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Validated Solutions of Nonlinear Equations

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Technical report No. 641

June 16, 1995

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# Validated Solutions of Nonlinear Equations<sup>1</sup>

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## **Abstract**

An existence and uniqueness check for systems of nonlinear equations is given.

## **Keywords**

Nonlinear equations, solution, existence

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# 1 Introduction

We consider here a system of  $n$  nonlinear equations in  $n$  unknowns

$$f_i(x_1, \dots, x_n) = 0 \quad (i = 1, \dots, n)$$

which we write simply as

$$f(x) = 0. \tag{1.1}$$

The main goal is to establish a verifiable sufficient condition for existence and uniqueness of solution of (1.1) in a given region  $X \subset \mathbb{R}^n$  (later chosen as an interval vector,  $X = [x - d, x + d]$ ). Let  $R(x) : X \rightarrow \mathbb{R}^{n \times n}$  be any mapping such that  $R(x)$  is a nonsingular matrix for each  $x \in X$ . Then (1.1) is equivalent to

$$x = x - R(x)f(x)$$

in  $X$ , and for

$$F(x) := x - R(x)f(x)$$

we obtain an equivalent fixed-point equation

$$x = F(x) \tag{1.2}$$

which we shall consider in the sequel. In the main result of this report (Theorem 1) we give an existence and uniqueness test for (1.2) together with a method for constructing a nested sequence of interval vectors containing the solution  $x^*$  of (1.2) and tending to  $x^*$  (hence, we have a verified enclosure of  $x^*$  at each iteration). The result is preceded by an auxiliary lemma which may be of independent interest. A simple nonexistence test is given in Theorem 2; a result of such type is necessary when using a branch-and-bound method (see e.g. Ratschek and Rokne [1]) for finding all solutions of (1.2) in some region  $X$  for which the conditions under which Theorem 1 works are not yet satisfied. In the last section we briefly consider the connection of these results to the material of our earlier report [2] on validated solutions of linear equations.

## 2 Lemma

**Lemma 1** *Let  $\{x_j\}_{j=0}^{\infty}$  and  $\{d_j\}_{j=0}^{\infty}$  be vector (or scalar) sequences satisfying*

$$|x_j - x_{j+1}| \leq d_j - d_{j+1} \tag{2.1}$$

*for each  $j$ , and let  $d_j \rightarrow d^*$ . Then we have:*

- 1)  $x_j \rightarrow x^*$ ,
- 2)  $x^* \in [x_j - d_j + d^*, x_j + d_j - d^*]$  for each  $j$ ,
- 3) the sequence of intervals  $\{[x_j - d_j + d^*, x_j + d_j - d^*]\}_{j=0}^{\infty}$  is nested.

*Proof.* 0) For each  $j \geq 0$  and  $m \geq 1$ , from (2.1) we have  $|x_j - x_{j+m}| \leq \sum_{k=j}^{j+m-1} |x_k - x_{k+1}| \leq \sum_{k=j}^{j+m-1} (d_k - d_{k+1}) = d_j - d_{j+m}$ , hence

$$|x_j - x_{j+m}| \leq d_j - d_{j+m}. \quad (2.2)$$

1) Let  $\varepsilon > 0$ . Since  $\{d_j\}$  is convergent, there exists a  $j$  such that  $|d_j - d_{j+m}| = d_j - d_{j+m} < \varepsilon$  for each  $m \geq 1$ . Then from (2.2) we have

$$|x_j - x_{j+m}| < \varepsilon$$

for each  $m \geq 1$ , hence  $\{x_j\}$  is a cauchian sequence, thus  $x_j \rightarrow x^*$ .

2) For each  $j \geq 0$ , taking  $m \rightarrow \infty$  in (2.2), we obtain

$$|x_j - x^*| \leq d_j - d^*,$$

which implies

$$x^* \in [x_j - d_j + d^*, x_j + d_j - d^*].$$

3) It follows from (2.1) that  $x_j - d_j \leq x_{j+1} - d_{j+1}$  and  $x_{j+1} + d_{j+1} \leq x_j + d_j$  for each  $j$ , hence the sequence of intervals  $[x_j - d_j + d^*, x_j + d_j - d^*]$  is nested. ■

### 3 Main result

**Theorem 1** *Let  $F$  map an interval vector  $[x - d, x + d] \subset \mathbb{R}^n$  into  $\mathbb{R}^n$ , and let there exist a nonnegative matrix  $H$  with the following properties:*

$$(i) \quad |F(x') - F(x'')| \leq H|x' - x''| \text{ for each } x', x'' \in [x - d, x + d],$$

$$(ii) \quad |x - F(x)| < (I - H)d.$$

Then the equation

$$\tilde{x} = F(\tilde{x}) \quad (3.1)$$

has a unique solution  $x^*$  in  $[x - d, x + d]$ , and the sequence of interval vectors  $\{[x_j - d_j, x_j + d_j]\}_{j=0}^{\infty}$  defined by

$$\begin{aligned} x_0 &= x, \\ d_0 &= d \end{aligned}$$

and

$$\begin{aligned} x_{j+1} &= F(x_j), \\ d_{j+1} &= Hd_j \end{aligned} \quad (3.2)$$

( $j = 0, 1, \dots$ ) is nested, satisfies

$$x^* \in [x_j - d_j, x_j + d_j]$$

for each  $j$ , and  $x_j \rightarrow x^*$ ,  $d_j \rightarrow 0$ .

