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INSTITUTE OF COMPUTER SCIENCE

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Checking Properties of Interval Matrices

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Technical report No. 686

September 1996

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Checking Properties of Interval Matrices[1]

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Abstract

We investigate complexity of checking various properties of interval matrices. The properties in question are regularity, positive definiteness, P–property, stability and Schur stability, all of which are shown to be NP–hard to check even in the class of interval matrices with uniform coefficient tolerances. Two additional sections handle complexity of computing eigenvalues and determinants. The common basis for all these results is the NP–hardness of computing the norm $\|A\|_{\infty,1}$. In most cases we also present finitely verifiable necessary and sufficient conditions to demonstrate the exponentiality inherent in all these problems. Several verifiable sufficient conditions are added to give some hints on how to proceed in solving practical examples.

Keywords
interval matrix, regularity, positive definiteness, P-matrix, stability, Schur stability, eigenvalue, determinant, necessary and sufficient condition, complexity

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[1]The text, or a part of it, will appear in the book by Kreinovich, Lakeyev, Rohn and Kahl [24]. This work was supported by the Charles University Grant Agency under grant GAUK 195/96.

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Contents

In this chapter we investigate complexity of checking various properties of interval matrices; an interval matrix is a set of matrices whose coefficients range independently of each other within prescribed bounds. The properties in question are regularity, positive definiteness, P–property, stability and Schur stability, all of which are shown to be NP–hard to check even in the class of interval matrices with uniform coefficient tolerances. Two additional sections handle complexity of computing eigenvalues and determinants. The common basis for all these results is the NP–hardness of computing the norm $\|A\|_{\infty,1}$, established in the first section. We have not restricted ourselves to proving the complexity results only, but in most cases we also present finitely verifiable necessary and sufficient conditions to demonstrate the exponentiality inherent in all these problems. In several cases we also add verifiable sufficient conditions to give some hints on how to proceed in solving practical examples.

We shall use the following notations. For two matrices A, B of the same size, inequalities like $A \leq B$ or $A < B$ are understood componentwise. A is called nonnegative if $A \geq 0$ and symmetric if $A^T = A$ (A^T is the transpose of A). The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$; properties like $|A + B| \leq |A| + |B|$ or $|AB| \leq |A||B|$ are easy to prove. The same notations also apply to vectors that are always considered one–column matrices. In particular, for $a = (a_i)$ and $b = (b_i)$, $a^T b = \sum_i a_i b_i$ is the scalar product whereas ab^T is the matrix $(a_i b_j)$. $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalue of a symmetric matrix A, respectively. As is well known, $\lambda_{\min}(A) = \min_{\|x\|_2=1} x^T A x$ and $\lambda_{\max}(A) = \max_{\|x\|_2=1} x^T A x$ hold. $\sigma_{\min}(A)$, $\sigma_{\max}(A)$ denote the minimal and maximal singular value of A, and $\varrho(A)$ is the spectral radius of A. I denotes the unit matrix, e_j is the jth column of I and $e = (1, \ldots, 1)^T$ is the vector of all ones. Z denotes the set of all ± 1 vectors, i.e., $Z = \{z \in R^n;\ |z| = e\}$.

1 The norm $\|A\|_{\infty,1}$

In this section we introduce the subordinate matrix norm $\|A\|_{\infty,1}$ and we prove that its computation is NP–hard. For the purposes of various applications to be given later, the result is presented in several different settings (Theorems 3 through 6).

1.1 Subordinate norms

Given two vector norms $\|x\|_\alpha$ in R^n and $\|x\|_\beta$ in R^m, a subordinate matrix norm in $R^{m\times n}$ is defined by

$$\|A\|_{\alpha,\beta} = \max_{\|x\|_\alpha=1} \|Ax\|_\beta$$

(see Golub and van Loan [18] or Higham [20]). $\|A\|_{\alpha,\beta}$ is a matrix norm, i.e., it possesses the three usual properties: 1) $\|A\|_{\alpha,\beta} \geq 0$ and $\|A\|_{\alpha,\beta} = 0$ if and only if $A = 0$, 2) $\|A+B\|_{\alpha,\beta} \leq \|A\|_{\alpha,\beta} + \|B\|_{\alpha,\beta}$, 3) $\|\lambda A\|_{\alpha,\beta} = |\lambda| \cdot \|A\|_{\alpha,\beta}$. However, generally it does not possess the property $\|AB\|_{\alpha,\beta} \leq \|A\|_{\alpha,\beta}\|B\|_{\alpha,\beta}$ (it does e.g. if $\alpha = \beta$).

By combining the three most frequently used norms

$$\|x\|_1 = \sum_i |x_i|,$$

$$\|x\|_2 = \sqrt{x^T x},$$
$$\|x\|_\infty = \max_i |x_i|,$$

we get nine subordinate norms, including the three usual norms

$$\|A\|_1 := \|A\|_{1,1} = \max_j \sum_i |a_{ij}|,$$
$$\|A\|_2 := \|A\|_{2,2} = \sqrt{\lambda_{\max}(A^T A)},$$
$$\|A\|_\infty := \|A\|_{\infty,\infty} = \max_i \sum_j |a_{ij}|.$$

Yet it turns out that one of these nine norms has an exceptional behavior in the sense that it is much more difficult to compute than the other ones: namely, the norm

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty = 1} \|Ax\|_1.$$

This norm can be computed by a finite formula which, however, involves maximization over the set Z of all ± 1–vectors (whose cardinality is 2^n):

Proposition 1 *For each $A \in R^{m\times n}$ we have*

$$\|A\|_{\infty,1} = \max_{z\in Z} \|Az\|_1 \tag{1.1}$$

where

$$Z = \{z \in R^n;\ z_j \in \{-1,1\} \text{ for each } j\}. \tag{1.2}$$

Moreover, if A is symmetric positive semidefinite, then

$$\|A\|_{\infty,1} = \max_{z\in Z} z^T A z. \tag{1.3}$$

Proof. 1) If $\|x\|_\infty = 1$, then x belongs to the unit cube $\{x;\ -e \le x \le e\}$, which is a convex polyhedron, therefore x can be expressed as a convex combination of its vertices which are exactly the points in Z:

$$x = \sum_{z\in Z} \lambda_z z, \tag{1.4}$$

where $\lambda_z \ge 0$ for each $z \in Z$ and $\sum_{z\in Z} \lambda_z = 1$. From (1.4) we have

$$\|Ax\|_1 = \|\sum_{z\in Z} \lambda_z Az\|_1 \le \max_{z\in Z} \|Az\|_1,$$

hence

$$\max_{\|x\|_\infty=1} \|Ax\|_1 \le \max_{z\in Z} \|Az\|_1 \le \max_{\|x\|_\infty=1} \|Ax\|_1$$

(since $\|z\|_\infty = 1$ for each $z \in Z$) and (1.1) follows.

2) Let A be symmetric positive semidefinite and let $z \in Z$. Define $y \in Z$ by $y_j = 1$ if $(Az)_j \ge 0$ and $y_j = -1$ if $(Az)_j < 0$ $(j = 1, \ldots, n)$, then

$$\|Az\|_1 = y^T A z.$$

Since A is symmetric positive semidefinite, we have

$$(y-z)^T A(y-z) \geq 0,$$

which implies

$$2y^T Az \leq y^T Ay + z^T Az \leq 2 \max_{z \in Z} z^T Az,$$

hence

$$\|Az\|_1 = y^T Az \leq \max_{z \in Z} z^T Az$$

and

$$\|A\|_{\infty,1} = \max_{z \in Z} \|Az\|_1 \leq \max_{z \in Z} z^T Az. \tag{1.5}$$

Conversely, for each $z \in Z$ we have

$$z^T Az \leq |z|^T \cdot |Az| = \|Az\|_1 \leq \max_{z \in Z} \|Az\|_1 = \|A\|_{\infty,1},$$

hence

$$\max_{z \in Z} z^T Az \leq \|A\|_{\infty,1},$$

which together with (1.5) gives (1.3). ■

In the next subsection we shall prove that computing $\|A\|_{\infty,1}$ is NP–hard. This will imply that unless P=NP, the formula (1.1) cannot be essentially simplified.

3

1.2 Computing $\|A\|_{\infty,1}$ is NP–hard

In order to prove the NP–hardness for a possibly narrow class of matrices, we introduce the following concept (first formulated in [40]):

Definition A real symmetric $n \times n$ matrix $A = (a_{ij})$ is called an MC–matrix[3] if it is of the form

$$a_{ij} \begin{cases} = n & \text{if } i = j \\ \in \{0, -1\} & \text{if } i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$.

Since an MC–matrix is symmetric by definition, there are altogether $2^{n(n-1)/2}$ MC–matrices of size n. The basic properties of MC–matrices are summed up in the following proposition:

Proposition 2 *An MC–matrix $A \in R^{n \times n}$ is symmetric positive definite, nonnegative invertible and satisfies*

$$\|A\|_{\infty,1} = \max_{z \in Z} z^T Az, \tag{1.6}$$

$$n \leq \|A\|_{\infty,1} \leq n(2n-1) \tag{1.7}$$

and

$$\|A^{-1}\|_1 \leq 1.$$

[3] from "maximum cut" (see the proof of Theorem 3 below)

Proof. A is symmetric by definition; it is positive definite since for $x \neq 0$,

$$x^T Ax \geq n\|x\|_2^2 - \sum_{i \neq j} |x_i x_j| = (n+1)\|x\|_2^2 - \|x\|_1^2 \geq \|x\|_2^2 > 0$$

($\|x\|_1 \leq \sqrt{n}\|x\|_2$ by Cauchy–Schwartz inequality [18]). Hence (1.6) holds by Proposition 1. Since $|a_{ij}| \leq 1$ for $i \neq j$, for each $z \in Z$ and $i \in \{1, \ldots, n\}$ we have

$$z_i(Az)_i = n + \sum_{j \neq i} a_{ij} z_i z_j \in [1,\, 2n-1],$$

hence

$$n \leq z^T Az \leq n(2n-1)$$

for each $z \in Z$, and (1.6) implies (1.7). By definition, A is of the form

$$A = nI - A_0 = n(I - \frac{1}{n}A_0)$$

where $A_0 = nI - A \geq 0$ and $\|\frac{1}{n}A_0\|_1 \leq \frac{n-1}{n} < 1$, hence

$$A^{-1} = \frac{1}{n}\sum_0^\infty (\frac{1}{n}A_0)^j \geq 0$$

and

$$\|A^{-1}\|_1 \leq \frac{1}{n - \|A_0\|_1} \leq 1.$$

■

The following basic result is due to Poljak and Rohn [33] (given there in a slightly different formulation without using the concept of an MC-matrix).

