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2016

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Technical report No. V-1236

October 2016

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Abstract:

We introduce a so-called cut language which contains the representations of numbers in a rational base that are less than a given threshold. The cut languages can be used to refine the analysis of neural net models between integer and rational weights. We prove a necessary and sufficient condition when a cut language is regular, which is based on the concept of a quasi-periodic power series. We show that any cut language with a rational threshold is context-sensitive while examples of non-context-free cut languages are presented.

Keywords:
Cut language, rational base, quassi-periodic power series

[1]Research was done wi institutional support RVO:67985807 and partially supported by the grant of the Czech Science Foundation No. P202/12/G061.

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1 Cut Languages

We study so-called cut languages which contain the representations of numbers in a rational base [1, 2, 5–7, 10, 12–15] that are less than a given threshold. Hereafter, let a be a rational number such that $0 < |a| < 1$, which is the inverse of a base (radix) $1/a$ where $|1/a| > 1$, and let $B \subset \mathbb{Q}$ be a finite set of rational digits. We say that $L \subseteq \Sigma^*$ is a *cut language* over a finite alphabet Σ if there is a mapping $b : \Sigma \longrightarrow B$ and a real threshold c such that

$$L = L_{<c} = \left\{ x_1 \ldots x_n \in \Sigma^* \;\middle|\; \sum_{i=0}^{n-1} b(x_{n-i}) a^i < c \right\} . \tag{1}$$

The cut languages can be used to refine the analysis of computational power of neural network models [17, 23]. This analysis is satisfactorily fine-grained in terms of Kolmogorov complexity when changing from rational to arbitrary real weights [4, 18]. In contrast, there is still a gap between integer and rational weights which results in a jump from regular to recursively enumerable languages in the Chomsky hierarchy. In particular, neural nets with *integer* weights, corresponding to binary-state networks, coincide with finite automata [3, 8, 9, 11, 16, 20, 24]. On the other hand, a neural network that contains *two* analog-state units with *rational* weights, can implement two stacks of pushdown automata, a model equivalent to Turing machines [19]. A natural question arises: what is the computational power of binary-state networks including one extra analog unit with rational weights? Such a model is equivalent to finite automata with a register [21], which accept languages that can be represented by some cut languages combined in a certain way by usual operations (e.g. intersection with a regular language, concatenation, union); see [22] for the exact representation.

In this paper we prove a necessary and sufficient condition when a given cut language is regular (Section 3). For this purpose, we introduce and characterize an a-quasi-periodic number within B whose all its representations in basis $1/a$ using the digits from B, are eventually quasi-periodic power series (Section 2). The concept of a quasi-periodic power series appears to be interesting on its own, allowing for different quasi-repetends even of unbounded length. In addition, we present examples of cut languages that are not context-free and we show that any cut language with a rational threshold is context-sensitive (Section 4). Finally, we summarize the results and present some open problems (Section 5).

2 Quasi-Periodic Power Series

In this section, we introduce and analyze a notion of a-quasi-periodic numbers within B which will be employed for characterizing the class of regular cut languages in Section 3. We say that a power series $\sum_{k=0}^{\infty} b_k a^k$ with coefficients $b_k \in B$ is *eventually quasi-periodic* with *period sum* P if there is an increasing infinite sequence of its term indices $0 \le k_1 < k_2 < \cdots$ such that for every $i \ge 1$,

$$\frac{\sum_{k=0}^{m_i - 1} b_{k_i + k}\, a^k}{1 - a^{m_i}} = P \tag{2}$$

where $m_i = k_{i+1} - k_i > 0$ is the length of *quasi-repetend* $b_{k_i}, \ldots, b_{k_{i+1}-1}$, while k_1 is the length of *preperiodic part* $b_0, \ldots, b_{k_1-1}$. For $k_1 = 0$, we call such a power series to be *quasi-periodic*. One can calculate the sum of any eventually quasi-periodic power series as

$$\sum_{k=0}^{\infty} b_k a^k = \sum_{k=0}^{k_1-1} b_k a^k + a^{k_1} P \tag{3}$$

since $\sum_{k=k_1}^{\infty} b_k a^k = \sum_{i=1}^{\infty} a^{k_i} \sum_{k=0}^{m_i-1} b_{k_i+k}\, a^k = P \cdot \sum_{i=1}^{\infty} a^{k_i}(1 - a^{m_i}) = P \cdot \sum_{i=1}^{\infty}(a^{k_i} - a^{k_{i+1}}) = a^{k_1} P$ is an absolutely convergent series. It follows that the sum (3) does not change if any quasi-repetend is removed from associated sequence $(b_k)_{k=0}^{\infty}$ or if it is inserted in between two other quasi-repetends, which means that the quasi-repetends can be permuted arbitrarily.

Example 1. A quasi-periodic power series can be composed of quasi-repetends having unbounded length. For example, for any rational period sum $P \neq 0$, we define three rational digits as $\beta_1 = (1 - a^2)P$, $\beta_2 = a(1 - a)P$, and $\beta_3 = 0$, that is, $B = \{\beta_1, \beta_2, \beta_3\}$. Then $\beta_1, \beta_2^n, \beta_3$ where β_2^n means β_2 repeated n times, creates a quasi-repetend of length $n + 2$ for every integer $n \geq 0$, because $(\beta_1 + \sum_{k=1}^{n} \beta_2 a^k + \beta_3 a^{n+1})/(1 - a^{n+2}) = P$ whereas for any integer r such that $0 \leq r < n$, it holds $(\beta_1 + \sum_{k=1}^{r} \beta_2 a^k)/(1 - a^{r+1}) \neq P$.

Furthermore, given a power series $\sum_{k=0}^{\infty} b_k a^k$, we define its *tail sequence* $(d_n)_{n=0}^{\infty}$ as $d_n = \sum_{k=0}^{\infty} b_{n+k}\, a^k$ for every $n \geq 0$.

