národní
úložiště
šedé
literatury

## Interval P-Matrices

Rohn, Jiří
1994
Dostupný z http://www.nusl.cz/ntk/nusl-33567

Dílo je chráněno podle autorského zákona č. 121/2000 Sb.

Tento dokument byl stažen z Národního úložiště šedé literatury (NUŠL).
Datum stažení: 03.04.2024
Další dokumenty můžete najít prostřednictvím vyhledávacího rozhraní nusl.cz .

# Interval $P$-Matrices 

Jiří Rohn and Georg Rex

Technical report No. 618

December 1994

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 18207 Prague 8 , Czech Republic phone: $(+422) 66414244$ fax: $(+422) 8585789$
e-mail: rohn@uivt.cas.cz

# Interval $P$-Matrices 

Jiří Rohn ${ }^{1}$ and Georg Rex ${ }^{2}$

Technical report No. 618
December 1994


#### Abstract

A characterization of interval $P$-matrices is given. The result implies that a symmetric interval matrix is a $P$-matrix if and only if it is positive definite (although nonsymmetric matrices may be involved). As a consequence it is proved that the problem of checking whether a symmetric interval matrix is a $P$-matrix is NP-hard.


## Keywords

interval matrix, $P$-matrix, positive definiteness, NP-hardness

[^0]
## 1 Introduction

As is well known, an $n \times n$ matrix $A$ is called a $P$-matrix if all its principal minors are positive. $P$-matrices play an important role in several areas, e.g. in the linear complementarity theory since they guarantee existence and uniqueness of the solution of a linear complementarity problem (see Murty [6]).

A basic characterization of $P$-matrices was given by Fiedler and Pták [3]: $A$ is a $P$-matrix if and only if for each $x \in R^{n}, x \neq 0$ there exists an $i$ such that $x_{i}(A x)_{i}>0$ holds. This result immediately implies that a symmetric matrix $A$ is a $P$-matrix if and only if it is positive definite. In fact, if $A$ is positive definite, then for each $x \neq 0$, from $\sum_{i} x_{i}(A x)_{i}=x^{T} A x>0$ it follows that $x_{i}(A x)_{i}>0$ for some $i$, hence $A$ is a $P$-matrix; conversely, if $A$ is a $P$-matrix, then it is positive definite in view of the Sylvester determinant criterion [6].

In this paper we focus our attention on interval $P$-matrices. An interval matrix

$$
A^{I}=[\underline{A}, \bar{A}]=\{A ; \underline{A} \leq A \leq \bar{A}\}
$$

where $\underline{A}$ and $\bar{A}$ are $n \times n$ matrices satisfying $\underline{A} \leq \bar{A}$ (componentwise), is said to be a $P-$ matrix if each $A \in A^{I}$ is a $P$-matrix. In section 2 we introduce a finite set of matrices $A_{z}$ in $A^{I}$ (whose cardinality is at most $2^{n-1}$ ) such that $A^{I}$ is a $P$-matrix if and only if all the matrices $A_{z}$ are $P$-matrices (Theorem 3). In view of a similar characterization of positive definiteness of $A^{I}$ via the matrices $A_{z}$ (Theorem 4), it is then proved in section 3 that a symmetric interval matrix $A^{I}$ (i.e., with symmetric bounds $\underline{A}, \bar{A}$ ) is a $P$-matrix if and only if it is positive definite (Theorem 6). This is a generalization of the above result for real symmetric matrices, but it is not a simple consequence of it since here nonsymmetric matrices may be involved. As a consequence of this result we obtain that the problem of checking whether a symmetric interval matrix is a $P-$ matrix is NP-hard (Theorem 8). This result shows that the exponential number of test matrices $A_{z}$ used in the necessary and sufficient condition of Theorem 3 is highly unlikely to be essentially reducible.