Theorem 3 *The following decision problem is NP-complete:*
Instance. *An MC-matrix A and a positive integer ℓ.*
Question. *Is $z^T Az \geq \ell$ for some $z \in Z$?*

Proof. Let (N, E) be a graph with $N = \{1, \ldots, n\}$. Let $A = (a_{ij})$ be given by $a_{ij} = n$ if $i = j$, $a_{ij} = -1$ if $i \neq j$ and the nodes i, j are connected by an edge, and $a_{ij} = 0$ if $i \neq j$ and i, j are not connected. Then A is an MC–matrix. For $S \subseteq N$, define the cut $c(S)$ as the number of edges in E whose one endpoint belongs to S and the other one to $N \setminus S$. We shall prove that

$$\|A\|_{\infty,1} = 4 \max_{S \subseteq N} c(S) - 2\mathrm{Card}(E) + n^2 \tag{1.8}$$

holds. Given a $S \subseteq N$, define a $z \in Z$ by

$$z_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S. \end{cases}$$

Then we have

$$\begin{aligned} z^T A z &= \sum_{i,j} a_{ij} z_i z_j = \sum_{i \neq j} a_{ij} z_i z_j + n^2 \\ &= \sum_{i \neq j} [-\frac{1}{2} a_{ij} (z_i - z_j)^2 + a_{ij}] + n^2 \\ &= -\frac{1}{2} \sum_{z_i z_j = -1} a_{ij} (z_i - z_j)^2 + \sum_{i \neq j} a_{ij} + n^2 \\ &= -\frac{1}{2} \sum_{z_i z_j = -1} 4 a_{ij} + \sum_{i \neq j} a_{ij} + n^2, \end{aligned}$$

hence

$$z^T A z = 4c(S) - 2\text{Card}(E) + n^2. \tag{1.9}$$

Conversely, given a $z \in Z$, then for $S = \{i \in N;\ z_i = 1\}$ the same reasoning implies (1.9). Taking maximum on both sides of (1.9), we obtain (1.8) in view of (1.6).

Hence, given a positive integer L, we have

$$c(S) \geq L \tag{1.10}$$

for some $S \subseteq N$ if and only if

$$z^T A z \geq 4L - 2\text{Card}(E) + n^2$$

for some $z \in Z$. Since the decision problem (1.10) is NP–complete ("simple max–cut problem", Garey, Johnson and Stockmeyer [17]), we obtain that the decision problem

$$z^T A z \geq \ell \tag{1.11}$$

(ℓ positive integer) is NP–hard. It is NP–complete since for a guessed solution $z \in Z$ the validity of (1.11) can be checked in polynomial time. ∎

In this way, we have also proved the following result:

Theorem 4 *Computing $\|A\|_{\infty,1}$ is NP–hard for MC–matrices.*

In a sharp contrast with this result, the norm $\|A\|_{1,\infty}$ (with indices swapped) can be computed in polynomial time:

$$\|A\|_{1,\infty} = \max_{i,j} |a_{ij}| \tag{1.12}$$

(Higham [20]).

To facilitate formulations of some subsequent results, it is advantageous to remove the integer parameter ℓ from the formulation of Theorem 3. This can be done by using M–matrices instead of MC–matrices. Let us recall that $A = (a_{ij})$ is called an M–matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$ (a number of equivalent formulations may be found in Berman and Plemmons [5]); hence each MC–matrix is an M–matrix (Proposition 2). Since a symmetric M–matrix is positive definite [5], this property is not explicitly mentioned in the following theorem:

Theorem 5 *The following decision problem is NP-hard:*
Instance. *An $n \times n$ symmetric rational M-matrix A with $\|A\|_1 \le 2n-1$.*
Question. *Is $\|A\|_{\infty,1} \ge 1$?*

Proof. Given an MC–matrix A and a positive integer ℓ, the assertion

$$z^T Az \ge \ell \text{ for some } z \in Z$$

is equivalent to $\|A\|_{\infty,1} \ge \ell$ and thereby also to

$$\left\|\frac{1}{\ell}A\right\|_{\infty,1} \ge 1,$$

where $\frac{1}{\ell}A$ is a symmetric rational M–matrix with $\|\frac{1}{\ell}A\|_1 \le \|A\|_1 \le 2n-1$. Hence the decision problem of Theorem 3 can be reduced in polynomial time to the current one, which is then NP–hard. ■

Finally we shall show that even computing a sufficiently close approximation of $\|A\|_{\infty,1}$ is NP–hard:

Theorem 6 *Suppose there exists a polynomial–time algorithm which for each MC–matrix A computes a rational number $\nu(A)$ satisfying*

$$|\nu(A) - \|A\|_{\infty,1}| < \frac{1}{2}.$$

Then P=NP.

Proof. If such an algorithm exists, then $\|A\|_{\infty,1} < \nu(A) + \frac{1}{2} < \|A\|_{\infty,1} + 1$, hence

$$\|A\|_{\infty,1} = \left\lfloor \nu(A) + \frac{1}{2} \right\rfloor$$

(since $\|A\|_{\infty,1}$ is integer for an MC–matrix A, see (1.6)), hence the NP–hard problem of Theorem 4 can be solved in polynomial time, 3implying P=NP. ■

2 Regularity

In the rest of this chapter we shall investigate complexity of checking various properties of square interval matrices. An interval matrix A^I is a set of matrices of the form

$$A^I = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \le A \le \overline{A}\}, \tag{2.1}$$

where the inequalities are understood componentwise and $\underline{A} \le \overline{A}$. Introducing the center matrix

$$A_c = \frac{1}{2}(\underline{A} + \overline{A})$$

and the nonnegative radius matrix

$$\Delta = \frac{1}{2}(\overline{A} - \underline{A}),$$

we can also write the interval matrix (2.1) in the form

$$A^I = [A_c - \Delta, A_c + \Delta] \tag{2.2}$$

which in many contexts turns out to be more useful than (2.1)[4].

A square interval matrix A^I is said to be regular if each $A \in A^I$ is nonsingular, and it is called singular otherwise (i.e., if it contains a singular matrix). Regularity of interval matrices plays an important role in theory of linear interval equations (Neumaier [30]), but it is also useful in some other respects since checking several properties of interval matrices (studied in the subsequent sections) can be reduced to checking regularity.

This section is devoted to the problem of checking regularity of interval matrices. We prove that the problem is NP–hard (Theorem 9) and describe some necessary and/or sufficient regularity conditions (subsection 2.2). In the last subsection it is proved that computing (even approximately) the radius of nonsingularity is NP–hard.

2.1 Checking regularity is NP–hard

Let us introduce the matrix of all ones

$$E = ee^T = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

The basic relationship of the current problem to the contents of the previous section is provided by the following equivalence:

Proposition 7 *For a symmetric positive definite matrix A, the following assertions are equivalent:*

(i) $\|A\|_{\infty,1} \geq 1$,

(ii) the interval matrix

$$[A^{-1} - E, A^{-1} + E] \tag{2.3}$$

is singular,

(iii) the interval matrix (2.3) contains a symmetric singular matrix A' of the form

$$A' = A^{-1} - \frac{zz^T}{z^T A z} \tag{2.4}$$

for some $z \in Z$.

[4]the "I" in A^I is an abbreviation of the word "interval" and has nothing to do with the unit matrix I

Proof. (i)⇒(iii): Due to Proposition 1, if (i) holds, then

$$\|A\|_{\infty,1} = \max_{z\in Z} z^T Az \geq 1,$$

hence $z^T Az \geq 1$ for some $z \in Z$. Since

$$\left|\frac{zz^T}{z^T Az}\right| \leq E,$$

the matrix A' defined by (2.4) belongs to $[A^{-1} - E, A^{-1} + E]$ and satisfies

$$A'Az = z - \frac{z(z^T Az)}{z^T Az} = 0,$$

where $Az \neq 0$ (A is nonsingular since it is positive definite), hence A' is singular, and obviously also symmetric.

(iii)⇒(ii) is obvious.

(ii)⇒(i): Let $A''x = 0$ for some $A'' \in [A^{-1} - E, A^{-1} + E]$ and $x \neq 0$. Define $z \in Z$ by $z_j = 1$ if $x_j \geq 0$ and $z_j = -1$ otherwise ($j = 1, \ldots, n$). Then we have

$$e^T|x| = z^T x = z^T A(A^{-1} - A'')x \leq |z^T A(A^{-1} - A'')x| \leq |z^T A|ee^T|x|,$$

hence

$$1 \leq |z^T A|e = \|Az\|_1 \leq \|A\|_{\infty,1},$$

which is (i). ■

The next result was published by Poljak and Rohn in a report form [32] in 1988 and in a journal form [33] in 1993:

Theorem 8 *The following problem is NP–complete:*
Instance. *A nonnegative symmetric positive definite rational matrix A.*
Question. *Is $[A - E, A + E]$ singular?*

Proof. For a symmetric rational M–matrix A (which is positive definite [5]),

$$\|A\|_{\infty,1} \geq 1 \tag{2.5}$$

is according to Proposition 7 equivalent to singularity of

$$[A^{-1} - E, A^{-1} + E],$$

where A^{-1} is rational, nonnegative and symmetric positive definite. Since computing A^{-1} can be done by Gaussian elimination in polynomial time (Edmonds [14]), we have a polynomial–time reduction of the NP–hard problem (2.5) (Theorem 5) to the current problem, which is thus also NP–hard.