Lemma 2. *A power series $\sum_{k=0}^{\infty} b_k a^k$ with $b_k \in B$ for all $k \geq 0$, is eventually quasi-periodic with period sum P iff its tail sequence $(d_n)_{n=0}^{\infty}$ contains a constant infinite subsequence $(d_{k_i})_{i=1}^{\infty}$ such that $d_{k_i} = P$ for every $i \geq 1$.*

Proof. Let $\sum_{k=0}^{\infty} b_k a^k$ be an eventually quasi-periodic power series with period sum P, which means there is an increasing infinite sequence of its term indices $0 \leq k_1 < k_2 < \cdots$ such that equation (2) holds for every $i \geq 1$. It follows that $a^{k_i}\, d_{k_i} = \sum_{k=k_i}^{\infty} b_k\, a^k = \sum_{j=i}^{\infty} a^{k_j} \sum_{k=0}^{m_j-1} b_{k_j+k}\, a^k = P \cdot \sum_{j=i}^{\infty} a^{k_j}(1 - a^{m_j}) = P \cdot \sum_{j=i}^{\infty}(a^{k_j} - a^{k_{j+1}}) = a^{k_i} P$, which implies $d_{k_i} = P$ for every $i \geq 1$.

Conversely, assume that $(d_n)_{n=0}^{\infty}$ contains a constant subsequence $(d_{k_i})_{i=1}^{\infty}$ such that $d_{k_i} = P$ for every $i \geq 1$. We have $\sum_{k=0}^{m_i-1} b_{k_i+k}\, a^k = d_{k_i} - a^{m_i} d_{k_{i+1}} = (1 - a^{m_i})\, P$ where $m_i = k_{i+1} - k_i > 0$, which implies (2) for every $i \geq 1$. □

Theorem 3. *A power series $\sum_{k=0}^{\infty} b_k a^k$ with $b_k \in B$ for all $k \geq 0$, is eventually quasi-periodic iff its tail sequence $(d_n)_{n=0}^{\infty}$ contains only finitely many values, that is, $D = \{d_n \,|\, n \geq 0\}$ is a finite set.*

Proof. Assume that D is a finite set, which means there must be a real number $P \in D$ such that $d_{k_i} = P$ for infinitely many indices $0 \leq k_1 < k_2 < \cdots$, that is, $(d_{k_i})_{i=1}^{\infty}$ creates a constant infinite subsequence of tail sequence $(d_n)_{n=0}^{\infty}$. According to Lemma 2, this ensures that $\sum_{k=0}^{\infty} b_k a^k$ is eventually quasi-periodic.

Conversely, let $\sum_{k=0}^{\infty} b_k a^k$ with $b_k \in B$ for all $k \geq 0$, be an eventually quasi-periodic power series with period sum P. Since $a \in \mathbb{Q}$ and $B \subset \mathbb{Q}$ is finite, P is a rational number by (2) and there exists a natural number $\beta > 0$ such that $B' = \{\beta(b - (1-a)P)/a \mid b \in B\} \subset \mathbb{Z}$ is a finite set of integers. According to Lemma 2, the tail sequence $(d_n)_{n=0}^{\infty}$ of $\sum_{k=0}^{\infty} b_k a^k$ contains a constant infinite subsequence $(d_{k_i})_{i=1}^{\infty}$ such that $d_{k_i} = P$ for every $i \geq 1$. Assume to the contrary that $D = \{d_n \mid n \geq 0\}$ is an infinite set.

We define a modified sequence $(d'_n)_{n=0}^{\infty}$ as $d'_n = \beta(d_{k_1+n} - P)$ for any $n \geq 0$, which satisfies $d'_{k'_i} = 0$ where $k'_i = k_i - k_1$, for every $i \geq 1$, and $D' = \{d'_n \mid n \geq 0\}$ is an infinite set. Furthermore, for each $n \geq 0$,

$$\frac{d'_n}{a} - d'_{n+1} = \frac{\beta(d_{k_1+n} - P)}{a} - \beta(d_{k_1+n+1} - P) = \beta\, \frac{b_{k_1+n} - (1-a)P}{a} \in B' \quad (4)$$

is an integer by the definition of B'. In addition, denote $1/a = \alpha/q \in \mathbb{Q}$ where natural number $\alpha > 0$ and integer $q \neq 0$ are coprime.

Lemma 4. *For every $n \geq 0$, there exists an integer δ and a natural number $p \geq 0$ such that $d'_n = \delta/q^p$.*

Proof. We proceed by induction on n. The assertion is obvious for $n = 0$ when $d'_0 = 0$. Assume that $d'_n = \delta/q^p$ for some $\delta \in \mathbb{Z}$ and $p \geq 0$. Then $d'_{n+1} = d'_n/a - b'$ for some integer $b' \in B' \subset \mathbb{Z}$ according to (4), which can be rewritten as $d'_{n+1} = (\alpha/q) \cdot (\delta/q^p) - b' = (\alpha\delta - b'q^{p+1})/q^{p+1} = \delta_1/q^{p+1}$ where $\delta_1 = \alpha\delta - b'q^{p+1} \in \mathbb{Z}$, completing the proof of Lemma 4. □

Lemma 5. *If $d'_{n+1} \in \mathbb{Z}$, then $d'_n \in \mathbb{Z}$.*

Proof. Let $d'_{n+1} \in \mathbb{Z}$. By (4) there is $b' \in B' \subset \mathbb{Z}$ such that $d'_n/a = d'_{n+1} + b' \in \mathbb{Z}$. According to Lemma 4, $d'_n = \delta/q^p$ for some $\delta \in \mathbb{Z}$ and $p \geq 0$, which gives $d'_n/a = \alpha\delta/q^{p+1} \in \mathbb{Z}$. Since α and q are coprime, q^{p+1} must be a factor of δ, which means $\delta = \delta' q^{p+1}$ for some $\delta' \in \mathbb{Z}$, and hence $d'_n = \delta/q^p = \delta' q \in \mathbb{Z}$, completing the proof of Lemma 5. □

We will show for each $n \geq 0$ that $d'_n \in \mathbb{Z}$. Let $i \geq 1$ be the least index such that $k'_i \geq n$ for which we know $d'_{k'_i} = 0 \in \mathbb{Z}$. By applying Lemma 5 $(k'_i - n)$ times we obtain $d'_{k'_i-1}, d'_{k'_i-2}, \ldots, d'_n \in \mathbb{Z}$.