## 2 Characterizations

Let us introduce an auxiliary set

$$
Z=\left\{z \in R^{n} ; z_{j} \in\{-1,1\} \text { for } j=1, \ldots, n\right\}
$$

i.e. the set of all $\pm 1$-vectors. The cardinality of $Z$ is obviously $2^{n}$. For an interval matrix

$$
A^{I}=[\underline{A}, \bar{A}],
$$

we define matrices $A_{z}, z \in Z$ by

$$
\left(A_{z}\right)_{i j}=\frac{1}{2}\left(\underline{A}_{i j}+\bar{A}_{i j}\right)-\frac{1}{2}\left(\bar{A}_{i j}-\underline{A}_{i j}\right) z_{i} z_{j}
$$

$(i, j=1, \ldots, n)$. Clearly, $\left(A_{z}\right)_{i j}=\underline{A}_{i j}$ if $z_{i} z_{j}=1$ and $\left(A_{z}\right)_{i j}=\bar{A}_{i j}$ if $z_{i} z_{j}=-1$, hence $A_{z} \in A^{I}$ for each $z \in Z$, and the number of mutually different matrices $A_{z}$ is at most
$2^{n-1}$ (since $A_{-z}=A_{z}$ for each $z \in Z$ ), and equal to $2^{n-1}$ if $\underline{A}<\bar{A}$. The properties in question ( $P$-property and positive definiteness) will be formulated below in terms of the finite set of matrices $A_{z}, z \in Z$. For a vector $x \in R^{n}$, let us define its sign vector

$$
z=\operatorname{sgn} x
$$

by

$$
z_{i}=\left\{\begin{aligned}
1 & \text { if } x_{i} \geq 0 \\
-1 & \text { if } x_{i}<0
\end{aligned}\right.
$$

$(i=1, \ldots, n)$, so that $\operatorname{sgn} x \in Z$. For a matrix $A=\left(A_{i j}\right)$ we introduce its absolute value by $|A|=\left(\left|A_{i j}\right|\right)$; a similar notation also applies to vectors.

The basic property of the matrices $A_{z}, z \in Z$, is summed up in the following auxiliary result; notice that no assumptions on $A^{I}$ are made.

Theorem 1 Let $A^{I}$ be an $n \times n$ interval matrix, $x \in R^{n}$, and let $z=\operatorname{sgn} x$. Then for each $A \in A^{I}$ and each $i \in\{1, \ldots, n\}$ we have

$$
\begin{equation*}
x_{i}(A x)_{i} \geq x_{i}\left(A_{z} x\right)_{i} \tag{1}
\end{equation*}
$$

Proof. Let $A \in A^{I}$ and $i \in\{1, \ldots, n\}$. Then

$$
\begin{aligned}
& \left|x_{i}(A x)_{i}-x_{i}\left(\left(\frac{1}{2}(\underline{A}+\bar{A})\right) x\right)_{i}\right|=\left|x_{i}\left(\left(A-\frac{1}{2}(\underline{A}+\bar{A})\right) x\right)_{i}\right| \\
& \quad \leq\left|x_{i}\right|\left(\left|A-\frac{1}{2}(\underline{A}+\bar{A})\right| \cdot|x|\right)_{i} \leq\left|x_{i}\right|\left(\frac{1}{2}(\bar{A}-\underline{A})|x|\right)_{i},
\end{aligned}
$$

hence

$$
x_{i}(A x)_{i} \geq x_{i}\left(\left(\frac{1}{2}(\underline{A}+\bar{A})\right) x\right)_{i}-\left|x_{i}\right|\left(\frac{1}{2}(\bar{A}-\underline{A})|x|\right)_{i} .
$$

Since $z=\operatorname{sgn} x$, we have $\left|x_{j}\right|=z_{j} x_{j}$ for each $j$, hence

$$
\begin{aligned}
x_{i}(A x)_{i} & \geq \sum_{j}\left(\frac{1}{2}\left(\underline{A}_{i j}+\bar{A}_{i j}\right)-\frac{1}{2}\left(\bar{A}_{i j}-\underline{A}_{i j}\right) z_{i} z_{j}\right) x_{i} x_{j} \\
& =\sum_{j}\left(A_{z}\right)_{i j} x_{i} x_{j}=x_{i}\left(A_{z} x\right)_{i}
\end{aligned}
$$

which concludes the proof.

