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APPROXIMATION OF THE MIXED-HYBRID FORMULATION OF THE POROUS MEDIA FLOW PROBLEM [1]

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Technical report No. 609

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Abstract

The porous media flow problem with a mixed boundary conditions is considered. Mixed-hybrid formulation of porous media flow problem is described and the existence and uniqueness of the solution is proved. Mixed-hybrid finite element method is used for approximate solutions. A-priori error estimates are derived with some conditions for convergence.

Keywords

Porous media flow problem, hybrid-mixed formulation, finite element method

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1 Introduction

Let Ω be a domain in Euclidean space $\mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$. The porous media flow problem is described by Darcy's law

$$\mathbf{u} = -\mathbf{A}^{-1}\nabla p \; ; \tag{1.1}$$

and the continuity equation for uncompressible fluid

$$\nabla \cdot \mathbf{u} \;=\; q, \tag{1.2}$$

where $\mathbf{u}$ is a velocity of the flow, p is a piezometric head, q is a density of sources and $\mathbf{A}^{-1}$ is positive definite tensor of permeability of porous media. (i.e. there exists $\alpha_0 > 0$ such that $\sum_{i,j\leq 3}[\mathbf{A}^{-1}(\mathbf{x})]_{ij}\xi_i\xi_j \geq \alpha_0 \sum_{i\leq 3}\xi_i\xi_i$ for all $\xi \in R^3$ and almost everywhere on Ω . Further we suppose $[\mathbf{A}^{-1}(\mathbf{x})]_{ij} \in L^\infty(\Omega)$, for $i,j \in \{1,2,3\}$). From macroscopic point of view it is necessary to involve the tectonical discontinuity into permeability tensor.

The boundary $\partial\Omega$ is composed from two parts. It holds $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. The following boundary conditions are considered

$$p = p_D \quad \text{on} \quad \partial\Omega_D \; , \tag{1.3}$$

$$\mathbf{u}\cdot\mathbf{n} = -\mathbf{A}^{-1}\nabla p\cdot\mathbf{n} = u_N \quad \text{on} \quad \partial\Omega_N \; , \tag{1.4}$$

where $\mathbf{n}$ denotes the unit outward normal to boundary $\partial\Omega$ (It exists almost everywhere).

2 Mixed-hybrid formulation of the porous media flow problem, existence and uniqueness of solution

Consider the decomposition $\mathcal{E}_h$ of domain Ω by elements e_i, $i \in I$, such that it is valid (see [8])

(i) $\quad \overline{\Omega} \;=\; \cup_{e\in\mathcal{E}_h}\overline{e}\;;$
(ii) $\quad e_i \cap e_j \;=\; \emptyset\;,\;\; \text{if}\;\; i \neq j\;;$
(iii) $\quad e \in \mathcal{E}_h$ is open subset Ω with a piecewise smooth boundary ∂e.

We shall denote by $\Gamma_h = \cup_{e\in\mathcal{E}_h}\partial e - \partial\Omega_D$ the structure of faces of elements from which we shall exclude the faces with Dirichlet boundary conditions $\partial\Omega_D$. Define following spaces on the decompositions $\mathcal{E}_h$, Γ_h :

$$\mathbf{H}(div,\mathcal{E}_h) = \{\,\mathbf{v}\in\mathbf{L}^2(\Omega);\;\nabla\cdot\mathbf{v}^e \in L^2(e),\;\forall e\in\mathcal{E}_h\} \tag{2.1}$$

with the norm

$$\|\,\mathbf{v}\,\|_{div,\mathcal{E}_h} = [\,\|\,\mathbf{v}\,\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_h}\|\,\nabla\cdot\mathbf{v}^e\,\|_{0,e}^2\,]^{\frac{1}{2}}, \tag{2.2}$$

where $\mathbf{v}^e$ denotes the restriction of vector function $\mathbf{v}$ on the element e and

$$H_D^{\frac{1}{2}}(\Gamma_h) = \{ \mu : \Gamma_h \to R ;\ \exists \varphi \in H_D^1(\Omega),\ \mu^e = \gamma_h \varphi^e,\ \forall e \in \mathcal{E}_h \}, \tag{2.3}$$

where $H_D^1(\Omega) = \{\varphi \in H^1(\Omega);\ \gamma\varphi = 0 \text{ on } \partial\Omega_D\}$, γ_h is the trace mapping of functions from $H_D^1(\Omega)$ on the structure of faces Γ_h and γ is the trace mapping on $\partial\Omega$. In the functional space $H_D^{\frac{1}{2}}(\Gamma_h)$ we define norm

$$\| \mu \|_{\frac{1}{2},\Gamma_h} = \inf_{\varphi \in H_D^1(\Omega)} \{ |\varphi|_{1,\Omega};\ \gamma_h \varphi = \mu \text{ na } \Gamma_h \}, \tag{2.4}$$

where $|\varphi|_{1,\Omega} = (\nabla\varphi, \nabla\varphi)_{0,\Omega}^{\frac{1}{2}}$. Further let

$$\mathbf{W}_{D,h} = \mathbf{H}(div, \mathcal{E}_h) \times L^2(\Omega) \times H_D^{\frac{1}{2}}(\Gamma_h) \tag{2.5}$$

be function space with standard norm

$$\| \mathbf{w} \|_{\mathbf{W},\Omega} = \left(\| \mathbf{v} \|_{div,\mathcal{E}_h}^2 + \| \phi \|_{0,\Omega}^2 + \| \mu \|_{\frac{1}{2},\Gamma_h}^2 \right)^{\frac{1}{2}}. \tag{2.6}$$