Thus, $D' \subset \mathbb{Z}$ and since D' is infinite, there exists an index $m \geq 0$ such that $|d'_m| \geq (|a| \cdot M)/(1 - |a|) > 0$ where $M = \max_{b' \in B'} |b'|$. Note that $M > 0$ since for $M = 0$, we would have $B = \{(1-a)P\}$ implying $D = \{P\}$ which contradicts that D is infinite. According to (4), $|d'_{m+1}| \geq |d'_m|/|a| - M$ which implies $|d'_{m+1}| - |d'_m| \geq (1/|a| - 1)|d'_m| - M \geq 0$ by the definition of m. Hence, $|d'_{m+1}| \geq |d'_m|$, and by induction we obtain $|d'_n| \geq (|a| \cdot M)/(1 - |a|) > 0$ for every $n \geq m$. On the other hand, we know that there is an index i such that $k'_i \geq m$ for which $d'_{k'_i} = 0$, which is a contradiction completing the proof of Theorem 3. □

We say that a real number c is *a-quasi-periodic within B* if any power series $\sum_{k=0}^{\infty} b_k a^k = c$ with $b_k \in B$ for all $k \geq 0$, is eventually quasi-periodic. Note that c that cannot not be written as a respective power series at all, or can, in addition, be expressed as a finite sum $\sum_{k=0}^{h} b_k a^k = c$ whereas $0 \notin B$, is also considered formally to be a-quasi-periodic. For example, the numbers from the complement of the Cantor set are formally $(1/3)$-quasi-periodic within $\{0, 2\}$.

Example 6. Example 1 can be extended to provide a nontrivial example of a-quasi-periodic numbers. Let $a \in \mathbb{Q}$ meet $0 < a < \frac{1}{2}$. We show that any positive rational number c is a-quasi-periodic within B where $B = \{\beta_1, \beta_2, \beta_3\}$ is defined in Example 1 so that $P = c$. Obviously, $\beta_1 > \beta_2 > \beta_3 = 0$. Assume that $c = \sum_{k=0}^{\infty} b_k a^k$ for some sequence $(b_k)_{k=0}^{\infty}$ where $b_k \in B$ for all $k \geq 0$. Observe first that it must be $b_0 = \beta_1$ since otherwise $c = \sum_{k=0}^{\infty} b_k a^k \leq \beta_2 + \sum_{k=1}^{\infty} \beta_1 a^k = a(1-a)c + (1-a^2)c \cdot a/(1-a) = 2ac < c$ due to $a < \frac{1}{2}$. Moreover, for any $n \geq 0$ such that $b_k = \beta_2$ for every $k = 1, \ldots, n$, it holds $b_{n+1} \neq \beta_1$ since otherwise $c = \sum_{k=0}^{\infty} b_k a^k \geq \beta_1 + \sum_{k=1}^{n} \beta_2 a^k + \beta_1 a^{n+1} = (1-a^2)c + a(1-a)c \cdot a(1-a^n)/(1-a) + (1-a^2)c \cdot a^{n+1} = c - a^{n+1}(a^2+a-1)c > c$ due to $a^2 + a - 1 < 0$ for $0 < a < \frac{1}{2}$.

First consider the case when there is $r \geq 1$ such that $b_k = \beta_2$ for all $k \geq r$. Then $b_0, \ldots, b_{r-1}$ is a preperiodic part and $b_k = \beta_2$ for $k \geq r$ represents a repetend of length $m_k = 1$, which proves $\sum_{k=0}^{\infty} b_k a^k$ to be eventually quasi-periodic. Further assume there is no such r, and thus $b_k = \beta_2$ for every $k = 1, \ldots, n_1$ and $b_{n_1+1} = \beta_3$, for some $n_1 \geq 0$. It follows that series $\sum_{k=0}^{\infty} b_k a^k = c$ starts with a quasi-repetend $\beta_1, \beta_2^{n_1}, \beta_3$ of length n_1+2 (cf. Example 1) which can be omitted as $\sum_{k=0}^{\infty} b_{n_1+2+k} a^k = (c - \sum_{k=0}^{n_1+1} b_k a^k)/a^{n_1+2} = c$ due to $\sum_{k=0}^{n_1+1} b_k a^k = c(1 - a^{n_1+2})$ by (2), and the argument can be repeated for its tail $\sum_{k=0}^{\infty} b_{n_1+2+k} a^k = c$ to reveal the next quasi-repetend $\beta_1, \beta_2^{n_2}, \beta_3$ for some $n_2 \geq 0$ etc. Hence, $\sum_{k=0}^{\infty} b_k a^k$ is quasi-periodic, which completes the proof that c is a-quasi-periodic within B.

Example 7. On the other hand, we present an example of an eventually quasi-periodic series $\sum_{k=0}^{\infty} b_k a^k = c$ with $b_k \in B$ for all $k \geq 0$, such that c is not a-quasi-periodic within B. Let $a = \frac{2}{3}$, $B = \{0, 1\}$, and define an eventually quasi-periodic series $\sum_{k=0}^{\infty} b_k a^k$ with a preperiodic part $b_0 = b_1 = 0$ and a repetend $b_{2+3k} = 0$, $b_{3+3k} = b_{4+3k} = 1$ for every $k \geq 0$, which sums to $c = ((\frac{2}{3})^3 + (\frac{2}{3})^4) \cdot \sum_{k=0}^{\infty} (\frac{2}{3})^{3k} = \frac{40}{57}$.

Furthermore, we employ a greedy approach to generate a series $\sum_{k=0}^{\infty} b'_k a^k = c$ with $b'_k \in \{0, 1\}$ for all $k \geq 0$, which is not eventually quasi-periodic. In particular, find minimal $k_1 \geq 0$ such that $a^{k_1} < c$ which gives $b'_0 = \cdots = b'_{k_1-1} = 0$, $b'_{k_1} = 1$, and remainder $c_1 = c/a^{k_1} - 1$. For $n > 1$, let $b'_0, \ldots, b'_{k_{n-1}}$ be 0s except for $b'_{k_1} = b'_{k_2} = \cdots = b'_{k_{n-1}} = 1$. Then find minimal $k_n > k_{n-1}$ such that $a^{k_n - k_{n-1}} < c_{n-1}$ which produces $b'_{k_{n-1}+1} = \cdots = b'_{k_n-1} = 0$, $b'_{k_n} = 1$, and remainder $c_n = c_{n-1}/a^{k_n - k_{n-1}} - 1$. It follows that $c_n = \sum_{k=0}^{\infty} b'_{k_n+k} a^k - 1 = (c - \sum_{i=1}^{n} a^{k_i})/a^{k_n}$ for $n \geq 1$. By plugging $a = \frac{2}{3}$ and $c = \frac{40}{57}$ into this formula, for which $k_1 = 1$ and $k_2 = 9$, we obtain

