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1996

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Technical report No. 688

October 1996

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Abstract

We describe a rectangle in the complex plane enclosing all eigenvalues of an interval matrix A^I. We give theoretical bounds (Theorem 1) that are exact for symmetric or skew-symmetric matrices (Theorem 2) and practical bounds (Theorem 3) requiring evaluation of 6 minimal or maximal eigenvalues of symmetric matrices. Some consequences are mentioned.

Keywords
Interval matrix, eigenvalue, bound

[1]This work was supported by the Czech Republic Grant Agency under grant GAČR 201/95/1484.

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1 Theoretical bounds

We consider square interval matrices in the form

$$A^I = [A_c - \Delta, A_c + \Delta] = \{A;\ A_c - \Delta \le A \le A_c + \Delta\}$$

where inequalities are understood componentwise; thus A_c is the center matrix and Δ is the radius matrix of A^I.

Theorem 1 *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a square interval matrix. Then for each eigenvalue λ of each $A \in A^I$ we have*

$$\underline{r} \le \operatorname{Re}\lambda \le \overline{r}, \tag{1.1}$$

$$\underline{i} \le \operatorname{Im}\lambda \le \overline{i}, \tag{1.2}$$

where

$$\begin{aligned}
\underline{r} &= \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|),\\
\overline{r} &= \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|),\\
\underline{i} &= \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|),\\
\overline{i} &= \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|).
\end{aligned}$$

Comments. Vectors are always considered column vectors, so that $x^T y$ is the scalar product whereas xy^T is the matrix $(x_i y_j)$. In the formulae for $\underline{i}$ and $\overline{i}$, for typographic reasons we write "$\|(x_1, x_2)\|_2 = 1$" in the subscript instead of the correct "$\|(x_1^T, x_2^T)^T\|_2 = 1$". For $A = (a_{ij})$ and $B = (b_{ij})$ we use

$$A \circ B = \sum_{ij} a_{ij} b_{ij}$$

("scalar product of matrices"). Then we have

$$x^T A y = \sum_{ij} x_i a_{ij} y_j = A \circ (xy^T).$$

Proof. Let $\lambda = \lambda_1 + \lambda_2 i$ be an eigenvalue of some $A \in A^I$. Then

$$A(x_1 + x_2 i) = (\lambda_1 + \lambda_2 i)(x_1 + x_2 i) \tag{1.3}$$

for some real vectors x_1, x_2, $x_1 \ne 0$ or $x_2 \ne 0$, which may be normalized to achieve

$$x_1^T x_1 + x_2^T x_2 = 1. \tag{1.4}$$

Premultiplying (1.3) by the complex conjugate vector $x_1 - x_2 i$, we obtain

$$\lambda_1 + \lambda_2 i = (x_1 - x_2 i)^T A (x_1 + x_2 i),$$

which yields

$$\mathrm{Re}\,\lambda = \lambda_1 = x_1^T A x_1 + x_2^T A x_2, \tag{1.5}$$

$$\mathrm{Im}\,\lambda = \lambda_2 = x_1^T A x_2 - x_2^T A x_1. \tag{1.6}$$

1) To prove that $\mathrm{Re}\,\lambda \leq \overline{r}$, denote $r(A) = \max_{\|x\|_2=1} x^T A x$, then we have

$$x_1^T A x_1 \leq r(A) x_1^T x_1,$$
$$x_2^T A x_2 \leq r(A) x_2^T x_2,$$

hence

$$x_1^T A x_1 + x_2^T A x_2 \leq r(A)(x_1^T x_1 + x_2^T x_2) = r(A) \tag{1.7}$$

due to (1.4), and

$$\begin{aligned} r(A) &= \max_{\|x\|_2=1} x^T A x = \max_{\|x\|_2=1} (x^T A_c x + x^T (A - A_c) x) \\ &\leq \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|) = \overline{r}. \end{aligned} \tag{1.8}$$

Then from (1.5), (1.7) and (1.8) we obtain

$$\mathrm{Re}\,\lambda \leq \overline{r},$$

which is the right-hand side inequality in (1.1).