We introduce a bilinear form $\mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w})$ on the product $\mathbf{W}_{D,h} \times \mathbf{W}_{D,h}$ by relations

$$\mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w}) = \sum_{e \in \mathcal{E}_h} \mathcal{B}_e(\tilde{\mathbf{w}}^e, \mathbf{w}^e), \tag{2.7}$$

$$\begin{aligned} \mathcal{B}_e(\tilde{\mathbf{w}}^e, \mathbf{w}^e) \ &= \ (\mathbf{A}\tilde{\mathbf{v}}^e, \mathbf{v}^e)_{0,e} - (\tilde{\phi}^e, \nabla \cdot \mathbf{v}^e)_{0,e} - (\nabla \cdot \tilde{\mathbf{v}}^e, \phi^e)_{0,e} + \\ &+ < \tilde{\mu}^e, \mathbf{n}^e \cdot \mathbf{v}^e >_{\partial e} + < \mathbf{n}^e \cdot \tilde{\mathbf{v}}^e, \mu^e >_{\partial e}, \end{aligned} \tag{2.8}$$

where $\mathbf{n}^e$ denotes the unit outward normal to ∂e. Further we define a linear continous functional on $\mathbf{W}_{D,h}$ by formula

$$\mathcal{Q}(\mathbf{w}) \ = \ \sum_{e \in \mathcal{E}_h} \{ -(q^e, \phi^e)_{0,e} - < p_D^e, \mathbf{n}^e \cdot \mathbf{v}^e >_{\partial e \cap \partial\Omega_D} + < u_N^e, \mu^e >_{\partial e \cap \partial\Omega_N} \} \ . \tag{2.9}$$

DEFINITION 1.1: The function $\mathbf{w}^* \in \mathbf{W}_{D,h}$ is said to be a weak solution of mixed-hybrid formulation of porous media flow problem described by equations (1.2), (1.1) using with boundary conditions (1.3), (1.4), the decomposition $\mathcal{E}_h$ of domain Ω and structure of faces Γ_h , if

$$\mathcal{B}(\mathbf{w}^*, \mathbf{w}) \ = \ \mathcal{Q}(\mathbf{w}) \ , \quad \forall \mathbf{w} \in \mathbf{W}_{D,h} \ . \tag{2.10}$$

Now we shall prove some lemmas, which introduce some functions important for showed of existence and uniqueness of weak solution.

LEMMA 1.1: *Let us choose $\mu \in H_D^{\frac{1}{2}}(\Gamma_h)$ and let $\varphi \in H_D^1(\Omega)$ be a function such that for all $e \in \mathcal{E}_h$ will be φ^e weak solution of problem*

$$-\ \nabla \cdot \nabla \varphi^e = 0 \quad in\ e, \tag{2.11}$$

with boundary condition

$$\varphi^e = \mu^e \quad on\ \partial e. \tag{2.12}$$

Then

$$|\varphi|^2_{1,\Omega} \;=\; \sum_{e\in\mathcal{E}_h} \int_{\partial e} \frac{\partial \varphi^e}{\partial \mathbf{n}^e}\mu^e\, dS \;=\; \|\ \mu\ \|^2_{\frac{1}{2},\Gamma_h}\ . \tag{2.13}$$

PROOF: Applying the Green formula to equation $-(\nabla\cdot\nabla\varphi^e,\varphi^e)_{0,e} = 0$ and considering boundary condition (2.12), we obtain

$$|\varphi^e|^2_{1,e} = \int_{\partial e} \frac{\partial \varphi^e}{\partial \mathbf{n}^e}\mu^e\, dS\,. \tag{2.14}$$

From (2.14) we get the left equality in (2.13). From the variational formulation of problem (2.11) with boundary condition (2.12) we can write:

$$|\varphi^e|^2_{1,e} = \inf_{\phi\in H^1_D(\Omega)} \{|\phi^e|^2_{1,e}\,;\ \phi^e = \mu^e \text{ on } \partial e\,\} = \|\ \mu^e\ \|^2_{\frac{1}{2},\partial e}, \tag{2.15}$$

which implies (2.13). □

Let us denote

$$|\mathbf{A}| = \sup_{\|\mathbf{v}\|_{0,\Omega}=1} (\mathbf{A}\mathbf{v},\mathbf{v})_{0,\Omega}. \tag{2.16}$$

REMARK 1.1: Let $\mu\in H^{\frac{1}{2}}_D(\Gamma_h)$ and let $\tilde{\varphi}$ be a function such that for all $e\in\mathcal{E}_h$ will be $\tilde{\varphi}^e$ weak solution of equation (2.11) with boundary condition $\tilde{\varphi}^e = \frac{1}{|\mathbf{A}|}\mu^e$ on the ∂e. Then

$$|\tilde{\varphi}|^2_{1,\Omega} = \sum_{e\in\mathcal{E}_h} |\tilde{\varphi}^e|^2_{1,e} = |\mathbf{A}|^{-2}\ \|\ \mu\ \|^2_{\frac{1}{2},\Gamma_h}\ . \tag{2.17}$$

LEMMA 1.2: *Let $\mu\in H^{\frac{1}{2}}_D(\Gamma_h)$ and $\phi\in L^2(\Omega)$. Let ψ be the solution of the following problem*

$$-\nabla\cdot\nabla\psi = \phi \quad in\ \ \Omega, \tag{2.18}$$

$$\psi = 0 \quad on\ \ \partial\Omega_D\ , \qquad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad on\ \ \partial\Omega_N. \tag{2.19}$$

