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Abstract

It is proved that computing enclosures of solutions of linear interval equations with overestimation bounded by a polynomial in the system size is NP-hard.

Keywords
Linear interval equations, enclosure, NP-hardness

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1 Introduction

Solving linear interval equations usually means computing enclosures. For a system of linear interval equations

$$A^I x = b^I \tag{1.1}$$

(A^I square), *enclosure* is defined as an interval vector $[\underline{y}, \overline{y}]$ satisfying

$$X \subseteq [\underline{y}, \overline{y}]$$

where X is the solution set:

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}.$$

Various enclosure methods can be found in Alefeld and Herzberger [1] or Neumaier [7]. If A^I is regular (i.e., each $A \in A^I$ is nonsingular), then there exists the narrowest (or: optimal) enclosure $[\underline{x}, \overline{x}]$ given by

$$\underline{x}_i = \min_X x_i,$$

$$\overline{x}_i = \max_X x_i$$

for each i. Computing $[\underline{x}, \overline{x}]$ was proved to be NP-hard (Rohn and Kreinovich [12]; also, Kreinovich, Lakeyev and Noskov [6] for the rectangular case). In the main result of this paper we show that computing enclosures with overestimation bounded by a polynomial in the system size is NP-hard. The result holds true even for a very restricted class of systems (1.1) with $A^I = [A_c - \Delta, A_c + \Delta]$ having nondegenerate interval coefficients in one row only and satisfying $\varrho(|A_c^{-1}|\Delta) = 0$. Hence, the problem of computing sufficiently narrow enclosures turns out to be more difficult than previously believed. Three case studies illustrate some implications of the result.

2 Preliminaries

A real symmetric $n \times n$ matrix $A = (a_{ij})$ is called an MC-matrix [10] if it is of the form

$$a_{ij} \begin{cases} = n & \text{if } \ i = j \\ \in \{0, -1\} & \text{if } \ i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$. In the proof of the main theorem we shall essentially utilize the following result [11, Corollary 7] concerning the norm

$$\|A\|_{\infty,1} = \max\{\|Ax\|_1;\ \|x\|_\infty = 1\}$$

(where $\|x\|_1 = \sum_i |x|_i$ and $\|x\|_\infty = \max_i |x_i|$; see Golub and van Loan [3, p. 15]):

Proposition 1 *Computing $\|A\|_{\infty,1}$ is NP-hard for MC-matrices.*

Next we introduce a class of systems (1.1) of a special form. For each rational number $\varepsilon > 0$, let us denote by H_ε the family of systems of linear interval equations

$$A^I x = b^I$$

with A^I of the form

$$A^I = \begin{pmatrix} 1 & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A^{-1} \end{pmatrix}, \tag{2.1}$$

where A is an $n \times n$ MC-matrix (n arbitrary, $n \geq 1$) and $e = (1, 1, \ldots, 1)^T \in R^n$ (i.e., A^I is $(n+1) \times (n+1)$), and

$$b^I = \begin{pmatrix} 0 \\ [-\beta e, \beta e] \end{pmatrix} \tag{2.2}$$

for some (but arbitrary) rational $\beta > 0$. If we write (2.1) as

$$A^I = [A_c - \Delta, A_c + \Delta],$$

then

$$A_c = \begin{pmatrix} 1 & 0^T \\ 0 & A^{-1} \end{pmatrix}$$

is symmetric positive definite [10, p. 795], the radius matrix

$$\Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix}$$

has nonzero coefficients in the first row only, and

$$|A_c^{-1}|\Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

hence

$$\varrho(|A_c^{-1}|\Delta) = 0. \tag{2.4}$$

Thus an interval matrix (2.1) is strongly regular (i.e. $\varrho(|A_c^{-1}|\Delta) < 1$, cf. [7]); problems with strongly regular interval matrices have been usually considered "tractable".

In order to be able to formulate a unifying complexity result, we introduce the following concept: *enclosure algorithm* is an algorithm which for each system $A^I x = b^I$ with rational data (and square A^I) in a finite number of steps either computes a rational enclosure, or fails (i.e., issues an error message). Failure of an enclosure algorithm may be caused by various reasons: 1) no enclosure exists since the solution set is unbounded (in case of a singular A^I), 2) the algorithm cannot be continued (e.g. in case of the interval Gaussian algorithm), 3) the algorithm works under some condition only (e.g., strong regularity), 4) a prescribed number of steps has been reached, etc.