If $[A - E, A + E]$ is singular, then it contains a rational singular matrix of the form

$$A - \frac{zz^T}{z^T A^{-1} z}$$

for some $z \in Z$ (Proposition 7, (ii)$\Leftrightarrow$(iii)) which can be guessed (generated by a nondeterministic polynomial–time algorithm) and then checked for singularity by Gaussian elimination in polynomial time [14]. Thus the problem is in the class NP, hence it it NP–complete. ∎

The result immediately implies NP–hardness of checking regularity:

Theorem 9 *The following problem is NP–hard:*
Instance. *A nonnegative symmetric positive definite rational matrix A.*
Question. *Is $[A-E, A+E]$ regular?*

This result was proved independently, also in 1993, by Nemirovskii [28] who employed a different approach based on another subclass of interval matrices.

As a by–product of the equivalence (ii)$\Leftrightarrow$(iii) of Proposition 7 we obtain that the problem of checking regularity of all *symmetric* matrices contained in $[A-E, A+E]$ is also NP–hard.

2.2 Necessary and/or sufficient conditions

In view of the NP–hardness result of Theorem 9, no easily verifiable necessary and sufficient regularity conditions may be expected. Indeed, 13 such conditions are proved in Theorem 5.1 in [37], all of which exhibit exponential behaviour. Probably the most easily implementable criterion is that one by Baumann [3] (Theorem 11 below) which employs matrices A_{yz}, $y, z \in R^n$, defined for an $n \times n$ interval matrix $A^I = [A_c - \Delta, A_c + \Delta] = [\underline{A}, \overline{A}]$ by

$$(A_{yz})_{ij} = (A_c)_{ij} - \Delta_{ij} y_i z_j \tag{2.6}$$

$(i, j = 1, \ldots, n)$. If $y, z \in Z$, then we have

$$(A_{yz})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases} \tag{2.7}$$

for each i, j, hence $A_{yz} \in A^I$ in this case. We shall first formulate an auxiliary result which will form a basis for proofs of the other results in this subsection. It is a consequence of the Oettli–Prager theorem [31].

Proposition 10 *An interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is singular if and only if the inequality*

$$|A_c x| \le \Delta |x| \tag{2.8}$$

has a nontrivial solution.

Proof. If A^I contains a singular matrix A, then $Ax = 0$ for some $x \neq 0$, which implies

$$|A_c x| = |(A_c - A)x| \le \Delta |x|.$$

Conversely, let (2.8) hold for some $x \neq 0$. Define $y \in R^n$ and $z \in Z$ by

$$y_i = \begin{cases} (A_c x)_i/(\Delta|x|)_i & \text{if } (\Delta|x|)_i > 0, \\ 1 & \text{if } (\Delta|x|)_i = 0 \end{cases}$$

and

$$z_j = \begin{cases} 1 & \text{if } x_j \geq 0, \\ -1 & \text{if } x_j < 0 \end{cases}$$

$(i, j = 1, \ldots, n)$. Then for the matrix A_{yz} given by (2.6) we have

$$(A_{yz}x)_i = (A_c x)_i - y_i(\Delta|x|)_i = 0$$

for each i, hence A_{yz} is singular, and since $|y_i| \leq 1$ for each i due to (2.8), from (2.6) it follows that $A_{yz} \in A^I$, hence A^I is singular. ∎

Baumann's criterion employs a finite set of test matrices A_{yz} for $y, z \in Z$ (of cardinality at most 2^{2n-1} since $A_{-y,-z} = A_{yz}$).

Theorem 11 *An interval matrix A^I is regular if and only if determinants of all the matrices A_{yz}, $y, z \in Z$ are nonzero and of the same sign.*

Proof. Let A^I be regular and assume that

$$(\det A_{yz})(\det A_{y'z'}) \leq 0$$

holds for some $y, z, y', z' \in Z$. Define a real function φ of one real variable by

$$\varphi(t) = \det(A_{yz} + t(A_{y'z'} - A_{yz})), \quad t \in [0, 1].$$

Then $\varphi(0)\varphi(1) \leq 0$, hence there exists a $\tau \in [0, 1]$ with $\varphi(\tau) = 0$. Thus the matrix $A_{yz} + \tau(A_{y'z'} - A_{yz})$ is singular and belongs to A^I (due to its convexity), which is a contradiction. Hence

$$(\det A_{yz})(\det A_{y'z'}) > 0$$

holds for each $y, z, y', z' \in Z$.

Conversely, let A^I be singular. From the proof of Proposition 10 we know that there exists a singular matrix of the form A_{yz} for some $|y| \leq e, z \in Z$. Let us introduce the function

$$f(s) = \det A_{sz}$$

for $s \in R^n$, and define a vector $\overline{y} = (\overline{y}_j) \in Z$ componentwise by induction on $j = 1, \ldots, n$ as follows: if the function of one real variable

$$f(\overline{y}_1, \ldots, \overline{y}_{j-1}, t, y_{j+1}, \ldots, y_n) \tag{2.9}$$

is increasing in t, set $\overline{y}_j := 1$, otherwise set $\overline{y}_j := -1$. Since the function (2.9) is linear in t due to (2.6), we have

$$f(\overline{y}_1, \ldots, \overline{y}_{j-1}, y_j, y_{j+1}, \ldots, y_n) \leq f(\overline{y}_1, \ldots, \overline{y}_{j-1}, \overline{y}_j, y_{j+1}, \ldots, y_n)$$

for each j, and by induction

$$0 = \det A_{yz} = f(y_1, \ldots, y_n) \le f(\overline{y}_1, \ldots, \overline{y}_n) = \det A_{\overline{y}z},$$

hence $0 \le \det A_{\overline{y}z}$, $\overline{y}, z \in Z$. In an analogous way we may construct a $\underline{y} \in Z$ satisfying $\det A_{\underline{y}z} \le 0$. Hence

$$(\det A_{\underline{y}z})(\det A_{\overline{y}z}) \le 0$$

for some $\underline{y}, \overline{y}, z \in Z$, which concludes the proof of the second implication. ■

In view of the exponentiality inherent in the necessary and sufficient conditions, in practical computations we must resort to verifiable sufficient conditions. We survey the most useful ones in the next theorem:

Theorem 12 *Each of the two conditions implies regularity of* $[A_c - \Delta, A_c + \Delta]$*:*

(i) $\varrho(|A_c^{-1}|\Delta) < 1$,

(ii) $\sigma_{\max}(\Delta) < \sigma_{\min}(A_c)$.

Furthermore, each of the following two conditions implies singularity of $[A_c - \Delta, A_c + \Delta]$*:*

(iii) $\max_j(\Delta|A_c^{-1}|)_{jj} \ge 1$,

(iv) $(\Delta - |A_c|)^{-1} \ge 0$.

Proof. (i) Assume to the contrary that A^I is singular, then

$$|A_c x| \le \Delta|x| \tag{2.10}$$

for some $x \ne 0$ (Proposition 10), hence

$$|x'| \le \Delta|A_c^{-1}x'| \le \Delta|A_c^{-1}||x'|$$

holds for $x' = A_c x \ne 0$, which implies

$$1 \le \varrho(\Delta|A_c^{-1}|) = \varrho(|A_c^{-1}|\Delta)$$

(Neumaier [30]), a contradiction.

(ii) Again assuming to the contrary that A^I is singular, we have that (2.10) holds for some $x \ne 0$ which may be normalized so that $\|x\|_2 = 1$, hence also

$$|A_c x|^T |A_c x| \le (\Delta|x|)^T(\Delta|x|),$$

which implies

$$\begin{aligned}
\sigma_{\min}^2(A_c) &= \lambda_{\min}(A_c^T A_c) = \min_{\|x\|_2=1} x^T A_c^T A_c x \le (A_c x)^T(A_c x) \\
&\le |A_c x|^T |A_c x| \le (\Delta|x|)^T(\Delta|x|) = |x|^T \Delta^T \Delta |x| \\
&\le \max_{\|x\|_2=1} x^T \Delta^T \Delta x = \lambda_{\max}(\Delta^T \Delta) = \sigma_{\max}^2(\Delta),
\end{aligned}$$

hence

$$\sigma_{\min}(A_c) \le \sigma_{\max}(\Delta),$$

which is a contradiction.

(iii) Let $(\Delta|A_c^{-1}|)_{jj} \ge 1$ for some j and let e_j denote the jth column of the unit matrix I. Then

$$e_j \le \Delta|A_c^{-1}|e_j = \Delta|A_c^{-1}e_j|$$

holds, hence for $x = A_c^{-1}e_j \ne 0$ we have

$$|A_c x| \le \Delta|x|$$

and A^I is singular due to Proposition 10.

(iv) Let $(\Delta - |A_c|)^{-1} \ge 0$. Then for $x = (\Delta - |A_c|)^{-1}e$ we have $x > 0$ and $(\Delta - |A_c|)x = e > 0$, hence

$$|A_c x| \le |A_c|x < \Delta x = \Delta|x|$$

and Proposition 10 implies singularity of A^I. ∎

The condition (i), which is most frequently used, is due to Beeck [4]; an interval matrix satisfying (i) is called strongly regular (Neumaier [30]). The second condition is due to Rump [44]. The condition (iii) is proved in [37], and (iv) comes from [42].

2.3 Radius of nonsingularity

Given an $n \times n$ matrix A and a nonnegative "directional" $n \times n$ matrix Δ, the radius of nonsingularity is defined by

$$d(A, \Delta) = \inf\{\varepsilon \ge 0;\ [A - \varepsilon\Delta, A + \varepsilon\Delta] \text{ is singular}\} \tag{2.11}$$

(i.e., $d(A, \Delta) = \infty$ if no such ε exists; if $d(A, \Delta) < \infty$, then the infimum is achieved as minimum). This notion was seemingly first formulated by Neumaier [29] and was since studied by Poljak and Rohn [32], [33], Demmel [12], Rohn [38] and Rump [46], [45] (Demmel and Rump use the term "componentwise distance to the nearest singular matrix"). A general formula for $d(A, \Delta)$ was given in [33]:

$$d(A, \Delta) = \frac{1}{\max\{\varrho_0(A^{-1}T_1\Delta T_2);\ |T_1| = |T_2| = I\}}, \tag{2.12}$$

with convention $\frac{1}{0} = \infty$. Here ϱ_0 denotes the real spectral radius defined by $\varrho_0(A) = \max\{|\lambda|;\ \lambda \text{ is a real eigenvalue of } A\}$ and $\varrho_0(A) = 0$ if no real eigenvalue exists. A matrix T satisfying $|T| = I$ is obviously a diagonal matrix with ± 1 entries on the diagonal. There are 2^n such matrices, hence the formula (2.12) is finite.

