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Technical report No. 644

June 15, 1995

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The Conjecture " $P \neq NP$ " and Overestimation in Bounding Solutions of Perturbed Linear Equations¹

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Abstract

It is proved that a classical bound on solutions of perturbed systems of linear equations may yield arbitrarily large polynomial overestimations for arbitrarily narrow perturbations provided the conjecture $P \neq NP$ is true.

Keywords

Linear equations, perturbation, error bound, overestimation, $P \neq NP$

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1 Introduction

For a system of linear equations

$$Ax = b \tag{1.1}$$

with an $n \times n$ nonsingular matrix A, consider a family of perturbed systems

$$A'x' = b' \tag{1.2}$$

with data satisfying

$$|A' - A| \le \Delta \tag{1.3}$$

and

$$|b' - b| \le \delta,\tag{1.4}$$

where $\Delta \geq 0$ and $\delta \geq 0$ are an $n \times n$ perturbation matrix and a perturbation n-vector, respectively, and the inequalities are understood componentwise. The classical numerical argument using Neumann series shows that if the spectral condition

$$\varrho(|A^{-1}|\Delta) < 1 \tag{1.5}$$

holds, then each A' satisfying (1.3) is nonsingular and the solution of each system (1.2) with data (1.3), (1.4) satisfies

$$|x' - x| \le d,\tag{1.6}$$

where

$$d = (I - |A^{-1}|\Delta)^{-1}|A^{-1}|(\Delta|x| + \delta)$$
(1.7)

and I is the unit matrix (see Skeel [8] or Rump [6]). To keep the paper self-contained, we give here another simple proof of this result: for the solutions x, x' of (1.1), (1.2) under (1.3), (1.4) we have

$$|x' - x| = |A^{-1}A(x' - x)| \le |A^{-1}| \cdot |(A - A')(x' - x) + (A - A')x + b' - b|$$

$$\le |A^{-1}|(\Delta |x' - x| + \Delta |x| + \delta),$$

hence

$$(I - |A^{-1}|\Delta)|x' - x| \le |A^{-1}|(\Delta|x| + \delta)$$

and premultiplying this inequality by $(I - |A^{-1}|\Delta)^{-1}$, which is nonnegative in view of (1.5), we obtain (1.6), where d is given by (1.7).

The quality of the estimation (1.6) has been paid little attention in the literature. Obviously, the bound d is exact if $\Delta = 0$. In fact, in this case, for each $i \in \{1, \ldots, n\}$, if we take $b'_j = b_j + \delta_j$ if $(A^{-1})_{ij} \geq 0$ and $b'_j = b_j - \delta_j$ otherwise, then b' satisfies (1.4) and for the solution x' of Ax' = b' we have

$$|x_i' - x_i| = \sum_{i} |(A^{-1})_{ij}| \delta_j = d_i,$$

hence the bound is achieved. However, this argument fails in the case $\Delta \neq 0$. In this paper we show that the famous conjecture "P \neq NP" (see Garey and Johnson [1] for details) shreds a surprising light on this problem: in the main result to follow we show that if the conjecture is true, then the formula (1.6) may yield an arbitrarily large polynomial overestimation for arbitrarily narrow perturbations Δ , δ . Hence, the conjecture deeply penetrates the area of numerical linear algebra as well.

2 Main result

We shall use the subordinate matrix norm

$$\|\Delta\|_m = \max_{i,j} |\Delta_{ij}|$$

and the vector norm

$$\|\delta\|_{\infty} = \max_{i} |\delta_{i}|.$$

Our main result is formulated as follows:

Theorem 1 If $P \neq NP$, then for each rational $\varepsilon > 0$, $\eta > 0$, $\alpha > 0$ and for each integer $k \geq 0$ there exist $n \times n$ matrices A, $\Delta \geq 0$ and n-vectors b, $\delta \geq 0$ for some $n \geq 2$ such that

$$\varrho(|A^{-1}|\Delta) = 0 \tag{2.1}$$

$$\|\Delta\|_m = \varepsilon \tag{2.2}$$

$$\|\delta\|_{\infty} = \eta \tag{2.3}$$

hold and the solution x' of each system (1.2) with data (1.3), (1.4) satisfies

$$|x_1' - x_1| + \alpha n^k \le d_1, \tag{2.4}$$

where x is the solution of (1.1) and d is given by (1.7).

Proof. Assume to the contrary that it is not so, so that there exist rational numbers $\varepsilon > 0$, $\eta > 0$, $\alpha > 0$ and an integer $k \geq 0$ such that for each $n \geq 2$ and all $n \times n$ matrices $A, \Delta \geq 0$ and all n-vectors $b, \delta \geq 0$ satisfying (2.1)-(2.3) we have

$$|x_1' - x_1| + \alpha n^k > d_1 \tag{2.5}$$

for the solution x' of some system (1.2) with data (1.3), (1.4).