$$c_n = \frac{20}{19}\left(\frac{3}{2}\right)^{k_n-1} - \sum_{i=1}^{n}\left(\frac{3}{2}\right)^{k_n-k_i} = \frac{3^{k_n-1} - 19\cdot 2\cdot \sum_{i=2}^{n} 2^{k_i-2}\cdot 3^{k_n-k_i}}{19\cdot 2^{k_n-1}} \tag{5}$$

which is an irreducible fraction since both 19 and 2 are not factors of 3^{k_n-1}. Hence, for any natural n_1, n_2 such that $0 < n_1 < n_2$ we know $c_{n_1} \neq c_{n_2}$. It follows that the tail sequence $(d'_n)_{n=0}^{\infty}$ of $\sum_{k=0}^{\infty} b'_k a^k = c$ contains infinitely many different values $d'_{k_n} = c_n + 1$ for $n \geq 1$, which implies that $\sum_{k=0}^{\infty} b'_k a^k$ is not an eventually quasi-periodic series, according to Theorem 3.

Theorem 8. *A real number c is a-quasi-periodic within B iff the tail sequences of all the power series satisfying $\sum_{k=0}^{\infty} b_k a^k = c$ with $b_k \in B$ for all $k \geq 0$, contain altogether only finitely many values, that is, $\mathcal{D} = \{\sum_{k=0}^{\infty} b_{n+k} a^k \mid n \geq 0\,;$ for any $\sum_{k=0}^{\infty} b_k a^k = c$, $b_k \in B$ for all $k \geq 0\}$ is a finite set.*

Proof. Let $\mathcal{D}$ be a finite set. Then the tail sequence of any power series $\sum_{k=0}^{\infty} b_k a^k = c$ with $b_k \in B$ for all $k \geq 0$, contains only finitely many values, which implies that any $\sum_{k=0}^{\infty} b_k a^k = c$ is eventually quasi-periodic according to Theorem 3. Hence, c is a-quasi-periodic within B.

Conversely, assume that c is a-quasi-periodic within B, which means any power series $\sum_{k=0}^{\infty} b_k a^k = c$ with $b_k \in B$ for all $k \geq 0$, is eventually quasi-periodic. For each such a series, denote by $\mathrm{pre}(\sum_{k=0}^{\infty} b_k a^k) = k_1$ the length of its shortest preperiodic part that meets (3). We define a directed rooted tree $T = (V, E)$ with vertex set $V = \{b_0 \cdots b_{k-1} \in B^* \mid 0 \leq k \leq \mathrm{pre}(\sum_{k=0}^{\infty} b_k a^k)\,,$ for any $\sum_{k=0}^{\infty} b_k a^k = c\}$, including an empty string as a root, and a set of directed edges

$$E = \{(b_0 \cdots b_{k-1}, b_0 \cdots b_{k-1} b_k) \mid b_0 \cdots b_{k-1}, b_0 \cdots b_{k-1} b_k \in V\}\ . \tag{6}$$

Clearly, T covers all the directed paths that start at the root and lead to $b_0 \cdots b_{k_1-1} \in V$ corresponding to a preperiodic part of some eventually quasi-periodic series $\sum_{k=0}^{\infty} b_k a^k = c$. Thus, the outdegree of T is bounded by $|B|$. Suppose that T is infinite. According to König's lemma, there exists an infinite directed path corresponding to a series $\sum_{k=0}^{\infty} b_k a^k = c$ whose shortest preperiodic part is infinite, which contradicts that $\sum_{k=0}^{\infty} b_k a^k$ is eventually quasi-periodic. It follows that there are only finitely many possible preperiodic parts over all the power series $\sum_{k=0}^{\infty} b_k a^k = c$.

Thus, for the proof that $\mathcal{D}$ is finite, it suffices to show that for any preperiodic part $b_0, \ldots, b_{k_1-1} \in B$ of length $k_1 = \mathrm{pre}(\sum_{k=0}^{\infty} b_k a^k)$, which starts a series $\sum_{k=0}^{\infty} b_k a^k = c$ with period sum P, the tail sequences of all quasi-periodic series $\sum_{k=0}^{\infty} b_{k_1+k} a^k = P$ contain altogether only finitely many values. On the contrary, suppose that these tail sequences include infinitely many values. According to (3), any such a value can be expressed as

$$\sum_{k=0}^{m_i-j-1} b_{k_i+j+k}\, a^k + a^{m_i} P \tag{7}$$

for $j \in \{0, \ldots, m_i - 1\}$, by using a quasi-repetend $b_{k_i}, \ldots, b_{k_{i+1}-1}$ of length $m_i = k_{i+1} - k_i$, taken from some series $\sum_{k=0}^{\infty} b_{k_1+k} a^k = P$. Thus, we can construct a new series $\sum_{k=0}^{\infty} b^*_{k_1+k} a^k = P$ that is composed of infinitely many quasi-repetends from the respective series so that each such quasi-repetend introduces a different value (7) in the tail sequence. It follows that series $\sum_{k=0}^{\infty} b^*_{k_1+k} a^k$ is quasi-periodic and its tail sequence contains infinitely many values, which contradicts Theorem 3. This completes the argument that $\mathcal{D}$ is finite. □

3 Regular Cut Languages

In this section we formulate a necessary and sufficient condition for a cut language $L_{<c}$ to be regular (Theorem 11), which is based on a-quasi-periodic thresholds c within B. The following Lemma 9 provides a technical characterization of the regular cut languages, which is proven by Myhill-Nerode theorem, while subsequent Lemma 10 separates the cases when threshold c is represented by a finite sum or when c has no representation in base $1/a$ using the digits from B.