*Proof.* Since  $H$  is nonnegative, from (ii) we have  $Hd < d$  and  $d > 0$ , hence  $\rho(H) < 1$  and  $H^j \rightarrow 0$ ,  $(I - H)^{-1} \geq 0$ . We shall prove by induction that the sequences  $\{x_j\}$  and  $\{d_j\}$  satisfy

$$|x_j - x_{j+1}| \leq d_j - d_{j+1} \quad (3.3)$$

for each  $j$ . For  $j = 0$  we have  $|x_0 - x_1| = |x - F(x)| < (I - H)d = d_0 - d_1$  due to (ii). Let (3.3) hold for some  $j \geq 0$ , then

$$|x_{j+1} - x_{j+2}| = |F(x_j) - F(x_{j+1})| \leq H|x_j - x_{j+1}| \leq H(d_j - d_{j+1}) = d_{j+1} - d_{j+2},$$

which concludes the inductive proof of (3.3). Since  $d_j = H^j d_0 \rightarrow 0$ , Lemma 1 implies that  $x_j \rightarrow x^*$ ,  $x^* \in [x_j - d_j, x_j + d_j]$  for each  $j$  (in particular,  $x^* \in [x - d, x + d]$ ), and the sequence  $\{[x_j - d_j, x_j + d_j]\}_{j=0}^\infty$  is nested. Since  $F$  is continuous in  $[x - d, x + d]$  due to (i), taking  $j \rightarrow \infty$  in (3.2) we obtain that  $x^*$  solves (3.1). Let  $\tilde{x}$  be any solution to (3.1). Then

$$|\tilde{x} - x^*| = |F(\tilde{x}) - F(x^*)| \leq H|\tilde{x} - x^*|,$$

hence

$$(I - H)|\tilde{x} - x^*| \leq 0$$

and premultiplying this inequality by the nonnegative matrix  $(I - H)^{-1}$  yields  $|\tilde{x} - x^*| \leq 0$ , hence  $\tilde{x} = x^*$ . Thus  $x^*$  is the unique solution of (3.1) in  $[x - d, x + d]$ . ■

The assumption (i) is satisfied if  $F$  is differentiable and

$$\left| \frac{\partial F_i}{\partial x_j}(x') \right| \leq H_{ij}$$

holds for each  $x' \in [x - d, x + d]$ . Thus the values  $H_{ij}$  can be computed using interval extensions of the partial derivatives, see [1].

## 4 Nonexistence test

**Theorem 2** *Let  $F$  map an interval vector  $[x - d, x + d] \subset \mathbb{R}^n$  into  $\mathbb{R}^n$ , and let it satisfy the assumption (i) of Theorem 1 for some nonnegative matrix  $H$ . If there exists an  $x' \in [x - d, x + d]$  satisfying*

$$|x'_i - F_i(x')| > 2((I + H)d)_i \quad (4.1)$$

for some  $i$ , then the equation

$$\tilde{x} = F(\tilde{x})$$

does not have a solution in  $[x - d, x + d]$ .

*Proof.* Assume to the contrary that  $x^* = F(x^*)$  holds for some  $x^* \in [x - d, x + d]$ . Then we have

$$|x' - F(x')| = |x' - x^* + F(x^*) - F(x')| \leq (I + H)|x' - x^*| \leq 2(I + H)d,$$

which contradicts (4.1). ■

## 5 The linear case

For a system of linear equations

$$Ax = b$$

( $A$  square), using a nonsingular matrix  $R$ , we can write equivalently

$$x = (I - RA)x + Rb$$

and for  $F(x) := (I - RA)x + Rb$  we have

$$|F(x') - F(x'')| \leq |I - RA| \cdot |x' - x''|,$$

hence the assumption (i) of Theorem 1 is satisfied for

$$H := |I - RA|.$$

Thus Theorem 1 generalizes both the "enclosure theorem" and the "refinement theorem" given for the linear case in our earlier paper [2]. Like in [2], Brouwer's fixed-point theorem has not been used in the proof of the main result.

# Bibliography

- [1] H. Ratschek and J. Rokne, *New Computer Methods for Global Optimization*, Ellis Horwood Ltd., Chichester (England) 1988
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