As the first consequence of this result, we prove a Fiedler-Pták type characterization of interval $P$-matrices. Notice that the inequality holds "uniformly" here:

Theorem 2 An interval matrix $A^{I}$ is a $P$-matrix if and only if for each $x \in R^{n}$, $x \neq 0$, there exists an $i \in\{1, \ldots, n\}$ such that

$$
\begin{equation*}
x_{i}(A x)_{i}>0 \tag{2}
\end{equation*}
$$

holds for each $A \in A^{I}$.
Proof. If (2) holds, then each $A \in A^{I}$ is a $P$-matrix by the Fiedler-Pták theorem. Conversely, let $A^{I}$ be a $P$-matrix and let $x \neq 0$. Put $z=\operatorname{sgn} x$, then $A_{z}$ is a $P$-matrix, hence by the Fiedler-Pták theorem we have $x_{i}\left(A_{z} x\right)_{i}>0$ for some $i$. Then (1) implies $x_{i}(A x)_{i} \geq x_{i}\left(A_{z} x\right)_{i}>0$ for each $A \in A^{I}$, and we are done.

The following characterization, however, turns out to be much more useful:

Theorem $3 A^{I}$ is a $P$-matrix if and only if each $A_{z}, z \in Z$, is a $P$-matrix.
Proof. If $A^{I}$ is a $P$-matrix, then each $A_{z} \in A^{I}$ is obviously also a $P$-matrix. Conversely, let each $A_{z}, z \in Z$, be a $P$-matrix. Let $x \in R^{n}, x \neq 0$, and let $z=\operatorname{sgn} x$. Since $A_{z}$ is a $P$-matrix, there exists an $i$ with $x_{i}\left(A_{z} x\right)_{i}>0$, then from Theorem 1 we ob$\operatorname{tain} x_{i}(A x)_{i} \geq x_{i}\left(A_{z} x\right)_{i}>0$ for each $A \in A^{I}$, hence $A^{I}$ is a $P$-matrix by Theorem 2 .

Another finite characterization of interval $P$-matrices, formulated in different terms, was proved by Białas and Garloff [1].

In analogy with the terminology introduced for $P$-matrices, an interval matrix $A^{I}$ is said to be positive definite if each $A \in A^{I}$ is positive definite (i.e., satisfies $x^{T} A x>0$ for each $x \neq 0$ ). The following theorem was proved in [9, Thm. 2]. We give here another proof of this result to make the paper self-contained and to demonstrate that it is a simple consequence of Theorem 1:

Theorem $4 A^{I}$ is positive definite if and only if each $A_{z}, z \in Z$, is positive definite.
Proof. The "only if" part is obvious since $A_{z} \in A^{I}$ for each $z \in Z$. To prove the "if" part, take an $A \in A^{I}$ and $x \in R^{n}, x \neq 0$. For $z=\operatorname{sgn} x$, from Theorem 1 we have

$$
x_{i}(A x)_{i} \geq x_{i}\left(A_{z} x\right)_{i}
$$

for each $i$, hence

$$
x^{T} A x=\sum_{i} x_{i}(A x)_{i} \geq \sum_{i} x_{i}\left(A_{z} x\right)_{i}=x^{T} A_{z} x>0
$$

so that $A$ is positive definite. Thus, by definition, $A^{I}$ is positive definite.
The last two theorems reveal that both the $P$-property and positive definiteness of interval matrices are characterized by the same finite subset of matrices $A_{z} \in A^{I}$, $z \in Z$. This relationship will become even more apparent in the case of symmetric interval matrices which we shall consider in the next section.