Lemma 9. *Let Σ be a finite alphabet, $b : \Sigma \longrightarrow B$ be a mapping, and c be a real number. Then the cut language $L_{<c} = \{x_1 \cdots x_n \in \Sigma^* \mid \sum_{i=0}^{n-1} b(x_{n-i}) a^i < c\}$ is regular iff the set*

$$C = \left\{ c(b_0, \ldots, b_{\kappa-1}) \,\middle|\, I_\kappa \le c - \sum_{k=0}^{\kappa-1} b_k a^k \le S_\kappa \,;\, b_0, \ldots, b_{\kappa-1} \in B \,;\, \kappa \ge 0 \right\} \tag{8}$$

is finite, where

$$I_\kappa = \inf_{\substack{b_\kappa, \ldots, b_{h-1} \in B \\ h \ge \kappa}} \sum_{k=\kappa}^{h-1} b_k a^k \,, \qquad S_\kappa = \sup_{\substack{b_\kappa, \ldots, b_{h-1} \in B \\ h \ge \kappa}} \sum_{k=\kappa}^{h-1} b_k a^k \,, \tag{9}$$

$$c(b_0, \ldots, b_{\kappa-1}) = \begin{cases} \inf C(b_0, \ldots, b_{\kappa-1}) & \text{if } a^\kappa > 0 \\ \sup C(b_0, \ldots, b_{\kappa-1}) & \text{if } a^\kappa < 0 \,, \end{cases} \tag{10}$$

$$C(b_0, \ldots, b_{\kappa-1}) = \left\{ \sum_{k=0}^{h-\kappa-1} b_{\kappa+k} a^k \,\middle|\, \sum_{k=0}^{h-1} b_k a^k \ge c \,;\, b_\kappa, \ldots, b_{h-1} \in B \,;\, h \ge \kappa \right\} . \tag{11}$$

Proof. Let $C = \{c_1, \ldots, c_p\}$ in (8) be a finite set such that $c_1 < c_2 < \cdots < c_p$. We introduce an equivalence relation $\sim$ on Σ^* as follows. For any $x, y \in \Sigma^*$ of length $n_x = |x|$ and $n_y = |y|$, respectively, we define $x \sim y$ iff both $z_x = \sum_{i=0}^{n_x-1} b(x_{n_x-i}) a^i$ and $z_y = \sum_{i=0}^{n_y-1} b(y_{n_x-i}) a^i$ belong either to one of the $p+1$ open intervals $(-\infty, c_1), (c_1, c_2), \ldots, (c_{p-1}, c_p), (c_p, \infty)$, or to one of the p singletons $\{c_1\}, \{c_2\}, \ldots, \{c_p\}$. Obviously, we have $2p+1$ equivalence classes. In order to prove that language $L_{<c}$ is regular we employ Myhill-Nerode theorem by showing that for any $x, y \in \Sigma^*$, if $x \sim y$, then for every $w \in \Sigma^*$, $xw \in L_{<c}$ iff $yw \in L_{<c}$. Thus, consider $x, y \in \Sigma^*$ such that $x \sim y$, and on the contrary,

suppose there is $w \in \Sigma^*$ of length $\kappa = |w|$ with $z_w = \sum_{i=0}^{\kappa-1} b(w_{\kappa-i})a^i$, such that $xw \in L_{<c}$ and $yw \notin L_{<c}$. This means $z_w + I_\kappa \le z_w + a^\kappa z_x < c \le z_w + a^\kappa z_y \le z_w + S_\kappa$ by (9), implying $I_\kappa < c - z_w \le S_\kappa$ which ensures $c_j = c(b(w_\kappa), \ldots, b(w_1)) \in C$ for some $j \in \{1, \ldots, p\}$, according to (8). It follows from (10) and (11) that $z_w + a^\kappa z_x < c \le z_w + a^\kappa c_j \le z_w + a^\kappa z_y$ which gives $a^\kappa z_x < a^\kappa c_j \le a^\kappa z_y$ contradicting $x \sim y$.

Conversely, let $L_{<c}$ be a regular languages. According to Myhill-Nerode theorem, there is an equivalence relation $\sim$ on Σ^* with a finite number p of equivalence classes such that for any $x, y \in \Sigma^*$, if $x \sim y$, then for every $w \in \Sigma^*$, $xw \in L_{<c}$ iff $yw \in L_{<c}$. Assume to the contrary that C in (8) is infinite. Choose $c_0, c_1, \ldots, c_{2p+2} \in C$ so that $c_0 < c_1 < \cdots < c_{2p+2}$, and for each $j \in \{0, \ldots, 2p+2\}$, let $c_j = c(b_{j0}, \ldots, b_{j,\kappa_j-1})$ for some $b_{j0}, \ldots, b_{j,\kappa_j-1} \in B$ and $\kappa_j \ge 0$, according to (8). Definition (10) and (11) ensures that for each odd $j \in \{1, 3, \ldots, 2p+1\}$, there exists $h_j \ge \kappa_j$ and $b_{j,\kappa_j}, \ldots, b_{j,h_j-1} \in B$ such that $c'_j = \sum_{k=0}^{h_j-\kappa_j-1} b_{j\kappa_j+k}a^k$ is sufficiently close to c_j so that $c_{j-1} < c'_j < c_{j+1}$. Since there are only p equivalence classes, there must be two odd indices $j_x, j_y \in \{1, 3, \ldots, 2p+1\}$, say $j_x < j_y$, determining $x, y \in \Sigma^*$ of length $n_x = |x| = h_{j_x} - \kappa_{j_x}$ and $n_y = |y| = h_{j_y} - \kappa_{j_y}$, respectively, by $b(x_{n_x-i}) = b_{j_x,\kappa_{j_x}+i}$ for $i = 0, \ldots, n_x - 1$ and $b(y_{n_y-i}) = b_{j_y,\kappa_{j_y}+i}$ for $i = 0, \ldots, n_y - 1$, such that $x \sim y$. Thus, $c'_{j_x} = \sum_{i=0}^{n_x-1} b(x_{n_x-i})a^i$ and $c'_{j_y} = \sum_{i=0}^{n_y-1} b(y_{n_y-i})a^i$. For $a^\kappa > 0$, choose $w \in \Sigma^*$ of length $\kappa = |w| = \kappa_{j_x+1}$ so that $c_{j_x+1} = c(b(w_\kappa), \ldots, b(w_1))$, and denote $z_w = \sum_{i=0}^{\kappa-1} b(w_{\kappa-i})a^i$. We know $c'_{j_x} < c_{j_x+1} < c'_{j_y}$. It follows that $z_w + a^\kappa c'_{j_x} < c \le z_w + a^\kappa c_{j_x+1} < z_w + a^\kappa c'_{j_y}$ since $z_w + a^\kappa c'_{j_x} \ge c$ would contradict that c_{j_x+1} is the infimum according to (10) and (11). Hence, $xw \in L_{<c}$ and $yw \notin L_{<c}$, which gives the contradiction. Similarly for $a^\kappa < 0$, choose $w \in \Sigma^*$ so that $c_{j_y-1} = c(b(w_\kappa), \ldots, b(w_1))$, which gives $z_w + a^\kappa c'_{j_y} < c \le z_w + a^\kappa c_{j_y-1} < z_w + a^\kappa c'_{j_x}$, leading to the contradiction $xw \notin L_{<c}$ and $yw \in L_{<c}$. □

