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## ACADEMY OF SCIENCES OF THE CZECH REPUBLIC


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### Abstract

A characterization of interval $P$–matrices is given. The result implies that a symmetric interval matrix is a $P$–matrix if and only if it is positive definite (although nonsymmetric matrices may be involved). As a consequence it is proved that the problem of checking whether a symmetric interval matrix is a $P$–matrix is NP–hard.



### Keywords

interval matrix, $P$–matrix, positive definiteness, NP–hardness



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# 1 Introduction

As is well known, an $n \times n$ matrix $A$ is called a $P$–matrix if all its principal minors are positive. $P$–matrices play an important role in several areas, e.g. in the linear complementarity theory since they guarantee existence and uniqueness of the solution of a linear complementarity problem (see Murty [6]).

A basic characterization of $P$–matrices was given by Fiedler and Pták [3]: $A$ is a $P$–matrix if and only if for each $x \in R^n, x \neq 0$ there exists an $i$ such that $x_i(Ax)_i > 0$ holds. This result immediately implies that a *symmetric* matrix $A$ is a $P$–matrix if and only if it is positive definite. In fact, if $A$ is positive definite, then for each $x \neq 0$, from $\sum_i x_i(Ax)_i = x^T Ax > 0$ it follows that $x_i(Ax)_i > 0$ for some $i$, hence $A$ is a $P$–matrix; conversely, if $A$ is a $P$–matrix, then it is positive definite in view of the Sylvester determinant criterion [6].

In this paper we focus our attention on interval $P$–matrices. An interval matrix

$$A^I = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}$ and $\overline{A}$ are $n \times n$ matrices satisfying $\underline{A} \leq \overline{A}$ (componentwise), is said to be a $P$–matrix if each $A \in A^I$ is a $P$–matrix . In section 2 we introduce a finite set of matrices $A_z$ in $A^I$ (whose cardinality is at most $2^{n-1}$) such that $A^I$ is a $P$–matrix if and only if all the matrices $A_z$ are $P$–matrices (Theorem 3). In view of a similar characterization of positive definiteness of $A^I$ via the matrices $A_z$ (Theorem 4), it is then proved in section 3 that a symmetric interval matrix $A^I$ (i.e., with symmetric bounds $\underline{A}$, $\overline{A}$) is a $P$–matrix if and only if it is positive definite (Theorem 6). This is a generalization of the above result for real symmetric matrices, but it is not a simple consequence of it since here nonsymmetric matrices may be involved. As a consequence of this result we obtain that the problem of checking whether a symmetric interval matrix is a $P$–matrix is NP–hard (Theorem 8). This result shows that the exponential number of test matrices $A_z$ used in the necessary and sufficient condition of Theorem 3 is highly unlikely to be essentially reducible.

# 2 Characterizations

Let us introduce an auxiliary set

$$Z = \{z \in R^n;\ z_j \in \{-1, 1\} \text{ for } j = 1, \ldots, n\},$$

i.e. the set of all $\pm 1$-vectors. The cardinality of $Z$ is obviously $2^n$. For an interval matrix

$$A^I = [\underline{A}, \overline{A}],$$

we define matrices $A_z, z \in Z$ by

$$(A_z)_{ij} = \frac{1}{2}(\underline{A}_{ij} + \overline{A}_{ij}) - \frac{1}{2}(\overline{A}_{ij} - \underline{A}_{ij})z_i z_j$$

$(i, j = 1, \ldots, n)$. Clearly, $(A_z)_{ij} = \underline{A}_{ij}$ if $z_i z_j = 1$ and $(A_z)_{ij} = \overline{A}_{ij}$ if $z_i z_j = -1$, hence $A_z \in A^I$ for each $z \in Z$, and the number of mutually different matrices $A_z$ is at most

$2^{n-1}$ (since $A_{-z} = A_z$ for each $z \in Z$), and equal to $2^{n-1}$ if $\underline{A} < \overline{A}$. The properties in question ($P$–property and positive definiteness) will be formulated below in terms of the finite set of matrices $A_z, z \in Z$. For a vector $x \in R^n$, let us define its sign vector

$$z = \operatorname{sgn} x$$

by

$$z_i = \begin{cases} \phantom{-}1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}$$

($i = 1, \ldots, n$), so that $\operatorname{sgn} x \in Z$. For a matrix $A = (A_{ij})$ we introduce its absolute value by $|A| = (|A_{ij}|)$; a similar notation also applies to vectors.

The basic property of the matrices $A_z, z \in Z$, is summed up in the following auxiliary result; notice that no assumptions on $A^I$ are made.