Then

$$\sum_{e\in\mathcal{E}_h} <\mu^e, \nabla\psi^e\cdot\mathbf{n}^e>_{\partial e}\; =\; 0\ . \tag{2.20}$$

PROOF: Because $\mu\in H^{\frac{1}{2}}_D(\Gamma_h)$ there exists $\varphi\in H^1_D(\Omega)$ such that $\gamma_h\varphi = \mu$. Using the Green formula we get:

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} <\mu^e, \nabla\psi^e\cdot\mathbf{n}^e>_{\partial e} &= \sum_{e\in\mathcal{E}_h} [(\varphi^e,\nabla\cdot\nabla\psi^e)_{0,e} + (\nabla\varphi^e,\nabla\psi^e)_{0,e}] = \\ &= (\varphi,\nabla\cdot\nabla\psi)_{0,\Omega} + (\nabla\varphi,\nabla\psi)_{0,\Omega} = <\gamma\varphi,\nabla\psi\cdot\mathbf{n}>_{\partial\Omega} = \\ &= <\gamma\varphi,\nabla\psi\cdot\mathbf{n}>_{\partial\Omega_N} = 0,\end{aligned} \tag{2.21}$$

□

REMARK 1.2: There exists C_Ω depending only on domain Ω such that

$$|\psi|_{1,\Omega} \leq C_\Omega \parallel \phi \parallel_{0,\Omega} . \tag{2.22}$$

For C_Ω and $|\mathbf{A}|$ (see also (2.16)) $\widetilde{\psi}$ is a weak solution of the problem

$$-\nabla \cdot \nabla \widetilde{\psi} = \frac{1}{|\mathbf{A}|C_\Omega^2} \phi \;\text{ in } \Omega \tag{2.23}$$

with boundary conditions (2.19). Then using (2.22) we get

$$|\widetilde{\psi}|_{1,\Omega} \leq \frac{1}{|\mathbf{A}|C_\Omega} \parallel \phi \parallel_{0,\Omega} . \tag{2.24}$$

THEOREM 1.1: *Mixed-hybrid formulation porous media flow problem defined by equation (2.10) has a unique solution* $\mathbf{w}^* \in \mathbf{W}_{D,h}$.
PROOF. First we estimate

$$\begin{aligned} < \mu^e, \mathbf{v}^e \cdot \mathbf{n}^e >_{\partial e} &= (\nabla \phi^e, \mathbf{v}^e)_{0,e} + (\phi^e, \nabla \cdot \mathbf{v}^e)_{0,e} \leq \\ &\leq \parallel \phi^e \parallel_{1,e} \parallel \mathbf{v}^e \parallel_{div,e} = \parallel \mu^e \parallel_{\frac{1}{2},\partial e} \parallel \mathbf{v}^e \parallel_{div,e} . \end{aligned} \tag{2.25}$$

Choosing $\mathbf{w}^e = (\mathbf{v}^e, \phi^e, \mu^e)$, $\tilde{\mathbf{w}}^e = (\tilde{\mathbf{v}}^e, \widetilde{\phi}^e, \widetilde{\mu}^e)$ and applying the Schwartz inequality to the bilinear form (2.8) we obtain

$$\begin{aligned} |\mathcal{B}_e(\tilde{\mathbf{w}}^e, \mathbf{w}^e)| &\leq |\mathbf{A}| \parallel \tilde{\mathbf{v}}^e \parallel_{0,e} \parallel \mathbf{v}^e \parallel_{0,e} + \parallel \widetilde{\phi}^e \parallel_{0,e} \parallel \nabla \cdot \mathbf{v}^e \parallel_{0,e} + \\ &+ \parallel \nabla \cdot \tilde{\mathbf{v}}^e \parallel_{0,e} \parallel \phi^e \parallel_{0,e} + \parallel \widetilde{\mu}^e \parallel_{\frac{1}{2},\partial e} \parallel \mathbf{v}^e \parallel_{div,e} + \\ &+ \parallel \mu^e \parallel_{\frac{1}{2},\partial e} \parallel \tilde{\mathbf{v}}^e \parallel_{div,e} \leq \\ &\leq 2 \max\{|\mathbf{A}|, 1\} \parallel \tilde{\mathbf{w}}^e \parallel_{\mathbf{W},e} \cdot \parallel \mathbf{w}^e \parallel_{\mathbf{W},e} . \end{aligned} \tag{2.26}$$

Introduce $C_1 = 2\max\{|\mathbf{A}|, 1\}$ and $a_0 = \inf_{(\mathbf{v},\mathbf{v})_{0,\Omega}=1} (\mathbf{A}\,\mathbf{v}, \mathbf{v})_{0,\Omega} > 0$. For any $\mathbf{w}^e = (\mathbf{v}^e, \phi^e, \mu^e)$ we choose $\tilde{\mathbf{w}}^e = (\tilde{\mathbf{v}}^e, \widetilde{\phi}^e, \widetilde{\mu}^e)$ in the following

$$\begin{aligned} \widetilde{\phi}^e &= -2\,\phi^e - 2\,\nabla \cdot \mathbf{v}^e, \\ \tilde{\mathbf{v}}^e &= 2\,\mathbf{v}^e + 2\,\nabla \widetilde{\varphi}^e + \nabla \widetilde{\psi}^e, \\ \widetilde{\mu}^e &= -2\,\mu^e, \end{aligned} \tag{2.27}$$