3 Main result

Theorem 1 *If $P \neq NP$ holds, then each polynomial-time enclosure algorithm has the following property: for each rational $\varepsilon > 0$,*

- *either it fails for some system in H_ε,*
- *or for each rational $\alpha > 0$ and each integer $k \geq 0$ there exists a system of size $n \geq 2$ in H_ε for which the enclosure $[\underline{y}, \overline{y}]$ computed by the algorithm satisfies*

$$\underline{y}_1 \leq \underline{x}_1 - \alpha n^k < \overline{x}_1 + \alpha n^k \leq \overline{y}_1. \tag{3.1}$$

Comments. 1) P and NP are the well-known complexity classes. The conjecture that P$\neq$NP, although unproved, is widely believed to be true (cf. Garey and Johnson [2]). 2) If the conjecture holds true, then each polynomial-time enclosure algorithm which works for at least one family H_ε may produce arbitrarily large overestimations (3.1); hence, no (even arbitrarily bad) accuracy can be guaranteed to be achievable by a polynomial-time enclosure algorithm.

Proof. Assume to the contrary that there exists a polynomial-time enclosure algorithm, rational numbers $\varepsilon > 0$, $\alpha > 0$ and an integer $k \geq 0$ such that for each system in H_ε the algorithm computes an enclosure $[\underline{y}, \overline{y}]$ satisfying either

$$\underline{x}_1 - \alpha n^k < \underline{y}_1$$

or

$$\overline{y}_1 < \overline{x}_1 + \alpha n^k$$

where n is the system size. Let A be an arbitrary MC-matrix of size m. Let us construct an $(m+1) \times (m+1)$ interval matrix

$$A^I = \begin{pmatrix} 1 & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A^{-1} \end{pmatrix}$$

and an $(m+1)$-dimensional interval vector

$$b^I = \begin{pmatrix} 0 \\ [-\frac{\gamma}{\varepsilon} e, \frac{\gamma}{\varepsilon} e] \end{pmatrix},$$

where

$$\gamma = \alpha (m+1)^k,$$

and apply the algorithm to the system

$$A^I x = b^I \tag{3.2}$$

(which obviously belongs to H_ε) to compute an enclosure $[\underline{y}, \overline{y}]$ which, according to the assumption, satisfies either

$$\underline{x}_1 - \gamma < \underline{y}_1 \tag{3.3}$$

or

$$\overline{y}_1 < \overline{x}_1 + \gamma. \tag{3.4}$$

This can be done in polynomial time. We shall prove that

$$\|A\|_{\infty,1} = \left[\frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\}\right] \tag{3.5}$$

holds, where $[\ldots]$ denotes the integer part. Hence, $\|A\|_{\infty,1}$ can be computed in polynomial time; but since this is an NP-hard problem (Proposition 1), P=NP will follow. To prove (3.5), first observe that the system (3.2) can be written as

$$x_1 + [-\varepsilon e^T, \varepsilon e^T]x' = 0,$$

$$-\frac{\gamma}{\varepsilon}e \le A^{-1}x' \le \frac{\gamma}{\varepsilon}e,$$

where $x' = (x_2, \ldots, x_m)^T$. Hence

$$\begin{aligned} \overline{x}_1 &= \max\{\varepsilon e^T|x'|;\ -\frac{\gamma}{\varepsilon}e \le A^{-1}x' \le \frac{\gamma}{\varepsilon}e\} \\ &= \gamma\max\{\|x''\|_1;\ -e \le A^{-1}x'' \le e\} \\ &= \gamma\max\{\|Ax'''\|_1;\ -e \le x''' \le e\} \\ &= \gamma\max\{\|Ax'''\|_1;\ \|x'''\|_\infty = 1\} \\ &= \gamma\|A\|_{\infty,1} \end{aligned}$$

and in a quite similar way,

$$\underline{x}_1 = -\gamma\|A\|_{\infty,1}.$$

Hence from (3.3) and (3.4) we obtain that either

$$-\frac{1}{\gamma}\underline{y}_1 < \|A\|_{\infty,1} + 1$$

or

$$\frac{1}{\gamma}\overline{y}_1 < \|A\|_{\infty,1} + 1$$

holds, in both the cases

$$\frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\} < \|A\|_{\infty,1} + 1. \tag{3.6}$$