2) Since $-\lambda$ is an eigenvalue of $-A$ which belongs to $[-A_c - \Delta, -A_c + \Delta]$, from the result proved in 1) applied to $[-A_c - \Delta, -A_c + \Delta]$ we obtain

$$-\mathrm{Re}\,\lambda = \mathrm{Re}\,(-\lambda) \leq \max_{\|x\|_2=1} (-x^T A_c x + |x|^T \Delta |x|),$$

which implies

$$\mathrm{Re}\,\lambda \geq -\max_{\|x\|_2=1} (-x^T A_c x + |x|^T \Delta |x|) = \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|) = \underline{r},$$

which is the left-hand side inequality in (1.1).

3) Since

$$\begin{aligned} x_1^T A x_2 - x_2^T A x_1 &= x_1^T A_c x_2 - x_2^T A_c x_1 + x_1^T (A - A_c) x_2 - x_2^T (A - A_c) x_1 \\ &= x_1^T (A_c - A_c^T) x_2 + (A - A_c) \circ (x_1 x_2^T - x_2 x_1^T) \\ &\leq x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|, \end{aligned}$$

from (1.6) and (1.4) we get

$$\mathrm{Im}\,\lambda \leq \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \overline{\imath},$$

which is the right-hand side inequality in (1.2).

4) Since $-\lambda$ is an eigenvalue of $-A \in [-A_c - \Delta, -A_c + \Delta]$, applying the result in 3) we obtain

$$-\mathrm{Im}\,\lambda = \mathrm{Im}\,(-\lambda) \leq \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|)$$

and thereby also

$$\mathrm{Im}\,\lambda \geq \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \underline{\imath},$$

which concludes the proof. ∎

2 The bounds are exact in special cases

A real matrix A is called symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$. An interval matrix A^I is said to be symmetric if

$$A^{IT} = A^I$$

and skew-symmetric if

$$A^{IT} = -A^I,$$

where

$$A^{IT} = \{A^T;\ A \in A^I\}$$

and

$$-A^I = \{-A;\ A \in A^I\}.$$

Hence, $A^I = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if $[A_c^T - \Delta^T, A_c^T + \Delta^T] = [A_c - \Delta, A_c + \Delta]$, which is equivalent to symmetry of both A_c and Δ. Similarly, A^I is skew-symmetric if and only if A_c is skew-symmetric and Δ is symmetric.

Theorem 2 *The bounds (1.1) are exact (i.e., achieved over A^I) if A^I is symmetric and the bounds (1.2) are exact if A^I is skew-symmetric.*

Proof. 1) Let A^I be symmetric, so that A_c and Δ are symmetric. Since the continuous mapping $x \mapsto x^T A_c x + |x|^T \Delta |x|$ achieves its maximum over the unit sphere $\{x;\ \|x\|_2 = 1\}$, there exists an x satisfying

$$\overline{r} = x^T A_c x + |x|^T \Delta |x| \tag{2.1}$$

and $\|x\|_2 = 1$. Define a diagonal matrix S by

$$S_{jj} = \begin{cases} 1 & \text{if } x_j \geq 0, \\ -1 & \text{if } x_j < 0 \end{cases}$$

