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Abstract

A method for enclosing solutions of overdetermined systems of linear interval equations is described. Various aspects of the problem (algorithm, improvement, optimal enclosure, complexity) are studied.

Keywords
Linear interval equations, overdetermined system, enclosure, algorithm

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1 Introduction

The problem (section 2) was proposed to the author by Prof. Dr. G. Heindl in April 1995 and independently by Dr. G. Lichtenberg in June 1995. We describe here the main result (sections 3–5), an algorithm (sections 9–11) and we discuss briefly some related issues (performability condition, enclosure improvement, optimal enclosure, special cases, complexity).

2 The problem

Given an overdetermined system of linear interval equations

$$A^I x = b^I \tag{2.1}$$

with an $m \times n$ interval matrix

$$A^I = \{A;\ A_c - \Delta \le A \le A_c + \Delta\}$$

where $m \ge n$ (in practice: m is essentially greater than n, see [1]), and an interval m-vector

$$b^I = \{b;\ b_c - \delta \le b \le b_c + \delta\}$$

(componentwise inequalities), find an interval vector $[\underline{x}, \overline{x}]$ satisfying

$$X \subseteq [\underline{x}, \overline{x}], \tag{2.2}$$

where

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}$$

is the so–called solution set of (2.1). An interval vector $[\underline{x}, \overline{x}]$ satisfying (2.2) is called an enclosure of X.

3 Enclosure theorem

Theorem 1 *Let R be an arbitrary $n \times m$ matrix[3] and let x_0 and $d > 0$ be arbitrary n-vectors such that*

$$Gd + g < d \tag{3.1}$$

holds, where

$$G = |I - RA_c| + |R|\Delta$$

and

$$g = |R(A_c x_0 - b_c)| + |R|(\Delta|x_0| + \delta).$$

Then

$$X \subseteq [x_0 - d, x_0 + d].$$

[3] notice the transposed size

4 Comment I

We recommend to take

$$R \approx (A_c^T A_c)^{-1} A_c^T$$

(an approximation of the Moore–Penrose inverse of A_c) and

$$x_0 \approx R b_c.$$

Then G and g can be computed from the initial data and from R, x_0 (I is the unit matrix), hence the problem reduces to solving the inequality (3.1). Since A_c, Δ are $m \times n$ and R is $n \times m$, the matrix G is a square matrix of size $n \times n$, where n is the *lower* of the two dimensions m, n.

5 Proof

Let $x \in X$, so that $Ax = b$ for some $A \in A^I$, $b \in b^I$. Then

$$x = x + R(-Ax + b) = (I - RA)x + Rb,$$

which implies

$$\begin{aligned} x - x_0 &= (I - RA)(x - x_0) + R(b - Ax_0) \\ &= (I - RA_c)(x - x_0) + R(A_c - A)(x - x_0) + R(b_c - A_c x_0) \\ &\quad + R(A_c - A)x_0 + R(b - b_c) \end{aligned}$$

and taking absolute values, we have

$$\begin{aligned} |x - x_0| &\le |I - RA_c| \cdot |x - x_0| + |R|\Delta|x - x_0| \\ &\quad + |R(b_c - A_c x_0)| + |R|\Delta|x_0| + |R|\delta \\ &= G|x - x_0| + g. \end{aligned}$$

Thus for a d satisfying (3.1) we obtain

$$(I - G)|x - x_0| \le g < (I - G)d. \tag{5.1}$$

Since $g \ge 0$, (3.1) implies $Gd < d$, which in view of $d > 0$ gives $\varrho(G) < 1$ (since G is nonnegative), hence $(I - G)^{-1} \ge 0$. Premultiplying (5.1) by $(I - G)^{-1}$, we obtain

$$|x - x_0| < d,$$

which proves $x \in [x_0 - d, x_0 + d]$. Hence $X \subseteq [x_0 - d, x_0 + d]$.

6 Comment II

The inequality $m \ge n$ has not been used in the proof. Therefore the proof may create an impression that the result is valid for arbitrary m, n. This is not the case, as the next corollary shows: if (3.1) holds, then it must be $m \ge n$; hence the inequality is implicitly contained in (3.1).

7 Corollary

Corollary 1 *If (3.1) holds for some R, x_0 and $d > 0$, then each $A \in A^I$ has linearly independent columns.*

8 Proof

Assume to the contrary that $Ax = 0$ for some $A \in A^I$, $x \neq 0$. Then $RAx = 0$, hence

$$x = x - RAx = (I - RA_c)x + R(A_c - A)x,$$

which implies

$$|x| \leq |I - RA_c| \cdot |x| + |R|\Delta|x| = G|x|$$

and consequently

$$1 \leq \varrho(G),$$

but from the proof of Theorem 1 we know that the existence of a solution to (3.1) implies

$$\varrho(G) < 1,$$

which is a contradiction.

9 Algorithm

The inequality (3.1) can be solved as an equation

$$d = Gd + g + f$$

where f is some positive vector. This observation suggests the following algorithm, which in fact is only a variant of the algorithm in [5]:

f:= a (small) positive vector;
$d' := 0$;
repeat
  $d := d'$;
  $d' := Gd + g + f$
until $|d' - d| < f$.
{then d is a positive solution to (3.1)}

10 Finite termination

Theorem 2 *The following conditions are equivalent:*

(i) $\varrho(G) < 1$,

(ii) the algorithm terminates in a finite number of steps for some $f > 0$,

(iii) the algorithm terminates in a finite number of steps for each $f > 0$.