But since $[\underline{y}_1, \overline{y}_1]$ encloses $[\underline{x}_1, \overline{x}_1]$, from $\underline{y}_1 \le \underline{x}_1$, $\overline{x}_1 \le \overline{y}_1$ we have

$$\|A\|_{\infty,1} \le \frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\}$$

which together with (3.6) gives

$$\|A\|_{\infty,1} \le \frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\} < \|A\|_{\infty,1} + 1. \tag{3.7}$$

However, the number

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty=1} \|Ax\|_1 = \max\{\|Ax\|_1;\ x_j \in \{-1,1\} \text{ for each } j\}$$

is integer for an MC-matrix A (which is integer by definition), hence from (3.7) we finally obtain

$$\|A\|_{\infty,1} = \left\lceil \frac{1}{\gamma} \min\{-\underline{y}_1, \overline{y}_1\} \right\rceil,$$

which is (3.5). Hence, $\|A\|_{\infty,1}$ can be computed in polynomial time for an MC-matrix A, which in view of Proposition 1 implies that P=NP. This concludes the proof by contradiction. □

4 Application 1: interval Gaussian algorithm

For each rational $\varepsilon > 0$, the interval Gaussian algorithm with partial pivoting [1], [7] (which is polynomial-time) is performable for each system in H_ε since all the pivots are real and nonzero due to the special form of the system matrix (2.1). Hence, if P$\neq$NP, then arbitrarily large overestimations (3.1) may occur for arbitrarily narrow system matrices (2.1).

5 Application 2: explicit bounds

For a system (1.1) with a strongly regular interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ and a right-hand side $b^I = [b_c - \delta, b_c + \delta]$, the classical approach using Neumann series (or the Oettli-Prager inequality) gives

$$|x - x_c| \le d$$

for each x in the solution set X, where $x_c = A_c^{-1} b_c$ and

$$d = (I - |A_c^{-1}|\Delta)^{-1} |A_c^{-1}| (\Delta |x_c| + \delta).$$

This enables us to construct the following polynomial-time enclosure algorithm: if $\| |A_c^{-1}|\Delta \|_1 < 1$, then the enclosure is $[x_c - d, x_c + d]$, otherwise it fails. In view of (2.3), the algorithm works for each H_ε with $\varepsilon \in (0,1)$. Hence, if P$\neq$NP, then for each rational $\varepsilon \in (0,1)$, $\alpha > 0$ and each integer $k \ge 0$ there exists a system (1.1) of size $n \ge 2$ with $\max_{ij} \Delta_{ij} = \varepsilon$ for which

$$|x - x_c| + \alpha n^k \le d$$

holds for each $x \in X$. Thus, an overestimation in d may get arbitrarily large for arbitrarily narrow interval matrices.

6 Application 3: preconditioning

For a system (1.1) with a strongly regular interval matrix A^I, the system matrix of the preconditioned system

$$A_c^{-1}A^I x = A_c^{-1}b^I \tag{6.1}$$

(multiplication performed in interval arithmetic) is regular and the *optimal* enclosure $[\underline{\underline{x}}, \overline{\overline{x}}]$ for (6.1), which encloses $[\underline{x}, \overline{x}]$ (cf. [7]), can be computed in polynomial time (Hansen [4], Rohn [9]). Hence, we can construct the following polynomial-time enclosure algorithm for (1.1): if $\| |A_c^{-1}|\Delta\|_1 < 1$, then the enclosure is $[\underline{\underline{x}}, \overline{\overline{x}}]$, otherwise it fails. Due to (2.3), the algorithm works for each H_ε, $\varepsilon \in (0, 1)$. Hence the main result implies that under the assumption P$\neq$NP, the optimal enclosure $[\underline{\underline{x}}, \overline{\overline{x}}]$ of the preconditioned system (6.1) may overestimate the optimal enclosure $[\underline{x}, \overline{x}]$ of the original system (1.1) by an arbitrary prescribed value even for arbitrarily narrow system matrices.

7 Concluding remark

Theorem 1 is a worst-case result which relies heavily on the fact that the right-hand side (2.2) of each system in H_ε has a zero midpoint. As a result, the solution set X stretches into all the 2^n orthants. This is not a typical situation. In practical computations the solution set is often a part of a single orthant; in this case the optimal enclosure $[\underline{x}, \overline{x}]$ can be computed by a linear programming technique in polynomial time (Oettli [8], Khachiyan [5]).

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