphi \to \Box^c_\beta\varphi$, for $\alpha < \beta$,
RN^o: From φ infer $\Box^o_\alpha\varphi$, $\forall\alpha \in G$
MP: From φ and $\varphi \to \psi$ infer ψ

Schemes K^i, T^i, B^i and 4^i, where i is either c or o, are direct counterpart, for the graded modal operators, of the well-known axioms of the classical $S5$ modal logic. Scheme C^c corresponds to the fact that, under the assumption of finite range G or finite set of worlds W, $I^{\mathcal{M}}_S(\varphi \mid w) = 1$ only if φ is true in w. Schemes N^i stand for the nested properties of the graded modal operators, while schemes EX^i set up the extremal conditions for them. Finally, schemes OC and CO establish the obvious relations between strict and non-strict inequalities.

It is very easy to check that $\mathbf{MS5(G,*)}$ is sound with respect to the class of structures $\mathcal{C}^G_*$, for any G and $*$. The question whether, in general, $\mathbf{MS5(G,*)}$ has not been addressed yet. However there is completeness in the following particular cases.

1. For any *finite* range G, the system $\mathbf{MS5^+(G,*)}$ obtained from $\mathbf{MS5(G,*)}$ by adding the axiom:
$$C^c : \varphi \to \Box^c_1\varphi.$$
is complete with respecte to the class of similarity models $\mathcal{C}^G_*$.

2. For any *dense* range G, and $* = minimum$, the system $\mathbf{MS5^{++}(G,min)}$ obtained from $\mathbf{MS5(G,min)}$ by adding the axioms:
B^c : $\varphi \to \Box^c_\alpha\Diamond^c_\alpha\varphi$, for $\alpha > 0$
C^c : $\varphi \to \Box^c_1\varphi$, and
4^o : $\Box^o_{\alpha\wedge\beta}\varphi \to \Box^o_\beta\Box^o_\alpha\varphi$, $\forall\alpha \in G$,
is complete with respect to the class of similarity models $\mathcal{FC}^G_{min}$.

Remark that, for the case of G being finite, one can define the set of open modal operators $\{\Box^o_\alpha\}_{\alpha\in G}$ in terms of the closed ones $\{\Box^c_\alpha\}_{\alpha\in G}$, and therefore the system $\mathbf{MS5^+(G,*)}$ admits the following much simpler axiom system:

PL: Tautologies of propositional logic,
K: $\Box_\alpha(\varphi \to \psi) \to (\Box_\alpha\varphi \to \Box_\alpha\psi)$,
T: $\Box_\alpha\varphi \to \varphi$,
B: $\varphi \to \Box_\alpha\Diamond_\alpha\varphi$,
4: $\Box_{\alpha*\beta}\varphi \to \Box_\beta\Box_\alpha\varphi$,
C: $A \to \Box_1\varphi$,
N: $\Box_\alpha\varphi \to \Box_\beta\varphi$, with $\beta \geq \alpha$,
EX: $\Diamond_0\varphi$,
where we have written $\Box_\alpha$ for $\Box^c_\alpha$.

As a kind of final remark, notice that it is clear that the similarity-based graded entailment relation $\models^\alpha_S$ introduced at the beginning of this section is fully captured inside the multi-modal systems. Namely, given a $*$-similarity S on the set of interpretations Ω of the propositional sublanguage, if φ and ψ are non-modal formulas, then we have that

$$\varphi \models^\alpha_S \psi \text{ iff } \mathcal{M}_\mathcal{L} \models \varphi \to \Diamond^c_\alpha\psi,$$

where $\mathcal{M}_\mathcal{L} = \langle\Omega, S, \| \ \|\rangle$.

6 Conclusion

We hope that we have shown the following:

- *Fuzzy logic is neither a poor man's logic nor poor man's probability.* Fuzzy logic (in the narrow sense) is a reasonably deep theory.
- *Fuzzy logic is a logic.* It has its syntax and semantic and notion of consequence. It is a study of consequence.
- *There are various systems of fuzzy logic, not just one.* The main two most developedp systems are those of Łukasiewicz and of Gödel, the first together with its extension à la Pavelka.

In addition, we claim the following:

- *Further logical investigations of fuzzy logic are possible.* In particular, one has to apply the theory of generalized quantifiers to fuzzy logic and go further in a strictly logical analysis of things pointed out by Zadeh as "particular agenda of fuzzy logic". Cf also [5].
- *To construct combined calculi of vagueness and of uncertainty is possible.* See [19, 20] for information; one gets many-valued modal logics.
- *Fuzzy logic in the narrow sense is a beautiful logic, but also is important for applications:* it offers foundations.

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