where $\widetilde{\varphi}^e$ and $\widetilde{\psi}$ were defined above. Now we have

$$\begin{aligned} \mathcal{B}_e(\tilde{\mathbf{w}}^e, \mathbf{w}^e) &= (\mathbf{A}\tilde{\mathbf{v}}^e, \mathbf{v}^e)_{0,e} - (\widetilde{\phi}^e, \nabla \cdot \mathbf{v}^e)_{0,e} - (\nabla \cdot \tilde{\mathbf{v}}^e, \phi^e)_{0,e} + \\ &+ < \widetilde{\mu}^e, \mathbf{v}^e \cdot \mathbf{n}^e >_{\partial e} + < \mu^e, \tilde{\mathbf{v}}^e \cdot \mathbf{n}^e >_{\partial e} \geq \\ &\geq \min\{\tfrac{a_0}{2}, 2, \tfrac{1}{|\mathbf{A}|}, \tfrac{1}{2|\mathbf{A}|C_\Omega^2}\} \cdot \parallel \mathbf{w}^e \parallel^2_{\mathbf{W},e} + < \mu^e, \nabla \widetilde{\psi}^e \cdot \mathbf{n}^e >_{\partial e} = \\ &= k(a_0, |\mathbf{A}|, C_\Omega) \cdot \parallel \mathbf{w}^e \parallel^2_{\mathbf{W},e} + < \mu^e, \nabla \widetilde{\psi}^e \cdot \mathbf{n}^e >_{\partial e}, \end{aligned}$$

where

$$k(a_0, |\mathbf{A}|, C_\Omega) = \min\left[\frac{a_0}{2}, 2, \frac{1}{|\mathbf{A}|}, \frac{1}{2|\mathbf{A}|C_\Omega^2}\right]. \tag{2.28}$$

For the components of the function $\tilde{\mathbf{w}}$ we get

$$(\tilde{\mathbf{v}}^e, \tilde{\mathbf{v}}^e)_{0,e} \leq 7 \parallel \mathbf{v}^e \parallel^2_{0,e} + \frac{7}{|\mathbf{A}|^2} \parallel \mu^e \parallel^2_{\frac{1}{2},\partial e} + \frac{3}{|\mathbf{A}|^2 C^2_\Omega} \parallel \phi^e \parallel^2_{1,e},$$

$$(\nabla \cdot \tilde{\mathbf{v}}^e, \nabla \cdot \tilde{\mathbf{v}}^e)_{0,e} \leq \left(4 + \frac{1}{|\mathbf{A}| C^2_\Omega}\right) \parallel \nabla \cdot \mathbf{v}^e \parallel^2_{0,e} + \frac{1}{|\mathbf{A}| C^2_\Omega} \left(1 + \frac{1}{|\mathbf{A}| C^2_\Omega}\right) \parallel \phi^e \parallel^2_{0,e},$$

$$(\tilde{\phi}^e, \tilde{\phi}^e)_{0,e} \leq 6 \parallel \phi^e \parallel^2_{0,e} + 2 \parallel \nabla \cdot \mathbf{v}^e \parallel^2_{0,e},$$

$$\parallel \tilde{\mu}^e \parallel^2_{\frac{1}{2},\partial e} = 4 \parallel \mu^e \parallel^2_{\frac{1}{2},\partial e}.$$

Consenquently,

$$\parallel \tilde{\mathbf{w}}^e \parallel^2_{\mathbf{W},e} \leq \; K^2(|\mathbf{A}|, C_\Omega) \parallel \mathbf{w}^e \parallel^2_{\mathbf{W},e},$$

with

$$\begin{aligned} K(|\mathbf{A}|, C_\Omega) \;\; = \;\; & \left\{ \max \left\{ \left[6 + \frac{3}{|\mathbf{A}|^2 C^2_\Omega} + \frac{1}{|\mathbf{A}| C^2_\Omega} \left(1 + \frac{1}{|\mathbf{A}| C^2_\Omega}\right)\right], \; 7, \right. \right. \\ & \left. \left. \left[6 + \frac{1}{|\mathbf{A}| C^2_\Omega}\right], \left[4 + \frac{7}{|\mathbf{A}|^2}\right] \right\} \right\}^{\frac{1}{2}}. \end{aligned}$$

Then

$$\mathcal{B}_e(\tilde{\mathbf{w}}^e, \mathbf{w}^e) \;\; \geq \;\; \frac{k}{K} \parallel \tilde{\mathbf{w}}^e \parallel_{\mathbf{W},e} \parallel \mathbf{w}^e \parallel_{\mathbf{W},e} + < \mu^e, \nabla \tilde{\psi}^e \cdot \mathbf{n}^e >_{\partial e}$$

and summing up we obtain

$$\mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w}) = \sum_{e \in \mathcal{E}_h} \mathcal{B}_e(\tilde{\mathbf{w}}^e, \mathbf{w}^e) \geq \frac{k}{K} \parallel \tilde{\mathbf{w}} \parallel_{\mathbf{W},\Omega} \parallel \mathbf{w} \parallel_{\mathbf{W},\Omega}.$$

Introducing

$$C_2 = \frac{k}{K} > 0 \tag{2.29}$$

we get

$$\inf_{\|\tilde{\mathbf{w}}\|_{\mathbf{W},h}=1} \; \sup_{\|\mathbf{w}\|_{\mathbf{W},h} \leq 1} \; \mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w}) \geq C_2. \tag{2.30}$$