($j = 1, \ldots, n$), then $|x| = Sx$ and from (2.1) we have

$$\overline{r} = x^T A_c x + x^T S \Delta S x = x^T (A_c + S\Delta S) x \leq \lambda_{\max}(A_c + S\Delta S), \tag{2.2}$$

where $\lambda_{\max}(A_c + S\Delta S)$ denotes the maximal eigenvalue of $A_c + S\Delta S$ (which is symmetric since both A_c and Δ are symmetric). Since $|S\Delta S| = \Delta$, the matrix $A_c + S\Delta S$ belongs to A^I, hence

$$\lambda_{\max}(A_c + S\Delta S) \leq \overline{r}$$

by Theorem 1, which combined with (2.2) gives

$$\overline{r} = \lambda_{\max}(A_c + S\Delta S),$$

hence $\overline{r}$ is achieved over A^I (even more, it is achieved at a symmetric matrix in A^I, cf. Hertz [1]). The proof for $\underline{r}$ is analogous; in this case we obtain

$$\underline{r} = \lambda_{\min}(A_c - S\Delta S).$$

2) Let A^I be skew-symmetric, so that A_c is skew-symmetric and Δ is symmetric. We have

$$\bar{\imath} = x_1^T(A_c - A_c^T)x_2 + \Delta \circ |x_1x_2^T - x_2x_1^T| \tag{2.3}$$

for some x_1, x_2 satisfying $\|(x_1, x_2)\|_2 = 1$. Define

$$z_{ij} = \begin{cases} -1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j < 0, \\ 0 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j = 0, \\ 1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j > 0 \end{cases}$$

$(i, j = 1, \ldots, n)$, then $z_{ij} = -z_{ji}$ for each i, j, hence the matrix $\tilde{\Delta}$ defined by

$$\tilde{\Delta}_{ij} = z_{ij}\Delta_{ij}$$

$(i, j = 1, \ldots, n)$ is skew-symmetric (since Δ is symmetric). Let

$$A = A_c + \tilde{\Delta},$$

then $A \in A^I$ and A is skew-symmetric (since both A_c and $\tilde{\Delta}$ are skew-symmetric). Next, from (2.3) we have

$$\begin{aligned} \bar{\imath} &= x_1^T(A_c - A_c^T)x_2 + \sum_{ij} \Delta_{ij} z_{ij} (x_1x_2^T - x_2x_1^T)_{ij} \\ &= x_1^T(A_c - A_c^T)x_2 + \tilde{\Delta} \circ (x_1x_2^T - x_2x_1^T) \\ &= x_1^T(A_c - A_c^T)x_2 + x_1^T\tilde{\Delta}x_2 - x_2^T\tilde{\Delta}x_1 \\ &= x_1^TAx_2 - x_2^TAx_1 \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{aligned}$$

where the matrix

$$\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \tag{2.4}$$

is symmetric since A is skew-symmetric. Denote by λ the maximal eigenvalue of (2.4) (which is real), then from the above expression for $\bar{\imath}$ we have

$$\bar{\imath} \le \lambda \tag{2.5}$$

and there exists a vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0$ satisfying

$$\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which implies

$$A(y_1 + y_2 i) = -\lambda y_2 + \lambda i y_1 = \lambda i(y_1 + y_2 i),$$

thus λi is an eigenvalue of A. Hence

$$\lambda \le \bar{\imath}$$

by Theorem 1, which combined with (2.5) gives

$$\bar{\imath} = \lambda = \mathrm{Im}\,(\lambda i),$$

hence $\bar{\imath}$ is achieved as the imaginary part of an eigenvalue of a matrix in A^I. To prove an analogous result for $\underline{i}$, let us apply the result just proved to the interval matrix $[-A_c - \Delta, -A_c + \Delta]$, which is also skew-symmetric. Then we have

$$\mathrm{Im}\,\lambda = \max_{\|(x_1,x_2)\|_2=1} (x_1^T(A_c^T - A_c)x_2 + \Delta \circ |x_1x_2^T - x_2x_1^T|)$$

for an eigenvalue λ of some $\tilde{A} \in [-A_c - \Delta, A_c + \Delta]$, hence

$$\mathrm{Im}\,(-\lambda) = -\mathrm{Im}\,\lambda = \min_{\|(x_1,x_2)\|_2=1} (x_1^T(A_c - A_c^T)x_2 - \Delta \circ |x_1x_2^T - x_2x_1^T|) = \underline{i}$$

for the eigenvalue $-\lambda$ of $-\tilde{A} \in [A_c - \Delta, A_c + \Delta]$, which shows that $\underline{i}$ is achieved as well. ■