Take an arbitrary $m \times m$ MC-matrix \hat{A} , $m \geq 1$, i.e. a matrix \hat{A} satisfying $\hat{A}_{ii} = m$ and $\hat{A}_{ij} \in \{0, -1\}$ if $i \neq j$ (i, j = 1, ..., m); \hat{A} is nonsingular (cf. [4]). Let us define

$$A = \begin{pmatrix} \frac{\varepsilon\eta}{\gamma} & 0^T \\ 0 & \tilde{A}^{-1} \end{pmatrix}, \tag{2.6}$$

$$\Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix}, \tag{2.7}$$

where $\gamma = \alpha(m+1)^k$ and $e = (1, ..., 1)^T \in \mathbb{R}^m$ (hence A and Δ are of size $(m+1) \times (m+1)$), and let

$$b = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.8}$$

and

$$\delta = \begin{pmatrix} 0 \\ \eta e \end{pmatrix} \tag{2.9}$$

be (m+1)-dimensional vectors. Then

$$|A^{-1}|\Delta = \begin{pmatrix} 0 & \frac{\gamma}{\eta}e^T \\ 0 & 0 \end{pmatrix},$$

hence (2.1), (2.2) and (2.3) hold, the solution of (1.1) is x=0 and for

$$\overline{x}_1 := \max\{x_1'; x' \text{ solves } (1.2) \text{ under } (1.3), (1.4)\}$$

we have (if we denote $\tilde{x} = (x_2, x_3, \dots, x_m)^T$) that

$$\overline{x}_{1} = \frac{\gamma}{\varepsilon \eta} \max\{\varepsilon e^{T} |\tilde{x}|; -\eta e \leq \tilde{A}^{-1} \tilde{x} \leq \eta e\}$$

$$= \gamma \max\{\|\tilde{A}x\|_{1}; x_{j} \in \{-1, 1\} \text{ for each } j\}$$

$$= \gamma \|\tilde{A}\|_{\infty, 1}$$

(see Golub and van Loan [2] for definition of $\|\tilde{A}\|_{\infty,1}$), and in a similar way for

$$\underline{x}_1 := \min\{x'_1; x' \text{ solves } (1.2) \text{ under } (1.3), (1.4)\}$$

we obtain

$$\underline{x}_1 = -\gamma \|\tilde{A}\|_{\infty,1}.$$

Let us now compute d by (1.7). Then in view of (2.5) we have (since x=0) that

$$\gamma \|\tilde{A}\|_{\infty,1} \ge |x_1'| > d_1 - \alpha (m+1)^k = d_1 - \gamma,$$

hence

$$d_1 < \gamma(\|\tilde{A}\|_{\infty,1} + 1). \tag{2.10}$$

But in view of (1.6) and of x = 0 we also have

$$\gamma \|\tilde{A}\|_{\infty,1} = \overline{x}_1 \le d_1, \tag{2.11}$$

hence (2.10) and (2.11) give

$$\|\tilde{A}\|_{\infty,1} \le \frac{d_1}{\gamma} < \|\tilde{A}\|_{\infty,1} + 1.$$
 (2.12)

Since the MC-matrix \tilde{A} is integer by definition, the number

$$\|\tilde{A}\|_{\infty,1} = \max\{\|\tilde{A}x\|_1; x_i \in \{-1,1\} \text{ for each } j\}$$

is also integer, hence from (2.12) we finally obtain

$$\|\tilde{A}\|_{\infty,1} = \left[\frac{d_1}{\gamma}\right],\tag{2.13}$$

where [...] denotes the integer part.

Summing up, we have proved the following: given an MC-matrix \tilde{A} , if we construct A, Δ , b and δ by (2.6)–(2.9) and then compute d by (1.7), then (2.13) holds. Since all these computations can be done in polynomial time (Schrijver [7]), we have a polynomial-time algorithm for computing $\|\tilde{A}\|_{\infty,1}$ for an MC-matrix \tilde{A} . However, computing $\|\tilde{A}\|_{\infty,1}$ was proved to be NP-hard for MC-matrices \tilde{A} ([5], Corollary 7, which is a simple consequence of Theorem 2.6 in [3]). Hence, an existence of a polynomial-time algorithm for solving an NP-hard problem implies P=NP, which contradicts our assumption.

3 Concluding remarks

We have proved that if $P\neq NP$, then for arbitrarily narrow perturbations (2.2), (2.3) the formula (1.7) may yield a catastrophic overestimation (2.4). This, of course, is a worst-case-type result. The conjecture " $P\neq NP$ " has not been proved to date, but it is widely believed to be true (Garey and Johnson [1]). In any case, we can see that the conjecture is closely related to one of the basic problems in numerical linear algebra; if the assertion concerning the overestimation (2.4) is not true, then a simple algorithm based on formulae (2.6), (2.7), (2.8), (2.9), (1.7) and (2.13) gives a polynomial-time algorithm for solving an NP-hard problem, thereby also solving in polynomial time all the problems in the class NP.

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