From the symmetry of the bilinear form $\mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w})$ follows immediately

$$\inf_{\|\mathbf{w}\|_{\mathbf{W},h}=1} \; \sup_{\|\tilde{\mathbf{w}}\|_{\mathbf{W},h} \leq 1} \; \mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w}) \geq C_2. \tag{2.31}$$

By [1] Theorem 2.1. there exists unique solution of problem (2.10), satisfying

$$\parallel \tilde{\mathbf{w}} \parallel_{\mathbf{W},h} \leq \frac{1}{C_2} \{ \parallel q \parallel^2_{0,\Omega} + \parallel p_D \parallel^2_{\frac{1}{2},\partial e} + \parallel u_N \parallel^2_{-\frac{1}{2},\partial e} \}^{\frac{1}{2}}, \tag{2.32}$$

where $\parallel u_N \parallel^2_{-\frac{1}{2},\partial e} = \inf_{\mathbf{v} \in \mathbf{H}_N(div,\Omega)} \{ \parallel \mathbf{v} \parallel_{div,\Omega}; \; u_N = \mathbf{v} \cdot \mathbf{n} \text{ on } \partial\Omega_N \}$.

Constants $C_1 = 2 \max\{|\mathbf{A}|, 1\}, \; C_2 = \frac{k}{K} > 0$ are independent on the decomposition $\mathcal{E}_h$ of domain Ω. □

REMARK 1.3: Let p^0 be a classical solution of equation $-\nabla \cdot \mathbf{A}^{-1} \nabla p = q$ with boundary conditions (1.3), (1.4). Consider $\mathbf{w}^0$ in the form $\mathbf{w}^0 = (\mathbf{u}^0, p^0, p^0|_{\Gamma_h})$. Then $\mathbf{w}^0 \in \mathbf{W}_{D,h}$ and for any $\Omega' \subset \Omega$ such that $\overline{\Omega'} \subset \Omega$ is $p^0 \in H^2(\Omega')$ and

$$(\mathbf{A}\, \mathbf{u}^{0e}, \mathbf{v}^e)_{0,e\cap\Omega'} - (p^{0e}, \nabla \cdot \mathbf{v}^e)_{0,e\cap\Omega'} + < p^{0e}|_{\Gamma_h}, \mathbf{n}^e \cdot \mathbf{v}^e >_{(\partial e\cap\Omega'_N)\cup(e\cap\partial\Omega'_N)}$$

$$= < p^{0e}, \mathbf{n}^e \cdot \mathbf{v}^e >_{e\cap\partial\Omega'_D} . \tag{2.33}$$

Here $\partial\Omega'_D$, resp. $\partial\Omega'_N$, approximate parts of boundary $\partial\Omega_D$, resp. $\partial\Omega_N$, and $\Omega'_N = \Omega' \cup \partial\Omega'_N$.
From the continuity of p^0 and ∇p^0 in the domain Ω and from the equation $-\mathbf{A}^{-1}\frac{\partial p^0}{\partial \mathbf{n}^e} = \mathbf{u}^{0e} \cdot \mathbf{n}^e$ on $\partial\Omega$ it follows for $\Omega' \to \Omega$:

$$(\mathbf{A}\, \mathbf{u}^{0e}, \mathbf{v}^e)_{0,e\cap\Omega'} \to (\mathbf{A}\, \mathbf{u}^{0e}, \mathbf{v}^e)_{0,e},$$

$$(p^{0e}, \nabla \cdot \mathbf{v}^e)_{0,e\cap\Omega'} \to (p^{0e}, \nabla \cdot \mathbf{v}^e)_{0,e},$$

$$< p^{0e}|_{\Gamma_h}, \mathbf{n}^e \cdot \mathbf{v}^e >_{(\partial e\cap\Omega'_N)\cup(e\cap\partial\Omega'_N)} \to < p_D, \mathbf{n}^e \cdot \mathbf{v}^e >_{\partial e\cap\partial\Omega_N},$$

$$\sum_{e\in\mathcal{E}_h} < \mathbf{u}^{0e} \cdot \mathbf{n}^e, \mu^e >_{e\cap\partial\Omega'} \to < \mathbf{u}^0 \cdot \mathbf{n}, \mu >_{\partial\Omega_N} = < u_N, \mu >_{\partial\Omega_N} . \tag{2.34}$$

Now for any $\mathbf{w} \in \mathbf{W}_{D,h}$ we obtain

$$\mathcal{B}(\mathbf{w}^0, \mathbf{w}) = \sum_{e\in\mathcal{E}_h} \mathcal{B}_e(\mathbf{w}^{0e}, \mathbf{w}^e) = \sum_{e\in\mathcal{E}_h} \{ (\mathbf{A}\, \mathbf{u}^{0e}, \mathbf{v}^e)_{0,e} -$$

$$-(p^{0e}, \nabla \cdot \mathbf{v}^e)_{0,e} - (\nabla \cdot \mathbf{u}^{0e}, \phi^e)_{0,e} + < p^{0e}, \mathbf{n}^e \cdot \mathbf{v}^e >_{\partial e} + < \mathbf{n}^e \cdot \mathbf{u}^{0e}, \mu^e >_{\partial e} \} =$$

$$= \sum_{e\in\mathcal{E}_h} \{ -(q^e, \phi^e)_{0,e} - < p^e_D, \mathbf{n}^e \cdot \mathbf{v}^e >_{\partial e\cap\partial\Omega_D} + < u^e_N, \mu^e >_{\partial e\cap\partial\Omega_N} \} = \mathcal{Q}(\mathbf{w}).$$

Consequently, $\mathbf{w}^0$ solves equation (2.10) and considering uniqueness we have $\mathbf{w}^0 = \mathbf{w}^*$.