Consider the special case of $\Delta = E$ and denote

$$d(A) := d(A, E).$$

$d(A)$ is always finite and $d(A) = 0$ if and only if A is singular. We have this result [33]:

Proposition 13 *For each nonsingular A there holds*

$$d(A) = \frac{1}{\|A^{-1}\|_{\infty,1}}. \tag{2.13}$$

Proof. Since $\|A\|_{1,\infty} = \max_{ij} |a_{ij}|$ (see (1.12)), Kahan's theorem [22] gives

$$\begin{aligned} d(A) &= \min\{\varepsilon \geq 0;\ [A - \varepsilon E, A + \varepsilon E] \text{ is singular}\} \\ &= \min\{\|A - A'\|_{1,\infty};\ A' \text{ is singular}\} \\ &= \frac{1}{\|A^{-1}\|_{\infty,1}}. \end{aligned}$$

■

The formula (2.13) implies this complexity result:

Proposition 14 *The following problem is NP–hard:*
Instance. *A nonnegative symmetric positive definite rational matrix A.*
Question. *Is $d(A) \leq 1$?*

Proof. For a symmetric M–matrix A,

$$\|A\|_{\infty,1} \geq 1$$

is according to (2.13) equivalent to

$$d(A^{-1}) \leq 1,$$

where A^{-1} is rational, nonnegative symmetric positive definite, hence the NP–hard problem of Theorem 5 can be reduced in polynomial time to the current one, which is thus NP–hard as well. ■

As an immediate consequence we obtain [33]:

Theorem 15 *Computing the radius of nonsingularity is NP–hard (even in the special case $\Delta = E$).*

In the next theorem we prove that even computing a sufficiently close approximation of the radius of nonsingularity is NP–hard.

Theorem 16 *Suppose there exists a polynomial–time algorithm which for each nonnegative symmetric positive definite rational matrix A computes a rational approximation $d'(A)$ of $d(A)$ satisfying*

$$\left| \frac{d'(A) - d(A)}{d(A)} \right| \leq \frac{1}{4n^2},$$

where n is the size of A. Then P=NP.

Proof. Let A be an $n \times n$ MC–matrix, then A^{-1} is rational nonnegative symmetric positive definite, hence we have

$$\left|\frac{d'(A^{-1}) - d(A^{-1})}{d(A^{-1})}\right| \le \frac{1}{4n^2}.$$

Since $\|A\|_{\infty,1} \le n(2n-1)$ by Proposition 2, there holds $2\|A\|_{\infty,1}+1 \le 4n^2-2n+1 < 4n^2$, hence

$$\left|\frac{d'(A^{-1})}{d(A^{-1})} - 1\right| \le \frac{1}{4n^2} < \frac{1}{2\|A\|_{\infty,1}+1} < \frac{1}{2\|A\|_{\infty,1}-1},$$

which implies

$$\frac{2\|A\|_{\infty,1}}{2\|A\|_{\infty,1}+1} = 1 - \frac{1}{2\|A\|_{\infty,1}+1} < \frac{d'(A^{-1})}{d(A^{-1})} < 1 + \frac{1}{2\|A\|_{\infty,1}-1} = \frac{2\|A\|_{\infty,1}}{2\|A\|_{\infty,1}-1}$$

and by (2.13),

$$\frac{2}{2\|A\|_{\infty,1}+1} < d'(A^{-1}) < \frac{2}{2\|A\|_{\infty,1}-1}$$

and

$$\left|\frac{1}{d'(A^{-1})} - \|A\|_{\infty,1}\right| < \frac{1}{2}.$$

Hence we have a polynomial–time algorithm for computing $\|A\|_{\infty,1}$ with accuracy better than $\frac{1}{2}$, which according to Theorem 6 implies that P=NP. ■

Bounds on the radius of nonsingularity can be derived from sufficient regularity or singularity conditions. E.g., from Theorem 12 we have

$$\frac{1}{\varrho(|A^{-1}|\Delta)} \le d(A,\Delta) \le \frac{1}{\max_j(\Delta|A^{-1}|)_{jj}}.$$

Using a sophisticated reasoning, Rump [46], [45] recently proved a "symmetric" estimation

$$\frac{1}{\varrho(|A^{-1}|\Delta)} \le d(A,\Delta) \le \frac{6n}{\varrho(|A^{-1}|\Delta)}.$$

R3elated to the radius of nonsingularity is the structured singular value introduced by Doyle [13]. The NP–hardness of its computation was proved by Braatz, Young, Doyle and Morari [8] and independently by Coxson and DeMarco [10].

3 Positive definiteness

A square matrix A (not necessarily symmetric) is called positive definite if $x^T Ax > 0$ for each $x \neq 0$. Since for the symmetric matrix

$$A_s = \frac{1}{2}(A + A^T)$$

there holds $x^T Ax = x^T A_s x$ for each x, we have that A is positive definite if and only if A_s is positive definite, and positive definiteness of a symmetric matrix A_s may be checked by Sylvester determinant criterion [27] using Gaussian elimination, hence it can be done in polynomial time [14].

An interval matrix A^I is said to be positive definite if each $A \in A^I$ is positive definite. In this section we show that due to a close relationship between positive definiteness and regularity (Theorem 17), the results of the previous section may be applied to prove that checking positive definiteness is NP–hard even for symmetric interval matrices (Theorem 20). In the last subsection we again give some necessary and/or sufficient conditions for positive definiteness of interval matrices.

3.1 Positive definiteness and regularity

For a square interval matrix

$$A^I = [\underline{A}, \overline{A}], \tag{3.1}$$

define

$$A^I_s = [\frac{1}{2}(\underline{A} + \underline{A}^T), \frac{1}{2}(\overline{A} + \overline{A}^T)]. \tag{3.2}$$

Hence, $A \in A^I$ implies $\frac{1}{2}(A + A^T) \in A^I_s$, and $(A^I_s)_s = A^I_s$. An interval matrix (3.1) is called *symmetric* if $A^I = A^I_s$. It can be easily seen that (3.1) is symmetric if and only if the bounds $\underline{A}$ and $\overline{A}$ are symmetric. Similarly, an interval matrix in the form $[A_c - \Delta, A_c + \Delta]$ is symmetric if and only if both A_c and Δ are symmetric. Hence, a symmetric interval matrix may contain nonsymmetric matrices (indeed, it is the case if and only if $\underline{A}_{ij} < \overline{A}_{ij}$ for some $i \neq j$).

In the next theorem we show that positive definiteness of interval matrices is closely related to regularity [39]:

Theorem 17 *An interval matrix A^I is positive definite if and only if A^I_s is regular and contains at least one positive definite matrix.*

Proof. Let $A^I = [A_c - \Delta, A_c + \Delta]$, so that $A^I_s = [A'_c - \Delta', A'_c + \Delta']$, where

$$A'_c = \frac{1}{2}(A_c + A_c^T)$$

and

$$\Delta' = \frac{1}{2}(\Delta + \Delta^T).$$

We shall first prove that if A^I is positive definite, then A^I_s is also positive definite. Assume to the contrary that A^I_s is not positive definite, so that $x^T A' x \leq 0$ for some $A' \in A^I_s$ and $x \neq 0$. Since $|x^T(A' - A'_c)x| \leq |x|^T \Delta' |x|$, we have

$$x^T A_c x - |x|^T \Delta |x| = x^T A'_c x - |x|^T \Delta' |x| \leq x^T A'_c x + x^T (A' - A'_c)x = x^T A' x \leq 0. \tag{3.3}$$

Define a diagonal matrix T by $T_{jj} = 1$ if $x_j \geq 0$ and $T_{jj} = -1$ otherwise. Then $|x| = Tx$, and from (3.3) we have

$$x^T (A_c - T\Delta T)x \leq 0,$$

where $|T\Delta T| = \Delta$, hence the matrix $A_c - T\Delta T$ belongs to A^I and is not positive definite. This contradiction shows that positive definiteness of A^I implies positive definiteness of A^I_s, and thereby also regularity of A^I_s.

Conversely, let A^I_s be regular and contain a positive definite matrix A_0. Assume to the contrary that some $A_1 \in A^I$ is not positive definite. Let $\tilde{A}_0 = \frac{1}{2}(A_0 + A_0^T)$, $\tilde{A}_1 = \frac{1}{2}(A_1 + A_1^T)$, hence both $\tilde{A}_0$ and $\tilde{A}_1$ are symmetric and belong to A^I_s, $\tilde{A}_0$ is positive definite whereas $\tilde{A}_1$ is not. Put

$$\tau = \sup\{t \in [0,1];\ \tilde{A}_0 + t(\tilde{A}_1 - \tilde{A}_0) \text{ is positive definite}\}.$$

Then $\tau \in (0,1]$, hence the matrix

$$A^* = \tilde{A}_0 + \tau(\tilde{A}_1 - \tilde{A}_0)$$

belongs to A^I_s (due to its convexity) and is symmetric positive semidefinite, but not positive definite, hence $\lambda_{\min}(A^*) = 0$, which shows that A^* is singular contrary to the assumed regularity of A^I_s. Hence A^I is positive definite, which completes the proof. ■

In the introduction of this section we mentioned that a real matrix A is positive definite if and only if A_s is positive definite. Theorem 17 now implies that the same relationship holds for interval matrices:

Proposition 18 *A^I is positive definite if and only if A^I_s is positive definite.*

Proof. According to Theorem 17, A^I is positive definite if and only if A^I_s is regular and contains a positive definite matrix. If we apply the same theorem to A^I_s instead of A^I, in view of the obvious fact that $(A^I_s)_s = A^I_s$ we obtain that A^I_s is positive definite if and only if A^I_s is regular and contains a positive definite matrix. These two equivalences show that A^I is positive definite if and only if A^I_s is positive definite. ■

In the next subsection we shall employ the relationship between positive definiteness and regularity established in Theorem 17 to prove NP–hardness of checking positive definiteness.