Lemma 10. *Assume the notation as in Lemma 9. Then the two subsets of* C, $C_1 = \{c(b_0, \ldots, b_{\kappa-1}) \in C \,|\, \sum_{k=0}^{\kappa-1} b_k a^k + a^\kappa c(b_0, \ldots, b_{\kappa-1}) > c\}$ *and* $C_2 = \{c(b_0, \ldots, b_{\kappa-1}) \in C \,|\, (\exists\, b_\kappa, \ldots, b_{h-1} \in B\,,\, h \ge \kappa) \sum_{k=0}^{h-1} b_k a^k = c \;\&\; (\forall\, b \in B)\; c(b_0, \ldots, b_{h-1}, b) \in C_1\}$ *are finite.*

Proof. We define a directed rooted tree $T = (V, E)$ with vertex set $V = \{b_0 \cdots b_{k-1} \in B^* \,|\, (\exists\, b_k, \ldots, b_{\kappa-1} \in B)\, c(b_0, \ldots, b_{k-1}, b_k \ldots, b_{\kappa-1}) \in C_1\}$, including an empty string as a root, and a set of directed edges (6). Clearly, T covers all the directed paths starting at the root and leading to $b_0 \ldots b_{\kappa-1} \in V$ such that $c(b_0, \ldots, b_{\kappa-1}) \in C_1$. This also guarantees that T includes all $b_0 \ldots b_{\kappa-1} \in V$ such that $c(b_0, \ldots, b_{\kappa-1}) \in C_2$, by the definition of C_2. For each vertex $b_0 \cdots b_{k-1} \in V$ we define a closed interval $I(b_0, \ldots, b_{k-1}) = [\sum_{i=0}^{k-1} b_i a^i + I_k\,,\, \sum_{i=0}^{k-1} b_i a^i + S_k]$ by using (9). Obviously, $I(b_0, \ldots, b_{k-1}, b_k) \subset I(b_0, \ldots, b_{k-1})$ for any edge $(b_0 \cdots b_{k-1}, b_0 \cdots b_{k-1}b_k) \in E$. Hence, $c \in I(b_0, \ldots, b_{k-1})$ for every vertex $b_0 \cdots b_{k-1} \in V$ since $b_0 \cdots b_{k-1} \cdots b_{\kappa-1} \in V$ such that $c(b_0, \ldots, b_{\kappa-1}) \in C_1$ satisfies $c \in I(b_0, \ldots, b_{\kappa-1}) \subset I(b_0, \ldots, b_{k-1})$ according to (8).

On the contrary, suppose that tree T whose outdegree is bounded by $|B|$, is infinite. According to König's lemma, there exists an infinite directed path corresponding to an infinite sequence $(b_k^*)_{k=0}^\infty$ with $b_k^* \in B$ for all $k \geq 0$, which contains infinitely many vertices $b_0^* \cdots b_{\kappa-1}^* \in V$ such that $c(b_0^*, \ldots, b_{\kappa-1}^*) \in C_1$. On the other hand, interval $I(b_0^*, \ldots, b_{k-1}^*)$ is a nonempty compact set satisfying $c \in I(b_0^*, \ldots, b_{k-1}^*) \supset I(b_0^*, \ldots, b_k^*)$ for every $k \geq 1$, which yields $c \in \bigcap_{k \geq 0} I(b_0^*, \ldots, b_{k-1}^*) \neq \emptyset$ by Cantor's intersection theorem. Hence, $\sum_{k=0}^\infty b_k^* a^k = c$ which implies $\sum_{k=0}^{\kappa-1} b_k^* a^k + a^\kappa c(b_0^*, \ldots, b_{\kappa-1}^*) = c$ for any $b_0^* \cdots b_{\kappa-1}^* \in V$ such that $c(b_0^*, \ldots, b_{\kappa-1}^*) \in C_1$, according to (10) and (11), which contradicts the definition of C_1. It follows that T is finite which implies that C_1, C_2 are finite. □

Theorem 11. *A cut language $L_{<c}$ is regular iff c is a-quasi-periodic within B.*

Proof. According to Lemma 9, language $L_{<c}$ is regular iff set C is finite which is equivalent to the condition that $C \setminus (C_1 \cup C_2)$ is finite, by Lemma 10. It follows from (8)–(11) that for any $b_0, \ldots, b_{\kappa-1} \in B$ and $\kappa \geq 0$, $c(b_0, \ldots, b_{\kappa-1}) \in C \setminus (C_1 \cup C_2)$ iff there exists sequence $(b_k)_{k=\kappa}^\infty$ with $b_k \in B$ for all $k \geq 0$, such that $\sum_{k=0}^{\kappa-1} b_k a^k + a^\kappa c(b_0, \ldots, b_{\kappa-1}) = c$ ($c(b_0, \ldots, b_{\kappa-1}) \notin C_1$) and $\sum_{k=0}^\infty b_k a^k = c$ ($c(b_0, \ldots, b_{\kappa-1}) \notin C_2$), which yields $c(b_0, \ldots, b_{\kappa-1}) = \sum_{k=0}^\infty b_{\kappa+k} a^k$. It follows that $C \setminus (C_1 \cup C_2) = \mathcal{D}$ by the definition of $\mathcal{D}$, which is finite iff c is a-quasi-periodic within B, according to Theorem 8. □

4 Non-Context-Free Cut Languages

The following Theorem 13 shows that the cut languages with a threshold whose greedy representation (in base $1/a$ using the digits from B) is not eventually quasi-periodic, are not context-free, which is proven by the pumping lemma. For this purpose, we present examples of rational numbers with no eventually quasi-periodic representations in Example 12. On the other hand, the cut languages with rational thresholds are shown to be context-sensitive in Theorem 14.