3 Practical bounds

Theorem 3 *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a square interval matrix. Then for each eigenvalue λ of each $A \in A^I$ we have*

$$\lambda_{\min}(A_c') - \lambda_{\max}(\Delta') \le \mathrm{Re}\,\lambda \le \lambda_{\max}(A_c') + \lambda_{\max}(\Delta'), \quad (3.1)$$

$$\lambda_{\min}(A_c'') - \lambda_{\max}(\Delta'') \le \mathrm{Im}\,\lambda \le \lambda_{\max}(A_c'') + \lambda_{\max}(\Delta''), \quad (3.2)$$

where

$$\begin{aligned} A_c' &= \frac{1}{2}(A_c + A_c^T), \\ \Delta' &= \frac{1}{2}(\Delta + \Delta^T), \\ A_c'' &= \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix}, \\ \Delta'' &= \begin{pmatrix} 0 & \Delta' \\ \Delta' & 0 \end{pmatrix}. \end{aligned}$$

Comments. $\lambda_{\min}, \lambda_{\max}$ denote the minimal and maximal eigenvalue of a symmetric matrix, respectively. Notice that all the matrices $A_c', \Delta', A_c'', \Delta''$ are symmetric by definition. Since $\lambda_{\max}(D) = \varrho(D)$ (spectral radius) holds for a nonnegative symmetric matrix D, the formulae (3.1), (3.2) may also be written in the form

$$\lambda_{\min}(A_c') - \varrho(\Delta') \le \mathrm{Re}\,\lambda \le \lambda_{\max}(A_c') + \varrho(\Delta'),$$

$$\lambda_{\min}(A_c'') - \varrho(\Delta'') \le \mathrm{Im}\,\lambda \le \lambda_{\max}(A_c'') + \varrho(\Delta'').$$

Proof. Let λ be an eigenvalue of a matrix $A \in A^I$.

1) Since

$$\begin{aligned}\overline{r} &= \max_{\|x\|_2=1}(x^TA_cx+|x|^T\Delta|x|)\\ &\le \max_{\|x\|_2=1} x^TA_cx+\max_{\|x\|_2=1}|x|^T\Delta|x|\\ &= \max_{\|x\|_2=1} x^TA_c'x+\max_{\|x\|_2=1}|x|^T\Delta'|x|\\ &= \lambda_{\max}(A_c')+\lambda_{\max}(\Delta'),\end{aligned}$$

by Theorem 1 there holds

$$\operatorname{Re}\lambda \le \lambda_{\max}(A_c')+\lambda_{\max}(\Delta'),$$

which is the right-hand side inequality in (3.1).

2) The proof of the left-hand side inequality is analogous since

$$\underline{r} \ge \min_{\|x\|_2=1} x^TA_cx-\max_{\|x\|_2=1}|x|^T\Delta|x| = \lambda_{\min}(A_c')-\lambda_{\max}(\Delta').$$