## 3 Symmetric interval matrices

For an interval matrix $A^{I}=[\underline{A}, \bar{A}]$, define an associated interval matrix $A_{s}^{I}$ by

$$
A_{s}^{I}=\left[\frac{1}{2}\left(\underline{A}+\underline{A}^{T}\right), \frac{1}{2}\left(\bar{A}+\bar{A}^{T}\right)\right]
$$

$A^{I}$ is called symmetric if $A^{I}=A_{s}^{I}$, which is clearly the case if and only if both $\underline{A}$ and $\bar{A}$ are symmetric. Hence, $A_{s}^{I}$ is always a symmetric interval matrix. The relationship between positive definiteness and $P$-property is provided by the following theorem:

Theorem $5 A^{I}$ is positive definite if and only if $A_{s}^{I}$ is a $P$-matrix.
Proof. For each $z \in Z$, let us denote by $A_{z}^{s}$ the matrix $A_{z}$ for $A_{s}^{I}$, i.e.

$$
\left(A_{z}^{s}\right)_{i j}=\frac{1}{4}\left(\underline{A}_{i j}+\underline{A}_{j i}+\bar{A}_{i j}+\bar{A}_{j i}\right)-\frac{1}{4}\left(\bar{A}_{i j}+\bar{A}_{j i}-\underline{A}_{i j}-\underline{A}_{j i}\right) z_{i} z_{j}
$$

$(i, j=1, \ldots, n)$. Then $A_{z}^{s}$ is symmetric and a direct computation shows that

$$
\begin{equation*}
x^{T} A_{z}^{s} x=x^{T} A_{z} x \tag{3}
\end{equation*}
$$

holds for each $x \in R^{n}$. Now, if $A^{I}$ is positive definite, then each $A_{z}, z \in Z$ is positive definite, hence each $A_{z}^{s}$ is positive definite due to (3), so that $A_{z}^{s}$ is a $P$-matrix, hence $A_{s}^{I}$ is a $P$-matrix by Theorem 3. Conversely, if $A_{s}^{I}$ is a $P$-matrix, then each $A_{z}^{s}, z \in Z$ is a $P$-matrix, hence it is positive definite due to its symmetry, thus each $A_{z}, z \in Z$ is positive definite by (3) and $A^{I}$ is positive definite by Theorem 4.

Our main result on symmetric interval matrices is now obtained as a simple consequence of Theorem 5.

Theorem $6 A$ symmetric interval matrix $A^{I}$ is a $P$-matrix if and only if it is positive definite.

Proof. The result follows immediately from Theorem 5 since a symmetric interval matrix $A^{I}$ satisfies $A^{I}=A_{s}^{I}$ by definition.

At the beginning of the Introduction we showed that a real symmetric matrix is a $P$-matrix if and only if it is positive definite. The result of Theorem 6 sounds verbally alike, but it is not a simple consequence of the real case since here nonsymmetric matrices may be involved. In fact, it can be immediately seen that a symmetric interval matrix $A^{I}=[\underline{A}, \bar{A}]$ contains nonsymmetric matrices if and only if $\underline{A}_{i j}<\bar{A}_{i j}$ holds for some $i \neq j$.

An interval matrix $A^{I}$ is called regular (cf. Neumaier [7]) if each $A \in A^{I}$ is nonsingular. The following result shows that for symmetric interval matrices the $P$-property is preserved by regularity. Several other results of this type are summed up in [10].

Theorem 7 A symmetric interval matrix $A^{I}$ is a $P$-matrix if and only if it is regular and contains at least one symmetric $P$-matrix.

Proof. A symmetric interval $P$-matrix $A^{I}$ is regular (each $A \in A^{I}$ has a positive determinant) and contains a symmetric $P$-matrix $\underline{A}$. If $A^{I}$ is regular and contains a symmetric $P$-matrix $A_{0}$, then $A_{0}$ is positive definite, hence $A^{I}$ is positive definite by Theorem 3 in [9], which in the light of Theorem 6 means that $A^{I}$ is a $P$-matrix.