Example 12. We generalize Example 7 to provide instances of rational numbers c such that any power series $\sum_{k=0}^\infty b'_k a^k = c$ with $b'_k \in B$ for all $k \geq 0$, is not eventually quasi-periodic. Let $B = \{0, 1\}$ and $a = \alpha_1/\alpha_2$, $c = \gamma_1/\gamma_2 \in \mathbb{Q}$ be irreducible fractions where $\alpha_1, \gamma_1 \in \mathbb{Z}$ and $\alpha_2, \gamma_2 \in \mathbb{N}$, such that $\alpha_1\gamma_2$ and $\alpha_2\gamma_1$ are coprime. Denote by $0 < k_1 < k_2 < \cdots$ all the indices of a (not necessarily greedy) representation of $c = \sum_{k=0}^\infty b'_k a^k$ such that $b'_{k_i} = 1$ for $i \geq 1$. Then formula (5) can be rewritten as

$$c_n = \frac{\gamma_1 \alpha_2^{k_n} - \gamma_2 \alpha_1 \sum_{i=1}^n \alpha_1^{k_i - 1} \alpha_2^{k_n - k_i}}{\gamma_2 \alpha_1^{k_n}} \tag{12}$$

which is still an irreducible fraction.

Theorem 13. *Let $B = \{0, 1\}$ and assume that the greedy representation of threshold $c = \sum_{k=0}^\infty b'_k a^k$ with $b'_k \in B$ for all $k \geq 0$, is not eventually quasi-periodic (see Example 12 for instances of such $c \in \mathbb{Q}$). Then the cut language $L_{<c}$ is not context-free.*

Proof. Consider a cut language $L_{<c}$ over alphabet $\Sigma = B = \{0,1\}$, where $b : \Sigma \longrightarrow B$ is the identity mapping. Let sequence $(b'_k)_{k=0}^{\infty}$ corresponding to power series $\sum_{k=0}^{\infty} b'_k a^k = c$ with $b'_k \in B$ for all $k \geq 0$, be generated by the greedy algorithm (see Example 7), which is assumed to be not eventually quasi-periodic. On the contrary, suppose that $L_{<c}$ is a context-free language, and hence the same holds for its reversal

$$L = L_{<c}^R = \left\{ b_0 \ldots b_{n-1} \in \{0,1\}^* \;\middle|\; \sum_{i=0}^{n-1} b_i a^i < c \right\} . \tag{13}$$

The greedy algorithm ensures that for any other $(b_k)_{k=0}^{\infty} \neq (b'_k)_{k=0}^{\infty}$ such that $c = \sum_{k=0}^{\infty} b_k a^k$ with $b_k \in \{0,1\}$ for all $k \geq 0$, there exists an index $n \geq 0$ such that $b_i = b'_i$ for every $i = 0, \ldots, n-1$, and $b_n < b'_n$, which means $(b'_k)_{k=0}^{\infty}$ is the greatest such sequence with respect to the lexicographic order $\preceq$. We will apply the pumping lemma to context-free language L, which guarantees there is an integer $p \geq 1$ such that every word $b_0 \ldots b_{n-1} \in L$ of length $n \geq p$, can be written as $uvwxy$ with substrings $u, v, w, x, y \in \{0,1\}^*$ satisfying $|vwx| \leq p$, $|vx| \geq 1$, and $uv^i wx^i y \in L$ for all $i \geq 0$.

Thus, consider a prefix $\beta_n = b'_0 \ldots b'_{n-1} \in \{0,1\}^*$ of the sequence $(b'_k)_{k=0}^{\infty}$, for any length $n = |\beta_n| \geq p$. It follows from (13) that $\beta_n \in L$ since $\sum_{i=0}^{n-1} b'_i a^i < c = \sum_{i=0}^{\infty} b'_i a^i$ due to $\sum_{i=n}^{\infty} b'_i a^i > 0$ as $b'_i = 1$ for some $i \geq n$, by the non-periodicity of $(b'_k)_{k=0}^{\infty}$. Hence, $\beta_n \in L$ can be written as $uvwxy$ with the respective substrings from the pumping lemma. Thus, we have $uwy = uv^0 wx^0 y \in L$, implying $uwy \prec uvwxy = \beta_n = b'_0 \ldots b'_{n-1} \prec (b'_k)_{k=0}^{\infty}$ in the strict lexicographic order due to $|uwy| < |\beta_n|$ because of $|vx| \geq 1$, which reduces to

$$w \prec vwx\,. \tag{14}$$

Furthermore, for every $i \geq 1$, we have $uv^i wx^i y \in L$, which implies that either $uv^i wx^i y \prec \beta_n$ in the lexicographic order or $\beta_n \in \mathrm{Pref}(uv^i wx^i y)$ where $\mathrm{Pref}(s) = \{s_1 \in \{0,1\}^* \,|\, (\exists\, s_2 \in \{0,1\}^*)\, s = s_1 s_2\}$ denotes the set of prefixes of a string $s \in \{0,1\}^*$. Suppose first that there exists $j \geq 2$ such that $uv^j wx^j y \prec \beta_n = uvwxy$ which reduces to $v^{j-1} wx^{j-1} \prec w$. By applying inequality (14), we further obtain $v^{j-2} wx^{j-2} \prec w$, which, repeated $(j-1)$ times, leads to $vwx \prec w$, contradicting (14).