**Theorem 1** *Let $A^I$ be an $n \times n$ interval matrix, $x \in R^n$, and let $z = \operatorname{sgn} x$. Then for each $A \in A^I$ and each $i \in \{1, \ldots, n\}$ we have*

$$x_i(Ax)_i \geq x_i(A_z x)_i. \tag{1}$$

*Proof.* Let $A \in A^I$ and $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} |x_i(Ax)_i - x_i((\frac{1}{2}(\underline{A} + \overline{A}))x)_i| &= |x_i((A - \frac{1}{2}(\underline{A} + \overline{A}))x)_i| \\ &\leq |x_i|((|A - \frac{1}{2}(\underline{A} + \overline{A})| \cdot |x|)_i \leq |x_i|(\frac{1}{2}(\overline{A} - \underline{A})|x|)_i, \end{aligned}$$

hence

$$x_i(Ax)_i \geq x_i((\frac{1}{2}(\underline{A} + \overline{A}))x)_i - |x_i|(\frac{1}{2}(\overline{A} - \underline{A})|x|)_i.$$

Since $z = \operatorname{sgn} x$, we have $|x_j| = z_j x_j$ for each $j$, hence

$$\begin{aligned} x_i(Ax)_i &\geq \sum_j (\frac{1}{2}(\underline{A}_{ij} + \overline{A}_{ij}) - \frac{1}{2}(\overline{A}_{ij} - \underline{A}_{ij})z_i z_j)x_i x_j \\ &= \sum_j (A_z)_{ij} x_i x_j = x_i(A_z x)_i, \end{aligned}$$

which concludes the proof. □

As the first consequence of this result, we prove a Fiedler–Pták type characterization of interval $P$–matrices. Notice that the inequality holds "uniformly" here:

**Theorem 2** *An interval matrix $A^I$ is a $P$–matrix if and only if for each $x \in R^n$, $x \neq 0$, there exists an $i \in \{1, \ldots, n\}$ such that*

$$x_i(Ax)_i > 0 \tag{2}$$

*holds for each $A \in A^I$.*

*Proof.* If (2) holds, then each $A \in A^I$ is a $P$–matrix by the Fiedler-Pták theorem. Conversely, let $A^I$ be a $P$–matrix and let $x \neq 0$. Put $z = \operatorname{sgn} x$, then $A_z$ is a $P$–matrix, hence by the Fiedler-Pták theorem we have $x_i(A_z x)_i > 0$ for some $i$. Then (1) implies $x_i(Ax)_i \geq x_i(A_z x)_i > 0$ for each $A \in A^I$, and we are done. □

The following characterization, however, turns out to be much more useful:

**Theorem 3** *$A^I$ is a $P$–matrix if and only if each $A_z, z \in Z$, is a $P$–matrix.*

*Proof.* If $A^I$ is a $P$–matrix, then each $A_z \in A^I$ is obviously also a $P$–matrix. Conversely, let each $A_z, z \in Z$, be a $P$–matrix. Let $x \in R^n$, $x \neq 0$, and let $z = \operatorname{sgn} x$. Since $A_z$ is a $P$–matrix, there exists an $i$ with $x_i(A_z x)_i > 0$, then from Theorem 1 we obtain $x_i(Ax)_i \geq x_i(A_z x)_i > 0$ for each $A \in A^I$, hence $A^I$ is a $P$–matrix by Theorem 2. $\Box$

Another finite characterization of interval $P$–matrices, formulated in different terms, was proved by Białas and Garloff [1].

In analogy with the terminology introduced for $P$–matrices, an interval matrix $A^I$ is said to be positive definite if each $A \in A^I$ is positive definite (i.e., satisfies $x^T Ax > 0$ for each $x \neq 0$). The following theorem was proved in [9, Thm. 2]. We give here another proof of this result to make the paper self-contained and to demonstrate that it is a simple consequence of Theorem 1:

**Theorem 4** *$A^I$ is positive definite if and only if each $A_z, z \in Z$, is positive definite.*

*Proof.* The "only if" part is obvious since $A_z \in A^I$ for each $z \in Z$. To prove the "if" part, take an $A \in A^I$ and $x \in R^n, x \neq 0$. For $z = \operatorname{sgn} x$, from Theorem 1 we have

$$x_i(Ax)_i \geq x_i(A_z x)_i$$

for each $i$, hence

$$x^T Ax = \sum_i x_i(Ax)_i \geq \sum_i x_i(A_z x)_i = x^T A_z x > 0,$$

so that $A$ is positive definite. Thus, by definition, $A^I$ is positive definite. $\Box$

The last two theorems reveal that both the $P$–property and positive definiteness of interval matrices are characterized by the same finite subset of matrices $A_z \in A^I$, $z \in Z$. This relationship will become even more apparent in the case of symmetric interval matrices which we shall consider in the next section.