3.2 Checking positive definiteness is NP–hard

Taking again into consideration the class of interval matrices of the form $[A-E, A+E]$, we arrive at this property:

Proposition 19 *Let A be a symmetric positive definite matrix. Then the interval matrix $[A-E, A+E]$ is positive definite if and only if it is regular.*

Proof. Under the assumption, the interval matrix $A^I = [A-E, A+E]$ satisfies $A^I_s = A^I$ and contains a symmetric positive definite matrix A. Hence according to Theorem 17, A^I is positive definite if and only if it is regular. ■

As a direct consequence we prove NP–hardness of checking positive definiteness [40]:

Theorem 20 *The following problem is NP–hard:*

Instance. *A nonnegative symmetric positive definite rational matrix A.*

Question. *Is $[A-E, A+E]$ positive definite?*

Proof. In view of Proposition 19, such an interval matrix is positive definite if and only if it is regular. Checking regularity was proved to be NP–hard for this class of interval matrices in Theorem 9. H3ence the same is true for checking positive definiteness. ∎

An interval matrix A^I is said to be positive semidefinite if each $A \in A^I$ is positive semidefinite. NP–hardness of checking positive semidefiniteness was proved by Nemirovskii [28] by another means.

3.3 Necessary and/or sufficient conditions

A finite characterization of positive definiteness of interval matrices was seemingly first given by Shi and Gao [47] who proved that a symmetric $A^I = [\underline{A}, \overline{A}]$ is positive definite if and only if each symmetric matrix $A \in A^I$ of the form $A_{ii} = \underline{A}_{ii}$, $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ for $i \neq j$, is positive definite. There are $2^{n(n-1)/2}$ such matrices. In [39] it was shown that the number of test matrices may be reduced down to 2^{n-1} if we employ instead the set of matrices A_{zz} defined for $z \in Z$ by

$$(A_{zz})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } z_i z_j = -1, \\ \underline{A}_{ij} & \text{if } z_i z_j = 1 \end{cases} \tag{3.4}$$

$(i, j = 1, \ldots, n)$. These are exactly the matrices A_{yz} (see (2.7)) used in the Baumann regularity criterion (Theorem 11), with $y = z$. Each A_{zz} is symmetric if A^I is symmetric.

Theorem 21 *A^I is positive definite if and only if each A_{zz}, $z \in Z$ is positive definite.*

Proof. The "only if" part is obvious since $A_{zz} \in A^I$ for each $z \in Z$. The "if" part was proved in the first part of the proof of Theorem 17 (a matrix $A_c - T\Delta T$ is of the form A_{zz} where z is the diagonal vector of T). ∎

In practical computations we may use the following sufficient condition [39] (where $\lambda_{\min}$ denotes the minimal eigenvalue of a symmetric matrix and ϱ is the spectral radius):

Theorem 22 *An interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is positive definite if*

$$\varrho(\Delta') < \lambda_{\min}(A_c')$$

holds, where $A_c' = \frac{1}{2}(A_c + A_c^T)$ and $\Delta' = \frac{1}{2}(\Delta + \Delta^T)$.

Proof. For each $A \in A^I$ and x with $\|x\|_2 = 1$ we have

$$\begin{aligned} x^T A x &= x^T A_c x + x^T (A - A_c) x \geq x^T A_c x - |x|^T \Delta |x| = x^T A_c' x - |x|^T \Delta' |x| \\ &\geq \lambda_{\min}(A_c') - \lambda_{\max}(\Delta') = \lambda_{\min}(A_c') - \varrho(\Delta') > 0, \end{aligned}$$

hence A^I is positive definite. ∎

4 P–property

An $n \times n$ matrix A is said to be a P–matrix (or, to have the P–property) if all its principal minors are positive; principal minors are determinants of square submatrices formed from rows and columns with the same indices (there are $2^n - 1$ of them). This definition is due to Fiedler and Pták who also proved the following characterization [15]: A is a P–matrix if and only if for each $x \neq 0$ there exists an $i \in \{1, \ldots, n\}$ such that $x_i(Ax)_i > 0$. P–matrices play important role in several areas, e.g. in the linear complementarity theory since they guarantee existence and uniqueness of the solution of a linear complementarity problem (see Murty [27]).

A symmetric matrix A is a P–matrix if and only if it is positive definite (Wilkinson [51]), hence it can be checked in polynomial time. However, the problem of checking nonsymmetric matrices for P–property is NP–hard, as it was proved by Coxson [9] (the proof of his result is added as an appendix in section 9).

An interval matrix A^I is called a P–matrix if each $A \in A^I$ is a P–matrix. In this section we show that due to a close relationship between P–property and positive definiteness (Proposition 24), the problem of checking P–property of interval matrices is NP–hard even in the symmetric case (Theorem 25).

4.1 Necessary and sufficient condition

First we give a characterization similar to that of Theorem 21. We shall again employ the matrices A_{zz}, $z \in Z$ defined in (3.4). The following theorem is due to Białas and Garloff [7], reformulation using matrices A_{zz} comes from Rohn and Rex [43].

Theorem 23 *A^I is a P–matrix if and only if each A_{zz}, $z \in Z$ is a P–matrix.*

Proof. If A^I is a P–matrix, then each A_{zz} is a P–matrix since $A_{zz} \in A^I$, $z \in Z$. Conversely, let each A_{zz}, $z \in Z$ be a P–matrix. Take $A \in A^I$, $x \neq 0$, and let $z \in Z$ be defined by $z_j = 1$ if $x_j \geq 0$ and $z_j = -1$ otherwise ($j = 1, \ldots, n$). Since A_{zz} is a P–matrix, according to the Fiedler–Pták theorem there exists an $i \in \{1, \ldots, n\}$ such that $x_i(A_{zz}x)_i > 0$. Then we have

$$\begin{aligned} x_i(Ax)_i &= \sum_j (A_c)_{ij} x_i x_j + \sum_j (A - A_c)_{ij} x_i x_j \geq \sum_j (A_c)_{ij} x_i x_j - \sum_j \Delta_{ij} |x_i||x_j| \\ &= \sum_j ((A_c)_{ij} - \Delta_{ij} z_i z_j) x_i x_j = x_i (A_{zz} x)_i > 0, \end{aligned}$$

hence A is a P–matrix by the Fiedler–Pták theorem. This proves that A^I is a P–matrix. ∎

4.2 P–property and positive definiteness

As quoted above, a symmetric matrix A is a P–matrix if and only if it is positive definite. The following result [43], although it sounds verbally alike, is not a trivial consequence of the previous statement since here nonsymmetric matrices may be involved.

Proposition 24 *A symmetric interval matrix A^I is a P–matrix if and only if it is positive definite.*

Proof. All the matrices A_{zz}, $z \in Z$ defined by (3.4) are symmetric for a symmetric interval matrix A^I. Hence, A^I is a P–matrix if and only if each A_{zz}, $z \in Z$ is a P–matrix, which is the case if and only if each A_{zz}, $z \in Z$ is positive definite, and this is equivalent to positive definiteness of A^I (Theorem 21). ∎

4.3 Checking P–property is NP–hard

In the introduction to this section we explained that checking a symmetric matrix for P–property can be performed in polynomial time. Unless P≠NP, this is not more true for symmetric interval matrices (Rohn and Rex [43]):

Theorem 25 *The following problem is NP–hard:*
Instance. *A nonnegative symmetric rational P–matrix A.*
Question. *Is $[A - E, A + E]$ a P–matrix?*

Proof. Since A is symmetric positive definite, $[A - E, A + E]$ is a P–matrix if and only if it is positive definite (Proposition 24). Checking positive definiteness of this class of interval matrices was proved to be NP–hard in Theorem 20. ∎

5 Stability

A square matrix A is called stable (sometimes, Hurwitz stable) if $\mathrm{Re}\,\lambda < 0$ for each eigenvalue λ of A. For symmetric matrices, this is equivalent to $\lambda_{\max}(A) < 0$. An interval matrix A^I is called stable if each $A \in A^I$ is stable.

Stability of interval matrices has been extensively studied in control theory due to its close connection to the problem of stability of the solution of a linear time invariant system $\dot{x}(t) = Ax(t)$ under data perturbations. Due to this fact, a number of sufficient stability conditions exist. We shall not make an attempt to survey them here, referring an interested reader to the survey paper by Mansour [26]. We shall focus our attention on the problem of stability of symmetric interval matrices since they admit a finite characterization (Theorem 27) and are a sufficient tool for proving NP–hardness of checking stability (Theorem 29) and Schur stability (Theorem 31).

5.1 Necessary and/or sufficient conditions

The following proposition [39] establishes a link to our previous results.

Proposition 26 *A symmetric interval matrix*

$$A^I = [\underline{A}, \overline{A}]$$

is stable if and only if the symmetric interval matrix

$$-A^I := [-\overline{A}, -\underline{A}]$$

is positive definite.

Proof. First notice that $A \in A^I$ if and only if $-A \in -A^I$. Let A^I be stable, and consider a symmetric matrix $A \in -A^I$. Then $-A \in A^I$ is symmetric and stable, hence $\lambda_{\max}(-A) = -\lambda_{\min}(A) < 0$, so that $\lambda_{\min}(A) > 0$, which means that A is positive definite. Hence each symmetric $A \in -A^I$ is positive definite, which in view of Theorem 21 implies that $-A^I$ is positive definite.