3) We have

$$\begin{aligned}\overline{\imath} &= \max_{\|(x_1,x_2)\|_2=1}(x_1^T(A_c-A_c^T)x_2+\Delta\circ|x_1x_2^T-x_2x_1^T|)\\ &\le \max_{\|(x_1,x_2)\|_2=1}(x_1^TA_cx_2-x_2^TA_cx_1)+\max_{\|(x_1,x_2)\|_2=1}(|x_1|^T\Delta|x_2|+|x_2|^T\Delta|x_1|)\\ &= \max_{\|(x_1,x_2)\|_2=1}\begin{pmatrix}x_1\\x_2\end{pmatrix}^T\begin{pmatrix}0 & A_c\\ -A_c & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\max_{\|(x_1,x_2)\|_2=1}\begin{pmatrix}|x_1|\\|x_2|\end{pmatrix}^T\begin{pmatrix}0 & \Delta\\ \Delta & 0\end{pmatrix}\begin{pmatrix}|x_1|\\|x_2|\end{pmatrix}\\ &= \max_{\|(x_1,x_2)\|_2=1}\begin{pmatrix}x_1\\x_2\end{pmatrix}^T\begin{pmatrix}0 & \frac{1}{2}(A_c-A_c^T)\\ \frac{1}{2}(A_c^T-A_c) & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ &\quad+\max_{\|(x_1,x_2)\|_2=1}\begin{pmatrix}|x_1|\\|x_2|\end{pmatrix}^T\begin{pmatrix}0 & \frac{1}{2}(\Delta+\Delta^T)\\ \frac{1}{2}(\Delta+\Delta^T) & 0\end{pmatrix}\begin{pmatrix}|x_1|\\|x_2|\end{pmatrix}\\ &= \lambda_{\max}(A_c'')+\lambda_{\max}(\Delta'').\end{aligned}$$

Hence Theorem 1 gives

$$\operatorname{Im}\lambda \le \overline{\imath} \le \lambda_{\max}(A_c'')+\lambda_{\max}(\Delta''),$$

which is the right-hand side inequality in (3.2).

4) An analogous reasoning gives

$$\begin{aligned}\underline{i} &= \min_{\|(x_1,x_2)\|_2=1}(x_1^T(A_c-A_c^T)x_2-\Delta\circ|x_1x_2^T-x_2x_1^T|)\\ &\ge \min_{\|(x_1,x_2)\|_2=1}\begin{pmatrix}x_1\\x_2\end{pmatrix}^T\begin{pmatrix}0 & \frac{1}{2}(A_c-A_c^T)\\ \frac{1}{2}(A_c^T-A_c) & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ &\quad-\max_{\|(x_1,x_2)\|_2=1}\begin{pmatrix}|x_1|\\|x_2|\end{pmatrix}^T\begin{pmatrix}0 & \frac{1}{2}(\Delta+\Delta^T)\\ \frac{1}{2}(\Delta+\Delta^T) & 0\end{pmatrix}\begin{pmatrix}|x_1|\\|x_2|\end{pmatrix}\\ &= \lambda_{\min}(A_c'')-\lambda_{\max}(\Delta''),\end{aligned}$$

which in view of Theorem 1 implies the left-hand side inequality in (3.2). ∎

4 Consequences

We keep the notations $A_c', \Delta', A_c'', \Delta''$ introduced in Theorem 3.

Corollary 1 *If*

$$\lambda_{\max}(A_c') + \lambda_{\max}(\Delta') < 0,$$

then A^I is (Hurwitz) stable.

Proof. Indeed, in this case Theorem 3 implies $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of each $A \in A^I$. ∎

We note that a symmetric interval matrix may contain nonsymmetric matrices with complex eigenvalues. Similarly, a skew-symmetric interval matrix may contain matrices with nonzero real parts.

Corollary 2 *If A^I is symmetric, then*

$$|\operatorname{Im}\lambda| \leq \lambda_{\max}(\Delta'')$$

for each eigenvalue λ of each $A \in A^I$.

Proof. The result follows from (3.2) since $A_c'' = 0$ in view of symmetry of A_c, hence $\lambda_{\min}(A_c'') = \lambda_{\max}(A_c'') = 0$. ∎

Corollary 3 *If A^I is skew-symmetric, then*

$$|\operatorname{Re}\lambda| \leq \lambda_{\max}(\Delta')$$

for each eigenvalue λ of each $A \in A^I$.

Proof. The assertion is a consequence of (3.1) since $A_c' = 0$. ∎

Acknowledgment

The author wishes to express his gratitude to Professor David Hertz who discovered through computer experiments an error in the original formulation of the bounds (3.2) contained in the preliminary announcement [2].

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