11 Proof

(i) $\Rightarrow$ (iii): if $\varrho(G) < 1$, then for each $f > 0$ the sequence

$$d_{j+1} = Gd_j + g + f$$

generated by the algorithm is Cauchian, hence convergent. Thus $d_{j+1} - d_j \to 0$, hence $|d_{j+1} - d_j| < f$ for some j. (iii) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (i): if the algorithm terminates for some $f > 0$, then from $|d' - d| < f$ we obtain $d' = Gd + g + f < d + f$, hence $Gd \leq Gd + g < d$ and since $d > 0$, we have $\varrho(G) < 1$.

12 Comment III

Hence, finite termination is independent of the choice of f (which, however, may influence the number of steps). For practical purposes it is recommendable to change the stopping rule of the algorithm to

$$\ldots k := k + 1 \textbf{ until } (|d' - d| < f \text{ or } k > k_{\max})$$

where $k_{\max}$ is a prescribed maximum number of steps. If $k > k_{\max}$, then existence of a positive solution to (3.1) has not been proved.

13 Sufficient performability condition

If the matrix A^I of the problem (2.1) satisfies

$$\varrho(|(A_c^T A_c)^{-1} A_c^T|\Delta) < 1, \tag{13.1}$$

then for $R := (A_c^T A_c)^{-1} A_c^T$ we have

$$\varrho(G) < 1,$$

hence the algorithm is finite (Theorem 2) and an enclosure can be computed by Theorem 1.

14 Enclosure improvement

Since R and x_0 in Theorem 1 may be chosen arbitrarily, we can try to sharpen the enclosure obtained by a repeated use of Theorem 1:

for $j := 1$ **to** $j_{\max}$ **do begin**
  generate randomly $A \in A^I$, $b \in b^I$;
  $R := (A^T A)^{-1} A^T$;
  $x_0 := Rb$;
  compute a $d > 0$ satisfying (3.1) by the algorithm;
  if $j = 1$ **then** $x^I := [x_0 - d, x_0 + d]$ **else** $x^I := x^I \cap [x_0 - d, x_0 + d]$
end.
$\{$then $X \subseteq x^I\}$

15 Optimal enclosure

Once an enclosure $x^I = [\underline{x}, \overline{x}]$ has been found, we may use the information contained therein to compute the optimal (narrowest) enclosure of X. Define

$$Z = \{z \in \mathbb{R}^n;\ z_j = 1 \text{ if } \underline{x}_j > 0,\ z_j = -1 \text{ if } \overline{x}_j < 0,\ |z_j| = 1 \text{ otherwise}\}$$

and for each $z \in Z$ let T_z denote the diagonal matrix with diagonal vector z. As a consequence of the Oettli–Prager theorem [2], if we solve the linear programming problems

$$\begin{aligned} \underline{x}_i^z &= \min\{x_i;\ b_c - \delta \le (A_c + \Delta T_z)x,\ (A_c - \Delta T_z)x \le b_c + \delta,\ T_z x \ge 0\}, \\ \overline{x}_i^z &= \max\{x_i;\ b_c - \delta \le (A_c + \Delta T_z)x,\ (A_c - \Delta T_z)x \le b_c + \delta,\ T_z x \ge 0\} \end{aligned}$$

for each $z \in Z$ and each $i \in \{1, \ldots, n\}$, then the *optimal* enclosure $[\underline{\underline{x}}, \overline{\overline{x}}]$ is given by

$$\begin{aligned} \underline{\underline{x}}_i &= \min\{x_i^z;\ z \in Z\}, \\ \overline{\overline{x}}_i &= \max\{x_i^z;\ z \in Z\} \end{aligned}$$

$(i = 1, \ldots, n)$. This procedure may prove disadvantageous if too many linear programming problems are involved. Therefore it can be recommended only if the cardinality of Z is moderate.

16 Special cases I

Theorem 1 works in particular in the square case $(m = n)$. Here the existence of a positive solution to (3.1) is equivalent to

$$\varrho(|A_c^{-1}|\Delta) < 1$$

(cf. [3]), i.e. to strong regularity of A^I.

17 Special cases II

In case of a real (noninterval) system $Ax = b$ we have $\Delta = 0$, $\delta = 0$, i.e. $G = |I - RA|$ and $g = |R(Ax_0 - b)|$, so that the inequality (3.1) takes on the form

$$|I - RA|d + |R(Ax_0 - b)| < d. \tag{17.1}$$

Hence the "enclosure theorem" in [5] is a special case of Theorem 1 here. The inequality (17.1) has been shown in [5] to be equivalent to Rump's inclusion based on Brouwer's fixed–point theorem [6].

18 Complexity of the problem

For linear interval systems with square matrices it was proved in [4] that the problem of computing an enclosure of X or verifying that no such enclosure exists (since X is unbounded) is NP-hard. Since systems with square matrices are a special case of overdetermined systems, the same is true for the problem treated in this paper. Hence an enclosure can be computed in polynomial time only for special classes of problems. The condition (13.1) under which our method works seems to be sufficiently general to cover most practical cases.

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