Conversely, let $-A^I$ be positive definite. Then a similar argument shows that each symmetric matrix in A^I is stable, and from Bendixson's theorem (see Stoer and Bulirsch [49]) we have that each eigenvalue λ of each $A \in A^I$ satisfies

$$\mathrm{Re}\,\lambda \leq \lambda_{\max}(\frac{1}{2}(A + A^T)) < 0$$

(since $\frac{1}{2}(A + A^T) \in A^I$), hence A^I is stable. ∎

Consider now the matrices A_{yz} defined by (2.7) with $y = -z$, i.e. the matrices satisfying

$$(A_{-z,z})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } z_i z_j = 1, \\ \underline{A}_{ij} & \text{if } z_i z_j = -1 \end{cases}$$

$(i, j = 1, \ldots, n)$. Each $A_{-z,z}$ is symmetric for a symmetric A^I.

Theorem 27 *A symmetric A^I is stable if and only if each $A_{-z,z}$, $z \in Z$ is stable.*

Proof. A^I is stable if and only if $-A^I$ is positive definite which in view of Theorem 21 is the case if and only if each $-A_{-z,z}$, $z \in Z$ is positive definite, and this is equivalent to stability of all $A_{-z,z}$, $z \in Z$. ∎

Each matrix $A_{-z,z}$, $z \in Z$ is a so–called vertex matrix, i.e., it satisfies $(A_{-z,z})_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ for each i, j. The first attempt to use vertex matrices for characterization of stability was made by Białas [6] who showed that a general interval matrix A^I is stable if and only if all the vertex matrices are stable. His result, however, was shown to be erroneous by Karl, Greschak and Verghese [23] and by Barmish and Hollot [2], see also Barmish, Fu and Saleh [1]. Soh proved later [48] that a symmetric interval matrix is stable if and only if all the $2^{n(n+1)/2}$ symmetric vertex matrices are stable. Theorem 27, where the number of vertex matrices to be tested is reduced to 2^{n-1} (since $A_{-z,z} = A_{z,-z}$), was proved in another form by Hertz [19] and Wang and Michel [50], in the present form in [39]. A branch–and–bound algorithm for checking stability of symmetric interval matrices, based on Theorem 27, was given in [41].

For practical purposes we may use the following sufficient condition valid for the nonsymmetric case [39], [11]:

Theorem 28 *An interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is stable if*

$$\lambda_{\max}(A'_c) + \varrho(\Delta') < 0 \tag{5.1}$$

holds, where $A'_c = \frac{1}{2}(A_c + A_c^T)$ *and* $\Delta' = \frac{1}{2}(\Delta + \Delta^T)$.

Proof. If (5.1) holds, then $\varrho(\Delta') < \lambda_{\min}(-A'_c)$, hence $[-A'_c - \Delta', -A'_c + \Delta']$ is positive definite by Theorem 22 and $[A'_c - \Delta', A'_c + \Delta']$ is stable by Proposition 26. Stability of $[A_c - \Delta, A_c + \Delta]$ then follows by using Bendixson's theorem as in the proof of Proposition 26. ∎

5.2 Checking stability is NP–hard

NP–hardness of checking stability now follows obviously [40]:

Theorem 29 *The following problem is NP–hard:*
Instance. *A nonpositive symmetric stable rational matrix* A.
Question. *Is* $[A - E, A + E]$ *stable?*

Proof. By Proposition 26, $[A - E, A + E]$ is stable if and only if $[-A - E, -A + E]$ is positive definite, where $-A$ is a nonnegative symmetric positive definite rational matrix. Hence the result follows from Theorem 20. ∎

Nemirovskii [28] proved NP–hardness of checking stability for general (nonsymmetric) interval matrices.

5.3 Schur stability

A square matrix A is called Schur stable if $\varrho(A) < 1$ (where ϱ denotes the spectral radius). In order to avoid difficulties caused by complex eigenvalues, we define Schur stability only for symmetric interval matrices in this way: a symmetric A^I is said to be Schur stable if each *symmetric* $A \in A^I$ is Schur stable. Hence, we do not take into account the nonsymmetric matrices contained in A^I. This definition is in accordance with the approach employed in [48] and [19]. Then we have this equivalence:

Proposition 30 *A symmetric interval matrix* $[\underline{A}, \overline{A}]$ *is stable if and only if the symmetric interval matrix*

$$[I + \alpha\underline{A}, I + \alpha\overline{A}]$$

is Schur stable, where

$$\alpha = \frac{2}{\|\underline{A}\|_1 + \|\overline{A} - \underline{A}\|_1 + 2}. \tag{5.2}$$

Proof. Let $[\underline{A}, \overline{A}]$ be stable. Then for each symmetric $A' \in [I + \alpha\underline{A}, I + \alpha\overline{A}]$ we have $A' = I + \alpha A$ for some symmetric $A \in [\underline{A}, \overline{A}]$, hence $\lambda_{\max}(A') = 1 + \alpha\lambda_{\max}(A) < 1$. Furthermore, from

$$|\lambda_{\min}(A)| \le \varrho(A) \le \|A\|_1 \le \|\underline{A}\|_1 + \|\overline{A} - \underline{A}\|_1 < \frac{2}{\alpha}$$

we have

$$\lambda_{\min}(A') = 1 + \alpha\lambda_{\min}(A) > -1,$$

hence A' is Schur stable and thereby $[I + \alpha\underline{A}, I + \alpha\overline{A}]$ is Schur stable.

Conversely, if $[I + \alpha\underline{A}, I + \alpha\overline{A}]$ is Schur stable, then each symmetric $A \in [\underline{A}, \overline{A}]$ is of the form $A = \frac{1}{\alpha}(A' - I)$ for some symmetric $A' \in [I + \alpha\underline{A}, I + \alpha\overline{A}]$, hence $\lambda_{\max}(A) = \frac{1}{\alpha}(\lambda_{\max}(A') - 1) < 0$, and A is stable. Stability of all symmetric matrices in $[\underline{A}, \overline{A}]$ implies stability of $[\underline{A}, \overline{A}]$ due to Theorem 27. ■

5.4 Checking Schur stability is NP–hard

As a consequence of Proposition 30 we obtain this NP–hardness result [40].

Theorem 31 *The following problem is NP–hard:*

Instance. *A symmetric Schur stable rational matrix A with $A \le I$, and a rational number $\alpha \in [0, 1]$.*

Question. *Is $[A - \alpha E, A + \alpha E]$ Schur stable?*

Proof. For a nonpositive symmetric stable rational matrix A, the symmetric interval matrix $[A - E, A + E]$ is stable if and only if $[(I + \alpha A) - \alpha E, (I + \alpha A) + \alpha E]$ is Schur stable, where α is given by (5.2). Here $I + \alpha A$ is a symmetric Schur stable rational matrix with $I + \alpha A \le I$, and $\alpha \in [0, 1]$. Hence we have a polynomial–time reduction of the NP–hard problem of Theorem 29 to the current problem, which shows that it is NP–hard as well. ■

This result differs from those of previous sections where NP–hardness was established for the class of interval matrices of the form $[A - E, A + E]$. This is explained by the fact that regularity, positive definiteness and stability are invariant under multiplication by a positive parameter whereas Schur stability is not.

5.5 Radius of stability

Similarly to the radius of nonsingularity $d(A, \Delta)$ introduced in subsection 2.3, we may define radius of stability by

$$s(A, \Delta) = \inf\{\varepsilon \ge 0;\ [A - \varepsilon\Delta, A + \varepsilon\Delta] \text{ is unstable}\}.$$

Hence, $[A - \varepsilon\Delta, A + \varepsilon\Delta]$ is stable if $0 \le \varepsilon < s(A, \Delta)$ and unstable if $\varepsilon \ge s(A, \Delta)$.

Proposition 32 *Let A be symmetric stable and Δ symmetric nonnegative. Then we have*

$$s(A,\Delta) = d(A,\Delta).$$

Proof. $[A-\varepsilon\Delta, A+\varepsilon\Delta]$ is stable if and only if $[-A-\varepsilon\Delta, -A+\varepsilon\Delta]$ is positive definite (Proposition 26) if and only if $[-A-\varepsilon\Delta, -A+\varepsilon\Delta]$ is regular (Theorem 17) if and only if $[A-\varepsilon\Delta, A+\varepsilon\Delta]$ is regular. Therefore the values of $s(A,\Delta)$ and $d(A,\Delta)$ are equal. ∎

Hence, we may apply the results of subsection 2.3 to the radius of stability. In particular, for a symmetric stable matrix A we have

$$s(A,E) = \frac{1}{\|A^{-1}\|_{\infty,1}}$$

(Proposition 13) and computing $s(A,E)$ is NP–hard (Theorem 15), even approximately (Theorem 16).

6 Eigenvalues

Since regularity, positive definiteness and stability can be formulated in terms of eigenvalues, the results of the previous sections may be applied to obtain some results regarding the eigenvalue problem for interval matrices.

6.1 Checking eigenvalues is NP–hard

Theorem 33 *The following problem is NP–hard:*

Instance. *A nonnegative symmetric positive definite rational matrix A and a rational number λ.*

Question. *Is λ an eigenvalue of some symmetric matrix in $[A-E, A+E]$?*

Proof. $[A-E, A+E]$ is singular if and only if 0 is an eigenvalue of some symmetric matrix in $[A-E, A+E]$ (Proposition 7). Hence the NP–hard problem of Theorem 9 can be reduced in polynomial time to the current problem, which is thereby NP–hard. ∎

It is interesting that rational eigen*vectors* can be checked in polynomial time, see [38].