3 Approximation of mixed-hybrid formulation

Assume the decomposition $\mathcal{E}_h$ of the domain Ω strongly regular, i.e. there exists constant C_0 independent on $\mathcal{E}_h$ such that

$$\max_{e\in\mathcal{E}_h} \frac{h_e}{\rho_e} \le C_0 , \tag{3.1}$$

where $h^e = diam\, e$ and ρ_e denotes the diameter of the spheres inscribed in e.
Let us introduce the class of polynomials of degree at most k in e $P_k(e)$ and for $k, r, t \in \mathbb{N}$ we define spaces

$$P_k(\mathcal{E}_h) = \{\phi_h \in L^2(\Omega);\ \phi^e_h \in P_k(e),\ \forall e \in \mathcal{E}_h\}, \tag{3.2}$$

$$\mathbf{H}_r(div, \mathcal{E}_h) = \{\mathbf{v}_h;\ \mathbf{v}^e_h \in \mathbf{H}(div, e),\ \mathbf{v}^e_h \in [P_r(e)]^3,\ \forall e \in \mathcal{E}_h\}, \tag{3.3}$$

$$H^{\frac{1}{2}}_{D,t}(\Gamma_h) = \{\mu_h \in H^{\frac{1}{2}}_D(\Gamma_h);\ \exists \varphi^e_h \in P_t(e),\ \mu^e_h = \gamma\varphi^e_h,\ \forall e \in \mathcal{E}_h\}, \tag{3.4}$$

$$\mathbf{W}_{D,h(k,r,t)} = \mathbf{H}_r(div,\mathcal{E}_h) \times P_k(\mathcal{E}_h) \times H^{\frac{1}{2}}_{D,t}(\Gamma_h) \subset \mathbf{W}_{D,h}. \tag{3.5}$$

DEFINITION 2.1: The function $\mathbf{w}_h^* \in \mathbf{W}_{D,h(k,r,t)}$ is the approximation of mixed-hybrid formulation, if holds

$$\mathcal{B}(\mathbf{w}_h^*, \mathbf{w}_h) = \mathcal{Q}(\mathbf{w}_h),\ \forall \mathbf{w}_h \in \mathbf{W}_{D,h(k,r,t)}, \tag{3.6}$$

where $\mathcal{B}(.,.)$, resp. $\mathcal{Q}(.)$, was defined (2.7), resp. (2.9).
We shall show some conditions for existence unique solution of (3.6). First we introduce necessary conditions.

LEMMA 2.1: *Assume there exists a unique solution of (3.6), then*
(i) $\forall\, \mu_h \in H^{\frac{1}{2}}_{D,t}(\Gamma_h)$ *:*

$$\sum_{e\in\mathcal{E}_h} < \mathbf{v}_h \cdot \mathbf{n}^e, \mu_h >_{\partial e} = 0,\ \forall \mathbf{v}_h \in \mathbf{H}_r(div,\mathcal{E}_h) \implies \mu_h = 0, \tag{3.7}$$

(ii) $\forall\, \varphi_h \in P_k(\mathcal{E}_h)$ *:*

$$(\nabla\cdot \mathbf{v}_h, \varphi_h)_{0,e} = 0,\ \forall \mathbf{v}_h \in \mathbf{H}_r(div,\mathcal{E}_h) \implies \varphi_h^e = 0. \tag{3.8}$$

PROOF. Let (i) be invalid, then there exists $\hat{\mu}_h \in H^{\frac{1}{2}}_{D,t}(\Gamma_h),\ \hat{\mu}_h \neq 0$ such that

$$\sum_{e\in\mathcal{E}_h} < \mathbf{v}_h \cdot \mathbf{n}^e, \tilde{\mu}_h >_{\partial e} = 0,\ \forall \mathbf{v}_h \in \mathbf{H}_r(div,\mathcal{E}_h).$$

Then $\mathcal{B}((\mathbf{0},0,\hat{\mu}_h),\mathbf{w}_h) = 0$ for all $\mathbf{w}_h \in \mathbf{W}_{D,h(k,r,t)}$. Therefore $\mathbf{w}_{0h}^* = (\mathbf{0},0,0)$ is not unique solution of (3.6) for $q=0,\ p_D=0,\ u_N=0$.
Let (ii) is invalid, then there exists $\hat{\varphi}_h \in P_k(\mathcal{E}_h),\ \hat{\varphi}_h^e \neq 0$ such that

$$(\nabla\cdot \mathbf{v}^e, \hat{\varphi}_h^e)_{0,e} = 0,\ \forall \mathbf{v}^e \in \mathbf{H}_r(div,\mathcal{E}_h).$$

Then $\mathcal{B}((\mathbf{0},\hat{\varphi}_h,0),\mathbf{w}_h) = 0$ for all $\mathbf{w}_h \in \mathbf{W}_{D,h(k,r,t)}$, and so $\mathbf{w}_{h0}^* = (\mathbf{0},0,0)$ is not unique solution of (3.6) for $q=0,\ p_D=0,\ u_N=0$. □