It follows that $\beta_n = uvwxy \in \mathrm{Pref}(uv^i wx^i y)$ for all $i \geq 1$. If $|v| \geq 1$, then there exists $j \geq 1$ such that $|uv^{j-1}| < |\beta_n| \leq |uv^j|$ and $\beta_n \in \mathrm{Pref}(uv^j)$, which means $\beta_n = uv^j v_1 = uv_1(v_2 v_1)^j$ where $v = v_1 v_2$ for some $v_1, v_2 \in \{0,1\}^*$. Thus, we can write $\beta_n = \mu\nu$ when denoting $\mu = uv_1\nu^{j-1}$ and $\nu = v_2 v_1$, which satisfies $1 \leq |\nu| = |v| \leq |vwx| \leq p$. In addition, $\beta_n \nu^i \sigma_i = uv^{j+i+1} wx^{j+i+1} y \in L$ where $\sigma_i = v_2 wx^{j+i+1} y$, for every $i \geq 0$. The same holds for $|v| = 0$, when $|x| \geq 1$ due to $|vx| \geq 1$, which ensures that there is $j \geq 1$ such that $|uwx^{j-1}| < |\beta_n| \leq |uwx^j|$ and $\beta_n \in \mathrm{Pref}(uwx^j)$. Thus, we can again write $\beta_n = uwx^j x_1 = uwx_1(x_2 x_1)^j = \mu\nu$ where $x = x_1 x_2$ for some $x_1, x_2 \in \{0,1\}^*$, and $\mu = uwx_1\nu^{j-1}$ and $\nu = x_2 x_1$, satisfying $1 \leq |\nu| = |x| \leq |vwx| \leq p$. Moreover, $\beta_n \nu^i \sigma_i = uv^{j+i+1} wx^{j+i+1} y \in L$ where $\sigma_i = x_2 y$, for every $i \geq 0$.

Since $(b'_k)_{k=0}^{\infty}$ is not periodic, there exists $\ell \geq 0$ and $\nu' \in \mathrm{Pref}(\nu)$ such that $b'_0 \dots b'_{r-1} = \beta_r = \beta_n \nu^\ell \nu' = \mu \nu^{\ell+1} \nu'$ and $\nu' 0 \in \mathrm{Pref}(\nu)$ where $0 \leq |\nu'| < m = |\nu| \leq p$ and $r = n + \ell\, m + |\nu'|$, while $\beta_r 0 \prec \beta_{r+1} = \beta_r 1$. It follows that $b'_{r-m} = 0$ due to $\beta_{r-m+1} = \mu \nu^\ell \nu' 0$, which implies

$$d_{r-m} = \frac{d_{n-m} - \sum_{k=0}^{r-n-1} b'_{n-m+k}\, a^k}{a^{r-n}} < 1 \tag{15}$$

where $(d_n)_{n=0}^{\infty}$ is the tail sequence of $(b'_k)_{k=0}^{\infty}$ that was generated by the greedy approach. On the other hand, the greedy algorithm ensures

$$d_r = \frac{d_n - \sum_{k=0}^{r-n-1} b'_{n+k}\, a^k}{a^{r-n}} > 1 \tag{16}$$

because of $b'_r = 1$. We have $b'_{n-m} \dots b'_{r-m-1} = \nu^\ell \nu' = b'_n \dots b'_{r-1}$ which yields $\sum_{k=0}^{r-n-1} b'_{n-m+k}\, a^k = \sum_{k=0}^{r-n-1} b'_{n+k}\, a^k$, and hence,

$$d_{n-m} < d_n \tag{17}$$

according to (15) and (16).

The preceding analysis is valid for prefix β_n of arbitrary length $n \geq p$. Thus, suppose that $\beta_n \in \mathrm{Pref}(uv^i w x^i y)$ with strings u, v, w, x, y specific to each β_n, holds for every $n \geq p$ and for all $i \geq 1$. Denote by $\mathbb{N}_p$ the set of natural numbers greater or equal p and define a mapping $\pi : \mathbb{N}_p \longrightarrow \mathbb{N}_0$ as $\pi(n) = n - m_n$ for every $n \geq p$, where $m_n = |\nu|$ is the length of the string ν specific to β_n, which satisfies $1 \leq m_n \leq p$. We introduce an infinite directed forest $T = (V, E)$ where $V = \pi(\mathbb{N}_p)$ and $E = \{(\pi(n), n) \,|\, n \in V\}$, which has the outdegree bounded by p due to $n - \pi(n) \leq p$. Observe that T is a disjoint union of at most p directed trees with the roots from $\{0, \dots, p-1\}$ having zero indegree, and thus one of these trees must be infinite containing an infinite path according to König's lemma. Hence, there is an infinite subsequence $(d_{k_n})_{n=0}^{\infty}$ such that $k_n = \pi(k_{n+1})$ for all $n \geq 1$, which is increasing according to (17) and upper bounded by $\sum_{k=0}^{\infty} a^k = 1/(1-a)$. It follows that d_{k_n} converges to some P when n tends to infinity, which implies

$$P_n = \frac{\sum_{i=0}^{m_n-1} b'_{k_{n-1}+i}\, a^i}{1 - a^{m_n}} = \frac{d_{k_{n-1}} - a^{m_n} d_{k_n}}{1 - a^{m_n}} \overset{n \to \infty}{\longrightarrow} P\,. \tag{18}$$

Nevertheless, the set $\{P_n \,|\, n \geq 1\}$ is finite due to $m_n \leq p$, which means $P_n = P$ for all sufficiently large n. Hence, $(b'_k)_{k=0}^{\infty}$ is eventually quasi-periodic which is a contradiction, completing the proof that $L_{<c}$ is not a context-free language. $\square$

Theorem 14. *Every cut language $L_{<c}$ with threshold $c \in \mathbb{Q}$ is context-sensitive.*

Proof. A corresponding (deterministic) linear bounded automaton M that accepts a given cut language $L_{<c} = L(M)$, evaluates (and stores) the sum $s_n = \sum_{i=0}^{n-1} b(x_{n-i}) a^i$ step by step when reading an input word $x_1 \dots x_n \in \Sigma^*$ from left to right. In particular, M starts with $s_0 = 0$ which updates to $s_i = a s_{i-1} + b(x_i)$

every time after M reads the next input symbol $x_i \in \Sigma$, for $i = 1, \ldots, n$. As the numbers $a, b(x_1), \ldots, b(x_n), c \in \mathbb{Q}$ can be represented within constant space, M needs only linear space in terms of input length n, for computing s_n and testing whether $s_n < c$. □

5 Conclusion

In this paper we have introduced the cut languages in rational bases and classified them within the Chomsky hierarchy, among others, by using the quasi-periodic power series. A natural direction for future research is to generalize the results to arbitrary real bases. For example, an open problem behind Theorem 3 can be formulated elementarily as follows. Let a be a real number such that $0 < |a| < 1$, and $(d_n)_{n=0}^{\infty}$ be a sequence of real numbers, containing a constant infinite subsequence (cf. Lemma 2), such that $B = \{d_n - ad_{n+1} \mid n \geq 0\}$ is finite. Is $D = \{d_n \mid n \geq 0\}$ a finite set?

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