6.2 Computing the maximal eigenvalue is NP–hard

For an interval matrix A^I define

$$\overline{\lambda}(A^I) = \max\{\operatorname{Re}\lambda;\ \lambda \text{ is an eigenvalue of some } A \in A^I\}.$$

If A^I is symmetric, then an obvious reasoning based on Bendixson's theorem as in section 5 shows that

$$\overline{\lambda}(A^I) = \max\{\lambda_{\max}(A);\ A \text{ symmetric}, A \in A^I\}.$$

We shall show that computing $\overline{\lambda}(A^I)$ approximately with relative error less than 1 is NP–hard already for symmetric interval matrices:

Theorem 34 *Suppose there exists a polynomial–time algorithm which for each interval matrix of the form $A^I = [A-E, A+E]$, A rational nonpositive symmetric stable, computes a rational number $\tilde{\lambda}(A^I)$ satisfying*

$$\left|\frac{\tilde{\lambda}(A^I) - \overline{\lambda}(A^I)}{\overline{\lambda}(A^I)}\right| < 1$$

if $\overline{\lambda}(A^I) \neq 0$ and $\tilde{\lambda}(A^I) \geq 0$ otherwise. Then P=NP.

Proof. Under the assumptions, $\tilde{\lambda}(A^I) < 0$ if and only if $\overline{\lambda}(A^I) < 0$, and this is equivalent to stability of A^I. Hence we have a polynomial–time algorithm for solving the NP–hard problem of Theorem 29, which implies P=NP. ■

6.3 Checking enclosures is NP–hard

Before formulating the result, we prove an auxiliary statement concerning the set of maximal eigenvalues of all symmetric matrices in A^I.

Proposition 35 *For a symmetric interval matrix A^I, the set*

$$\lambda^I_{\max}(A^I) := \{\lambda_{\max}(A);\ A \text{ symmetric}, A \in A^I\}$$

is a compact interval.

Proof. Let

$$\begin{aligned}\underline{\lambda}(A^I) &= \min\{\lambda_{\max}(A);\ A \text{ symmetric}, A \in A^I\},\\ \overline{\lambda}(A^I) &= \max\{\lambda_{\max}(A);\ A \text{ symmetric}, A \in A^I\}.\end{aligned}$$

By continuity argument, both bounds are achieved, hence

$$\begin{aligned}\underline{\lambda}(A^I) &= \lambda_{\max}(A_1),\\ \overline{\lambda}(A^I) &= \lambda_{\max}(A_2)\end{aligned}$$

for some symmetric $A_1, A_2 \in A^I$. Define a real function φ of one real variable by

$$\varphi(t) = f(A_1 + t(A_2 - A_1)), \quad t \in [0,1],$$

where

$$f(A) = \max_{\|x\|_2=1} x^T A x.$$

φ is continuous since $f(A)$ is continuous [39], and $\varphi(0) = f(A_1) = \lambda_{\max}(A_1) = \underline{\lambda}(A^I)$, $\varphi(1) = f(A_2) = \lambda_{\max}(A_2) = \overline{\lambda}(A^I)$, hence for each $\lambda \in [\underline{\lambda}(A^I), \overline{\lambda}(A^I)]$ there exists a $t_\lambda \in [0,1]$ such that

$$\lambda = \varphi(t_\lambda) = f(A_1 + t_\lambda(A_2 - A_1)) = \lambda_{\max}(A_1 + t_\lambda(A_2 - A_1)).$$

Hence each $\lambda \in [\underline{\lambda}(A^I), \overline{\lambda}(A^I)]$ is the maximal eigenvalue of some symmetric matrix in A^I, and we have

$$\lambda^I_{\max}(A^I) = [\underline{\lambda}(A^I), \overline{\lambda}(A^I)].$$

∎

In the last result of this section we show that checking enclosures of $\lambda^I_{\max}(A^I)$ is NP–hard:

Theorem 36 *The following problem is NP–hard:*

Instance. *A nonpositive symmetric stable rational matrix A, and rational numbers a, b, $a < b$.*

Question. *Is $\lambda^I_{\max}([A - E, A + E]) \subset (a, b)$?*

Proof. For each symmetric $A' \in [A - E, A + E]$ we have

$$|\lambda_{\max}(A')| \leq \varrho(A') \leq \|A'\|_1 \leq \|A\|_1 + \|E\|_1 = \|A\|_1 + n < \alpha := \|A\|_1 + n + 1.$$

Hence due to Theorem 27, $[A - E, A + E]$ is stable if and only if

$$\lambda^I_{\max}([A - E, A + E]) \subset (-\alpha, 0)$$

holds. This shows that the NP–hard problem of checking stability of $[A - E, A + E]$ (Theorem 29) can be reduced in polynomial time to the current problem, which is thus NP–hard. ∎

7 Determinants

Determinants of interval matrices have been scarcely studied in the literature so far. We include here some results that might be of interest.

7.1 Edge theorem

The following theorem was proved in [35]:

Theorem 37 *Let $A^I = [\underline{A}, \overline{A}]$ be an interval matrix. Then for each $A \in A^I$ there exists an $A' \in A^I$ of the form*

$$A'_{ij} \in \begin{cases} \{\underline{A}_{ij}, \overline{A}_{ij}\} & \text{if } (i,j) \neq (k,m), \\ [\underline{A}_{ij}, \overline{A}_{ij}] & \text{if } (i,j) = (k,m) \end{cases} \tag{7.1}$$

for some (k, m) such that

$$\det A = \det A'.$$

Proof. For each $\tilde{A} \in A^I$ denote by $h(\tilde{A})$ the number of entries with $\tilde{A}_{ij} \notin \{\underline{A}_{ij}, \overline{A}_{ij}\}$, $i, j = 1, \ldots, n$. Given an $A \in A^I$, let A' be a matrix satisfying $A' \in A^I, \det A' = \det A$ and

$$h(A') = \min\{h(\tilde{A});\ \tilde{A} \in A^I, \det \tilde{A} = \det A\}. \tag{7.2}$$

If $h(A') \geq 2$, then there exist indices $(p,q), (r,s), (p,q) \neq (r,s)$ such that $A'_{pq} \in (\underline{A}_{pq}, \overline{A}_{pq}), A'_{rs} \in (\underline{A}_{rs}, \overline{A}_{rs})$. Then we can move these two entries within their intervals in such a way that at least one achieves its bound, and the determinant is kept unchanged. Then the resulting matrix A'' satisfies $h(A'') < h(A')$, which is a contradiction. Hence A' defined by (7.2) satisfies $h(A') \leq 1$, which shows that it is of the form (7.1), and $\det A = \det A'$ holds. ∎

A matrix of the form (7.1) belongs to an edge of the interval matrix A^I considered a hyperrectangle in R^{n^2}. Hence the theorem says that the range of the determinant over A^I is equal to its range over the edges of A^I. In particular, for zero values of the determinant we have this "normal form" theorem [37].

Theorem 38 *If A^I is singular, then it contains a singular matrix of the form (7.1).*

As a consequence we obtain that *real* eigenvalues of matrices in A^I are achieved at the edge matrices of A^I.

Theorem 39 *If a real number λ is an eigenvalue of some $A \in A^I$, then it is also an eigenvalue of some matrix of the form (7.1).*

Proof. If λ is a real eigenvalue of some $A \in A^I = [\underline{A}, \overline{A}]$, then $A - \lambda I$ is a singular matrix belonging to $[\underline{A} - \lambda I, \overline{A} - \lambda I]$, which is thus singular, hence by Theorem 38 it contains a singular matrix $A' - \lambda I$, where A' is of the form (7.1). Then λ is an eigenvalue of A'. ∎

A general "edge theorem" for complex eigenvalues was proved by Hollot and Bartlett in [21].

7.2 Computing extremal values of determinants is NP–hard

For an interval matrix A^I, consider the extremal values of the determinant over A^I given by

$$\begin{aligned} \overline{\det}(A^I) &= \max\{\det A;\ A \in A^I\}, \\ \underline{\det}(A^I) &= \min\{\det A;\ A \in A^I\}. \end{aligned}$$

Since the determinant is linear in each entry, Theorem 37 implies that the extremal values are achieved at some of the 2^{n^2} vertex matrices, i.e. matrices of the form

$$A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}, \quad i, j = 1, \ldots, n.$$

We have this result:

Theorem 40 *Computing $\underline{\det}(A^I), \overline{\det}(A^I)$ is NP–hard for the class of interval matrices of the form $A^I = [A - E, A + E]$, A rational nonnegative.*

Proof. For an interval matrix of the form $A^I = [A - E, A + E]$, where A is a nonnegative symmetric positive definite rational matrix, singularity of A^I is equivalent to

$$\overline{\det}(A_0^I) \geq 0, \tag{7.3}$$

where $A_0^I = A^I$ if $\det A \leq 0$ and A_0^I is constructed by swapping the first two rows of A^I otherwise (which changes the sign of the determinant). Here $A_0^I = [A_0 - E, A_0 + E]$, where A_0 is a nonnegative rational matrix. Hence the NP–hard problem of checking regularity (Theorem 9) can be reduced in polynomial time to the decision problem (7.3) which shows that computing $\overline{\det}(A^I)$ is NP–hard in this class of interval matrices. The proof for $\underline{\det}(A^I)$ is analogous. ∎

8 Nonnegative invertibility and M–matrices

So far we have shown a number of properties of interval matrices that are NP–hard to check. Finally we present two useful properties whose checking may be done in polynomial time.

8.1 Nonnegative invertibility

An interval matrix is said to be nonnegative invertible if $A^{-1} \geq 0$ for each $A \in A^I$. The following result is due to Kuttler [25]; we use here the elementary proof from [36].

Theorem 41 *An interval matrix $A^I = [\underline{A}, \overline{A}]$ is nonnegative invertible if and only if $\underline{A}^{-1} \geq 0$ and $\overline{A}^{-1} \geq 0$.*

Proof. The "only if" part is obvious. To prove the "if" part, denote $D_0 = \overline{A}^{-1}(\overline{A} - \underline{A})$, then $D_0 \geq 0$ and

$$(I - D_0)^{-1} = (\overline{A}^{-1}\underline{A})^{-1} = \underline{A}^{-1}\overline{A} = I + \underline{A}^{-1}(\overline{A} - \underline{A}) \geq 0,$$

hence $\varrho(D_0) < 1$. Then for each $A \in A^I$ we have $\varrho(\overline{A}^{-1}(\overline{A} - A)) \leq \varrho(D_0) < 1$, and from the identity

$$A = \overline{A}(I - \overline{A}^{-1}(\overline{A} - A))$$

we obtain

$$A^{-1} = \sum_{j=0}^{\infty}(\overline{A}^{-1}(\overline{A} - A))^j\overline{A}^{-1} \geq 0.$$

∎

Hence, checking nonnegative invertibility of an interval matrix A^I with rational bounds can be performed in polynomial time [14].