For (3.7) is necessary $\{\mathbf{v}_h\cdot\mathbf{n}^e;\ \mathbf{v}_h \in \mathbf{H}_r(div,\mathcal{E}_h),\ e\in\mathcal{E}_h\}$ generated complete system of functionals on $H^{\frac{1}{2}}_{D,t}(\Gamma_h)$. Let us choose $\mu_h \in H^{\frac{1}{2}}_{D,t}(\Gamma_h)$ and consider the problem

$$-\nabla\cdot\nabla\widetilde{\varphi}_h = 0 \ \text{ in } e,\ \ \widetilde{\varphi}_h = \mu_h \ \text{ on } \partial e.$$

For $\mathbf{v}_h = \nabla\widetilde{\varphi}_h \in [P_{t-1}(e)]^3$ for all $e\in\mathcal{E}_h$ we get

$$\sum_{e\in\mathcal{E}_h} < \nabla\widetilde{\varphi}_h \cdot \mathbf{n}^e, \mu_h >_{\partial e} = |\widetilde{\varphi}_h|^2_{1,\Omega} = \parallel \mu_h \parallel^2_{\frac{1}{2},\Gamma_h}. \tag{3.9}$$

For fulfilment (3.7) it is necessary to satisfy $r \geq t-1$.
For (3.8) it is necessary $\{\nabla\cdot\mathbf{v}_h;\ \mathbf{v}_h \in \mathbf{H}_r(div,\mathcal{E}_h)\}$ to generate the complete system of functionals on $P_k(\mathcal{E}_h)$. From that we have the condition $r \geq k+1$.

Let $\phi_h \in P_k(\mathcal{E}_h)$ and $\widetilde{\psi}_h \in H^1(\Omega)$ be a solution of the equation

$$- \nabla \cdot \nabla \widetilde{\psi}_h = \frac{1}{|\mathbf{A}| C_\Omega^2} \phi_h \quad \text{in } \Omega \tag{3.10}$$

with mixed boundary conditions

$$\widetilde{\psi}_h = 0 \quad \text{on } \partial\Omega_D \,, \quad \frac{\partial \widetilde{\psi}_h}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega_N. \tag{3.11}$$

Then

$$\nabla \widetilde{\psi}_h \in \mathbf{H}_r(div, \mathcal{E}_h). \tag{3.12}$$

We introduce

$$\mathbf{H}_0(div, e) = \{ \mathbf{v}^e \in [P_r(e)]^3; \ \nabla \cdot \mathbf{v}^e = 0 \}. \tag{3.13}$$

THEOREM 2.1: *Let $r = k + 1 \geq t - 1$. Then there exists unique solution of (3.6).*

PROOF. For $\mathbf{w}_h^e = (\mathbf{v}_h^e, \phi_h^e, \mu_h^e)$ we choose $\tilde{\mathbf{w}}_h^e = (\tilde{\mathbf{v}}_h^e, \widetilde{\phi}_h^e, \tilde{\mu}_h^e)$ in the form

$$\begin{aligned} \tilde{\mathbf{v}}_h^e &= 2\mathbf{v}_h^e + 2\nabla\widetilde{\varphi}_h^e + \nabla\widetilde{\psi}_h^e, \\ \widetilde{\phi}_h^e &= -2\phi_h^e - 2\nabla \cdot \mathbf{v}_h^e, \\ \tilde{\mu}_h^e &= -2\mu_h^e. \end{aligned}$$

We calculate

$$\begin{aligned} \mathcal{B}_e(\tilde{\mathbf{w}}_h^e, \mathbf{w}_h^e) = 2(\mathbf{A}\mathbf{v}_h^e, \mathbf{v}_h^e)_{0,e} + 2(\mathbf{A}\nabla\widetilde{\varphi}_h^e, \mathbf{v}_h^e)_{0,e} + (\mathbf{A}\nabla\widetilde{\psi}_h^e, \mathbf{v}_h^e)_{0,e} + \\ +(\nabla \cdot \mathbf{v}_h^e, \nabla \cdot \mathbf{v}_h^e)_{0,e} - (\nabla \cdot \nabla\widetilde{\varphi}_h^e, \phi_h^e)_{0,e} - (\nabla \cdot \nabla\widetilde{\psi}_h^e), \phi_h^e)_{0,e} + \\ +2 < \mu_h^e, \nabla\widetilde{\varphi}_h^e \cdot \mathbf{n}^e >_{\partial e} + < \mu_h^e, \nabla\widetilde{\psi}_h^e \cdot \mathbf{n}^e >_{\partial e} . \end{aligned}$$

$\nabla\widetilde{\varphi}_h^e \in \mathbf{H}_0(e)$ and therefore $(\nabla \cdot \nabla\widetilde{\varphi}_h^e, \phi_h^e)_{0,e} = 0$.
Further analogously to the Theorem 1.1 we get

$$\mathcal{B}(\tilde{\mathbf{w}}_h, \mathbf{w}_h) \geq k \parallel \mathbf{w}_h \parallel_{\mathbf{W},h}^2, \tag{3.14}$$

and because $\parallel \tilde{\mathbf{w}}_h \parallel_{\mathbf{W},h} \leq K \parallel \mathbf{w}_h \parallel_{\mathbf{W},h}$, we obtain

$$\inf_{\|\mathbf{w}_h\|_{\mathbf{W},h}=1} \sup_{\|\tilde{\mathbf{w}}_h\|_{\mathbf{W},h}\leq 1} |\mathcal{B}(\tilde{\mathbf{w}}_h, \mathbf{w}_h)| \geq C_2. \tag{3.15}$$