Another relationship between regularity and $P$-property of interval matrices was established in $\left[8\right.$, Thm. 5.1, assert. (B1)]: an interval matrix $A^{I}=[\underline{A}, \bar{A}]$ is regular if and only if $(\underline{A}+\bar{A}-S(\bar{A}-\underline{A}))^{-1}(\underline{A}+\bar{A}+S(\bar{A}-\underline{A}))$ is a $P$-matrix for each signature
matrix $S$ (i.e., a diagonal matrix with $\pm 1$ diagonal elements). This topic was recently studied by Johnson and Tsatsomeros [5].

The necessary and sufficient condition of Theorem 3 employs up to $2^{n-1}$ test matrices $A_{z}, z \in Z$. There is a natural question whether an essentially simpler criterion could be found. The last theorem gives an indirect answer to this question: it implies that an existence of a polynomial-time algorithm for checking the $P$-property of symmetric interval matrices would imply that the complexity classes $P$ and NP are equal, thereby running contrary to the current (unproved) conjecture that $\mathrm{P} \neq \mathrm{NP}$. We refer the reader to the classical book by Garey and Johnson [4] for a detailed discussion of the problem " $\mathrm{P}=\mathrm{NP}$ " and related issues.

Theorem 8 The following problem is NP-hard:
Instance. A symmetric interval matrix $A^{I}=[\underline{A}, \bar{A}]$ with rational bounds $\underline{A}, \bar{A}$.
Question. Is $A^{I}$ a $P$-matrix?
Proof. By Theorem 6, $A^{I}$ is a $P$-matrix if and only if it is positive definite; checking positive definiteness of symmetric interval matrices was proved to be NP-hard in [11].

Coxson [2] proved that the $P$-matrix problem for real matrices is co-NP-complete. His result concerns nonsymmetric matrices since the symmetric case can be solved by Sylvester determinant criterion which can be performed in polynomial time (Schrijver [12]). Theorem 8 shows that for interval matrices even the symmetric case is NP-hard.

Acknowledgment. Correspondence with Prof. J. Garloff on the subject of this paper is highly appreciated.

## Bibliography

[1] S. Białas and J. Garloff, Intervals of $P$-matrices and related matrices, Linear Algebra Appl., 58(1984), pp. 33-41.
[2] G. E. Coxson, The $P$-matrix problem is co-NP-complete, Math. Program., 64(1994), pp. 173-178.
[3] M. Fiedler and V. Pták, On matrices with non-positive off-diagonal elements and positive principal minors, Czech. Math. J., 12(1962), pp. 382-400.
[4] M. E. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
[5] C. R. Johnson and M. J. Tsatsomeros, Convex sets of nonsingular and $P$-matrices, to appear in Linear Multilinear Algebra.
[6] K. G. Murty, Linear Complementarity, Linear and Nonlinear Programming, Heldermann, Berlin, 1988.
[7] A. Neumaier, Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, 1990.
[8] J. Rohn, Systems of linear interval equations, Linear Algebra Appl., 126(1989), pp. 39-78.
[9] J. Rohn, Positive definiteness and stability of interval matrices, SIAM J. Matrix Anal. Appl., 15(1994), pp. 175-184.
[10] J. Rohn, On some properties of interval matrices preserved by nonsingularity, Z. Angew. Math. Mech., 74(1994), p. T688.
[11] J. Rohn, Checking positive definiteness or stability of symmetric interval matrices is NP-hard, Commentat. Math. Univ. Carol., 35(1994), pp. 795-797.
[12] A. Schrijver, Theory of Integer and Linear Programming, Wiley, Chichester, 1986.


[^0]:    ${ }^{1}$ This work was supported in part by the Charles University Grant Agency under grant GAUK 237. Part of the work was done during the first author's stay at the Center of Theoretical Sciences of the University of Leipzig.
    ${ }^{2}$ Institute of Mathematics, University of Leipzig, Augustusplatz 10-11, D-04109 Leipzig, Germany (rex@mathematik.uni-leipzig.d400.de).