8.2 M–matrices

An interval matrix A^I is called an M–matrix if each $A \in A^I$ is an M–matrix (i.e., $A_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$). As a consequence of Kuttler's theorem we have this characterization:

Theorem 42 *An interval matrix $A^I = [\underline{A}, \overline{A}]$ is an M–matrix if and only if $\underline{A}$ and $\overline{A}$ are M–matrices.*

Proof. The "only if" part is obvious. Conversely, if both $\underline{A}$ and $\overline{A}$ are M–matrices, then $\underline{A}^{-1} \geq 0$ and $\overline{A}^{-1} \geq 0$, hence each $A \in A^I$ satisfies $A^{-1} \geq 0$ (Theorem 41) and $A_{ij} \leq \overline{A}_{ij} \leq 0$ for $i \neq j$, i.e. A is an M–matrix. ∎

9 Appendix: Regularity and P–property

In section 4 we mentioned Coxson's NP–hardness result for checking P–property of real (noninterval) matrices. We add the result here as an appendix since it is of independent interest and is based on a nice equivalence of regularity of interval matrices with P–property of associated real matrices, which is also due to Coxson [9].

9.1 Regularity and P–property I

Consider an $n \times n$ interval matrix $A^I = [\underline{A}, \overline{A}]$ which we shall write in the form $A^I = [\underline{A}, \underline{A} + 2\Delta]$, where $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ as before. Assuming nonsingularity of $\underline{A}$, for each $i, j \in \{1, \ldots, n\}$ define the vector

$$c_{ij} = 2(\Delta_{i1}\underline{A}^{-1}_{1j}, \Delta_{i2}\underline{A}^{-1}_{2j}, \ldots, \Delta_{in}\underline{A}^{-1}_{nj})^T$$

(where we write $\underline{A}^{-1}_{kj}$ for $(\underline{A}^{-1})_{kj}$), and the matrix

$$C_{ij} = c_{ij}e^T,$$

where e is the n–vector of all ones. Hence, C_{ij} is an $n \times n$ matrix whose all columns are identical and equal to the vector c_{ij}. Finally, define the real matrix

$$C(A^I) = \begin{pmatrix} I & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I \end{pmatrix} + \begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{pmatrix}$$

whose all blocks are $n \times n$ matrices, hence $C(A^I)$ is of size $n^2 \times n^2$. For each $y, z \in Z$, let us define the yz–minor of $C(A^I)$ as the determinant of the principal submatrix of $C(A^I)$ consisting of rows and columns with indices $(i-1)n + j$, where $y_i z_j = -1$. The equivalence (i)⇔(ii) of the following theorem is due to Coxson [9], equivalence (i)⇔(iii) is added here as a consequence of the Baumann theorem 11 to show that the number of determinants to be checked for positivity can be decreased from $2^{n^2} - 1$ to $2^{2n-1} - 1$. The specific feature of this result consists in the fact that regularity of an $n \times n$ interval matrix A^I is characterized in terms of an $n^2 \times n^2$ real matrix $C(A^I)$. Nevertheless, the number of operations involved still remains exponential in n.

Theorem 43 *For an interval matrix A^I, the following conditions are equivalent:*

(i) A^I is regular,

(ii) $\underline{A}$ is nonsingular and $C(A^I)$ is a P–matrix,

(iii) $\underline{A}$ is nonsingular and each yz–minor of $C(A^I)$ is positive, $y, z \in Z$.

Proof. (i)⇔(ii): Put

$$F = \begin{pmatrix} e^{3}T & 0^T & \dots & 0^T \\ 0^T & e^T & \dots & 0^T \\ \vdots & \vdots & \ddots & \vdots \\ 0^T & 0^T & \dots & e^T \end{pmatrix},$$

where all the blocks are n–dimensional vectors, hence F is of size $n \times n^2$, and

$$G = \begin{pmatrix} \Delta_{11}e_1 & \Delta_{12}e_2 & \dots & \Delta_{1n}e_n \\ \Delta_{21}e_1 & \Delta_{22}e_2 & \dots & \Delta_{2n}e_n \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1}e_1 & \Delta_{n2}e_2 & \dots & \Delta_{nn}e_n \end{pmatrix},$$

where e_j denotes the jth column of the $n \times n$ unit matrix I, hence G is of size $n^2 \times n$. Consider any vertex matrix A of A^I, i.e. a matrix satisfying

$$A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}, \quad i, j = 1, \dots, n.$$

A straightforward computation shows that A can be written in the form

$$A = \underline{A} + 2FDG,$$

where D is the $n^2 \times n^2$ diagonal matrix satisfying

$$D_{(i-1)n+j,(i-1)n+j} = \begin{cases} 1 & \text{if } A_{ij} = \overline{A}_{ij}, \\ 0 & \text{if } A_{ij} = \underline{A}_{ij} \end{cases}$$

$(i, j = 1, \dots, n)$. Then we have

$$\det A = (\det \underline{A})(\det(I + 2\underline{A}^{-1}FDG)). \tag{9.1}$$

Since

$$\det(I + 2\underline{A}^{-1}FDG) = \det(I_{n^2} + 2DG\underline{A}^{-1}F) \tag{9.2}$$

(see Gantmacher [16]; I_{n^2} is the $n^2 \times n^2$ unit matrix), and since

$$2G\underline{A}^{-1}F = C(A^I) - I_{n^2} \tag{9.3}$$

(as it can be easily verified), from (9.1)–(9.3) we obtain

$$\det A = (\det \underline{A})(\det(I_{n^2} + D(C(A^I) - I_{n^2}))), \tag{9.4}$$

where

$$\det(I_{n^2} + D(C(A^I) - I_{n^2})) \tag{9.5}$$

is obviously the determinant of the principal submatrix formed from rows and columns of $C(A^I)$ with indices $(i-1)n+j$ for which $A_{ij} = \overline{A}_{ij}$.

Now, if A^I is regular, then each principal minor of $C(A^I)$ can be written in the form (9.5) for an appropriately chosen vertex matrix A. Since $(\det A)(\det \underline{A}) > 0$ due to regularity, (9.4) implies that (9.5) is positive. Conversely, if each principal minor of $C(A^I)$ is positive, then $(\det A)(\det \underline{A}) > 0$ for each vertex matrix A of A^I due to (9.4), which implies that A^I is regular (Theorem 11). Hence (i) and (ii) are equivalent.

To prove (i)$\Leftrightarrow$(iii), notice that each matrix $A_{yz} \in A^I, y, z \in Z$ defined by (2.6) satisfies

$$(A_{yz})_{ij} = \underline{A}_{ij} + (1 - y_i z_j)\Delta_{ij}, \quad i, j = 1, \ldots, n,$$

hence it can be written as

$$A_{yz} = \underline{A} + F D_{yz} G,$$

where F and G are as above and D_{yz} is the $n^2 \times n^2$ diagonal matrix satisfying

$$(D_{yz})_{(i-1)n+j,(i-1)n+j} = 1 - y_i z_j, \quad i, j = 1, \ldots, n.$$

Then we obtain as before that

$$\det A_{yz} = (\det \underline{A})(\det(I_{n^2} + \frac{1}{2} D_{yz}(C(A^I) - I_{n^2}))),$$

where

$$\det(I_{n^2} + \frac{1}{2} D_{yz}(C(A^I) - I_{n^2}))$$

is exactly the yz–minor of $C(A^I)$ defined earlier in this section. Hence an obvious reasoning based on Baumann's theorem 11 leads to the conclusion that A^I is regular if and only if all the yz–minors of $C(A^I)$ are positive, $y, z \in Z$. ■

9.2 Checking P–property is NP–hard for real matrices

Coxson's result [9] is obtained as an immediate consequence of the previous characterization.

Theorem 44 *Checking P–property of real matrices is NP–hard.*

Proof. According to the equivalence (i)$\Leftrightarrow$(ii) of Theorem 43, the problem of checking regularity of an interval matrix A^I with rational bounds can be reduced in polynomial time to the problem of checking P–property of a rational matrix $C(A^I)$. Since the former problem is NP–hard (Theorem 9), the same is true for the latter one as well. ■

9.3 Regularity and P–property II

It should be noted that there also exists another relationship between regularity and the P–property, which proved to be a very useful tool for deriving some nontrivial properties of inverse interval matrices and of systems of linear interval equations. The following theorem was published in a report form [34] in 1984 and in a journal form [37] in 1989.

Theorem 45 *If A^I is regular, then $A_1^{-1}A_2$ is a P–matrix for each $A_1, A_2 \in A^I$.*

Proof. Assume to the contrary that $A_1^{-1}A_2$ is not a P–matrix for some $A_1, A_2 \in A^I = [A_c - \Delta, A_c + \Delta]$. Then according to the Fiedler–Pták theorem [15] (quoted at the beginning of section 4) there exists an $x \neq 0$ such that $x_i(A_1^{-1}A_2x)_i \leq 0$ for each i. Put $x' = A_1^{-1}A_2x$, then

$$x_i x_i' \leq 0 \quad (i = 1, \ldots, n) \tag{9.6}$$

and

$$x \neq x' \tag{9.7}$$

holds. In fact, since $x \neq 0$, there exists a j with $x_j \neq 0$; then $x_j^2 > 0$ whereas (9.6) implies $x_j x_j' \leq 0$, hence $x_j \neq x_j'$. Now we have

$$|A_c(x' - x)| = |(A_c - A_1)x' + (A_2 - A_c)x| \leq \Delta|x'| + \Delta|x| = \Delta|x' - x| \tag{9.8}$$

since $|x'| + |x| = |x' - x|$ due to (9.6). Hence Proposition 10 in the light of (9.8) and (9.7) implies that A^I is singular, which is a contradiction. ∎

For applications of this result, see [37].

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