The bilinear form $\mathcal{B}(\tilde{\mathbf{w}}_h, \mathbf{w}_h)$ is symmetric and so

$$\inf_{\|\tilde{\mathbf{w}}_h\|_{\mathbf{W},h}=1} \sup_{\|\mathbf{w}_h\|_{\mathbf{W},h}\leq 1} |\mathcal{B}(\tilde{\mathbf{w}}_h, \mathbf{w}_h)| \geq C_2. \tag{3.16}$$

According to [1] there exists unique solution of (3.6) and following estimate is valid

$$\parallel \mathbf{w}^* - \mathbf{w}_h^* \parallel_{\mathbf{W}_{D,h}} \leq \left(1 + \frac{C_1}{C_2}\right) \inf_{\mathbf{w}_h \in \mathbf{W}_{D,h(r,k,t)}} \parallel \mathbf{w}^* - \mathbf{w}_h \parallel_{\mathbf{W}_{D,h}} \tag{3.17}$$

$\square$

Let s is integer and we introduce the Sobolev space $H^s(\Omega)$ (see [7]). We denote by $\|\cdot\|_{s,\Omega}$ the norm of the space $H^s(\Omega)$.

LEMMA 2.2: *Let $s \in \mathbb{N}$. Then for any $\varphi \in H^s(\Omega)$, $s \geq 0$, $s \in \mathbb{N}$ there exists constant $K_1 > 0$ indepedent on h and function $\varphi_h \in P_k(\mathcal{E}_h)$ such that*

$$\| \varphi - \varphi_h \|_{0,\Omega} \leq K_1 h^{\alpha_1} \| \varphi \|_{s,e}, \tag{3.18}$$

where $\alpha_1 = min\{k+1, s\}$.

LEMMA 2.3: *Let $m \in \mathbb{N}$. Then for any $\mathbf{u}^e \in [H^m(e)]^3$, $m \geq 1$, $m \in \mathbb{N}$, there exists constant $K_2 > 0$ independent on h and function $\mathbf{u}_h \in \mathbf{H}_r(div, \mathcal{E}_h)$ such that*

$$\| \mathbf{u} - \mathbf{u}_h \|_{div,\Omega} \leq K_2 h_e^{\alpha_2} \| \mathbf{u}^e \|_{m,\Omega}, \tag{3.19}$$

where $\alpha_2 = min\{r, m-1\}$.

LEMMA 2.4: *Let $\ell \in \mathbb{N}$.*

$$H^{\frac{1}{2}}_{D,t}(\Gamma_h) = \{\mu_h \in H^{\frac{1}{2}}_D(\Gamma_h);\ \exists \varphi_h^e \in P_t(e),\ \mu_h^e = \gamma_h \varphi_h^e,\ \forall e \in \mathcal{E}_h\}. \tag{3.20}$$

Then for any $\varphi \in H^\ell(\Omega)$, $\ell \geq 1$, $\ell \in \mathbb{N}$ such that $\mu = \varphi$ on Γ_h there exists constant $K_3 > 0$ indepedent on h and function $\mu_h \in H^{\frac{1}{2}}_{D,t}(\Gamma_h)$ such that

$$\| \mu - \mu_h \|_{\frac{1}{2},\Gamma_h} \leq K_3 h^{\alpha_3} \| \varphi \|_{\ell,\Omega}, \tag{3.21}$$

where $\alpha_3 = min\{t, \ell - 1\}$.

Proof of inequalities (3.18), (3.19) is introduced in [2]. Inequality (3.21) follows immediately from the definition of norm $\|\cdot\|_{\frac{1}{2},\Gamma_h}$.

If $\varphi_h \in H^1_D(\Omega) \cap P_t(e)$, for all $e \in \mathcal{E}_h$ and $\varphi_h|_{\Gamma_h} = \mu_h \in H^{\frac{1}{2}}_{D,t}(\Gamma_h)$, then we get the inequality

$$\| \mu - \mu_h \|_{\frac{1}{2},\Gamma_h} \leq |\varphi - \varphi_h|_{1,\Omega} \tag{3.22}$$

THEOREM 2.2: *Let $p^0 \in H^\ell(\Omega)$, $\ell \geq 2$ be the exact solution of equation $-\nabla\cdot\nabla p = q$ and let $r = k+1 \geq t-1$.*
Let $\mathbf{w}_h^* = (\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{W}_{D(r,k,t)}$ be the solution of problem (3.6).
Let us denote

$$\varepsilon_{\mathbf{w}} = (\varepsilon_{\mathbf{u}}, \varepsilon_p, \varepsilon_\lambda), = (-\mathbf{A}^{-1}\nabla p^0 - \mathbf{u}_h\,,\ p^0 - p_h\,,\ p^0|_{\Gamma_h} - \lambda_h\,),$$

$$\alpha = min\{\, r\,,\ t\,,\ \ell - 1\,\},$$

$$C^* = \left(1 + \frac{C_1}{C_2}\right) \max\{K_1, K_2, K_3\}.$$

Then

$$\| \varepsilon_{\mathbf{w}} \|_{\mathbf{W},\Omega} \leq C^* h^{\alpha} \left[\sum_{e \in \mathcal{E}_h} \| p^{0e} \|^2_{\ell,e} \right]^{\frac{1}{2}} . \tag{3.23}$$

PROOF. The statement follows from [1], assertion of the Theorem 2.1. and from lemmas 2.2 , 2.3 and 2.4. □

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