# 3 Symmetric interval matrices

For an interval matrix $A^I = [\underline{A}, \overline{A}]$, define an associated interval matrix $A^I_s$ by

$$A^I_s = [\frac{1}{2}(\underline{A} + \underline{A}^T), \frac{1}{2}(\overline{A} + \overline{A}^T)].$$

$A^I$ is called *symmetric* if $A^I = A^I_s$, which is clearly the case if and only if both $\underline{A}$ and $\overline{A}$ are symmetric. Hence, $A^I_s$ is always a symmetric interval matrix. The relationship between positive definiteness and $P$–property is provided by the following theorem:

**Theorem 5** *$A^I$ is positive definite if and only if $A^I_s$ is a $P$–matrix.*

*Proof.* For each $z \in Z$, let us denote by $A^s_z$ the matrix $A_z$ for $A^I_s$, i.e.

$$(A^s_z)_{ij} = \frac{1}{4}(\underline{A}_{ij} + \underline{A}_{ji} + \overline{A}_{ij} + \overline{A}_{ji}) - \frac{1}{4}(\overline{A}_{ij} + \overline{A}_{ji} - \underline{A}_{ij} - \underline{A}_{ji})z_i z_j$$

$(i, j = 1, \ldots, n)$. Then $A^s_z$ is symmetric and a direct computation shows that

$$x^T A^s_z x = x^T A_z x \tag{3}$$

holds for each $x \in R^n$. Now, if $A^I$ is positive definite, then each $A_z, z \in Z$ is positive definite, hence each $A^s_z$ is positive definite due to (3), so that $A^s_z$ is a $P$–matrix, hence $A^I_s$ is a $P$–matrix by Theorem 3. Conversely, if $A^I_s$ is a $P$–matrix, then each $A^s_z, z \in Z$ is a $P$–matrix, hence it is positive definite due to its symmetry, thus each $A_z, z \in Z$ is positive definite by (3) and $A^I$ is positive definite by Theorem 4. □

Our main result on symmetric interval matrices is now obtained as a simple consequence of Theorem 5.

**Theorem 6** *A symmetric interval matrix $A^I$ is a $P$–matrix if and only if it is positive definite.*

*Proof.* The result follows immediately from Theorem 5 since a symmetric interval matrix $A^I$ satisfies $A^I = A^I_s$ by definition. □

At the beginning of the Introduction we showed that a real symmetric matrix is a $P$–matrix if and only if it is positive definite. The result of Theorem 6 sounds verbally alike, but it is not a simple consequence of the real case since here nonsymmetric matrices may be involved. In fact, it can be immediately seen that a symmetric interval matrix $A^I = [\underline{A}, \overline{A}]$ contains nonsymmetric matrices if and only if $\underline{A}_{ij} < \overline{A}_{ij}$ holds for some $i \neq j$.

An interval matrix $A^I$ is called regular (cf. Neumaier [7]) if each $A \in A^I$ is nonsingular. The following result shows that for symmetric interval matrices the $P$–property is preserved by regularity. Several other results of this type are summed up in [10].

**Theorem 7** *A symmetric interval matrix $A^I$ is a $P$–matrix if and only if it is regular and contains at least one symmetric $P$–matrix.*

*Proof.* A symmetric interval $P$–matrix $A^I$ is regular (each $A \in A^I$ has a positive determinant) and contains a symmetric $P$–matrix $\underline{A}$. If $A^I$ is regular and contains a symmetric $P$–matrix $A_0$, then $A_0$ is positive definite, hence $A^I$ is positive definite by Theorem 3 in [9], which in the light of Theorem 6 means that $A^I$ is a $P$–matrix. □

Another relationship between regularity and $P$–property of interval matrices was established in [8, Thm. 5.1, assert. (B1)]: an interval matrix $A^I = [\underline{A}, \overline{A}]$ is regular if and only if $(\underline{A} + \overline{A} - S(\overline{A} - \underline{A}))^{-1}(\underline{A} + \overline{A} + S(\overline{A} - \underline{A}))$ is a $P$–matrix for each signature matrix $S$ (i.e., a diagonal matrix with $\pm 1$ diagonal elements). This topic was recently studied by Johnson and Tsatsomeros [5].

The necessary and sufficient condition of Theorem 3 employs up to $2^{n-1}$ test matrices $A_z, z \in Z$. There is a natural question whether an essentially simpler criterion could be found. The last theorem gives an indirect answer to this question: it implies that an existence of a polynomial-time algorithm for checking the $P$–property of symmetric interval matrices would imply that the complexity classes P and NP are equal, thereby running contrary to the current (unproved) conjecture that P≠NP. We refer the reader to the classical book by Garey and Johnson [4] for a detailed discussion of the problem "P=NP" and related issues.

**Theorem 8** *The following problem is NP-hard:*
Instance. *A symmetric interval matrix* $A^I = [\underline{A}, \overline{A}]$ *with rational bounds* $\underline{A}, \overline{A}$*.*
Question. *Is* $A^I$ *a* $P$*–matrix?*

*Proof.* By Theorem 6, $A^I$ is a $P$–matrix if and only if it is positive definite; checking positive definiteness of symmetric interval matrices was proved to be NP-hard in [11]. □

Coxson [2] proved that the $P$–matrix problem for real matrices is co-NP-complete. His result concerns nonsymmetric matrices since the symmetric case can be solved by Sylvester determinant criterion which can be performed in polynomial time (Schrijver [12]). Theorem 8 shows that for interval matrices even the symmetric case is NP-hard.

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