Weakly Implicative (Fuzzy) Logics

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Abstract:

This paper presents two classes of propositional logics (understood as a consequence relation). First of them is called *weakly implicative logics*. This class of logics generalizes the well-known Rasiowa's class of implicative logics. This class is broad enough to contain many "usual" logics, yet easily manageable with nice logical properties. Then we introduce a subclass of weakly implicative logics—the class of *weakly implicative fuzzy logics*. This class contains majority of logics studied in the literature under the name fuzzy logic. We present many general theorems for both classes, demonstrating their usefulness and importance. We also provide a uniform way how to define first-order calculi for weakly implicative logics and an additional (different) way how to do it for fuzzy logics.

Keywords:
Implicative Logic, Fuzzy Logic, Predicate Logic, Consequence Relation, Matrix Semantics

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This technical reports roughly sums up my work in the last few months about weakly implicative logics and weakly implicative fuzzy logics. There are many open problems and a lot of work is to be done. However, I think that the main ideas and directions towards further research are presented here. Any comments, remarks, suggestions or objections are welcomed.

Introduction

This paper presents two interesting classes of propositional logics (by logic we understand a consequence relation). First of them is called *weakly implicative logics*. This class of logics generalizes the well-known class of implicative logics of Rasiowa (see [7]). We will see that this class is broad enough to contain many "usual" logics (of course this class is rather narrow from the perspective of abstract algebraic logic), yet easily manageable with nice logical properties. Then we introduce a subclass of weakly implicative logics—the class of *fuzzy logics* (or better: of weakly implicative fuzzy logics).

Fuzzy logic is a very fancy term. Many "things" are known as fuzzy logic, some of them are very distant from the subject usually understood under the name "logic". In this work we identify a class of objects which "deserve" the name fuzzy logic. Of course, such a bold statement can be easily objected and so we restrict ourselves a little and we will try to describe which weakly implicative logic could (and should) be considered fuzzy. It turns out that "fuzzy logics" is the class of logics which is complete w.r.t. *linearly ordered* matrices. Then we show several equivalent definition of this class, and so we demonstrate the "robustness" of this class.

However, the goal of this paper is not to present any philosophical, pragmatical or methodological reasons to support this claim. We concentrate on the mathematical properties of *our* fuzzy logics. We only notice that nearly all particular logics studied in literature under the name fuzzy logics which are weakly implicative are fuzzy logics in our formal sense as well. Thus our development of general tools to work with our class of fuzzy logics is applicable in these particular logics as well.

Anyway, we present many general theorems for both classes, which, we hope, will demonstrate usefulness and importance of studying them. We also provide a uniform way how to define first-order calculi for weakly implicative logics and an additional (different) way how to do it for fuzzy logics.

1 General theory for propositional logic

We start with some obligatory introductory definitions. However, we will use rather non-standard approach towards propositional logic. Of course, it turns out to be equivalent (or nearly equivalent) to the known ones, but here we put stress on some issues, concerning mainly the notion of proof and meta-rule. For the comprehensive survey (with the standard terminology) into the problematic of general approach towards logic, consequence relation, logical matrices, etc. see the survey to Abstract algebraic logic (AAL) [3].

In the first subsection we introduce *weakly implicative logics*. Then, in the second subsection, we present the semantic for them (using well-known notion of logical matrix). The third subsection is the core part of this work. There we introduce the class of fuzzy logic (or better: weakly implicative fuzzy logics).

1.1 Syntax

Definition 1.1 (Propositional language) *A* propositional language $\mathcal{L}$ *is a triple* $(\mathbf{VAR}, \mathbf{C}, \mathbf{a})$*, where* $\mathbf{VAR}$ *is a non-empty set of the (propositional)* variables*,* $\mathbf{C}$ *is a non-empty set of the (propositional)* connectives*, and* $\mathbf{a}$ *is a function assigning to each element of* C *a natural number. A connective* c *for which* $a(c) = 0$ *is called a truth constant.*

The set $\mathbf{VAR}$ is usually taken as fixed countable set, and so we usually define the propositional language $\mathcal{L}$ is a pair $(\mathbf{C}, \mathbf{a})$. Later on we fix symbols for some basic connectives $(\rightarrow, \wedge, \perp)$ together with their arities and then we define propositional language just as a set of connectives.

Definition 1.2 (Formula) *Let $\mathcal{L}$ be a propositional language. The set of (propositional)* formulas $\mathbf{FOR}_\mathcal{L}$ *is the smallest set which contains the set* $\mathbf{VAR}$ *and is closed under connectives from* $\mathbf{C}$*, i.e., for each $c \in \mathbf{C}$, such that $\mathbf{a}(c) = n$, and for each $\varphi_1, \dots, \varphi_n \in \mathbf{FOR}_\mathcal{L}$ we have $c(\varphi_1, \dots, \varphi_n) \in \mathbf{FOR}_\mathcal{L}$.*

Definition 1.3 (Substitution) *Let $\mathcal{L}$ be a propositional language. A* substitution *is a mapping $\sigma : \mathbf{FOR}_\mathcal{L} \to \mathbf{FOR}_\mathcal{L}$, such that $\sigma c(\varphi_1, \dots, \varphi_n) = c(\sigma(\varphi_1), \dots \sigma(\varphi_n))$. The set of all substitutions will be denoted as* $\mathbf{SUB}_\mathcal{L}$

Of course a substitution is fully determined by its values on propositional variables. Let $v \in \mathbf{VAR}$ and $\varphi, \psi \in \mathbf{FOR}_\mathcal{L}$, by $\psi[v := \varphi]$ we understand the formula $\sigma(\psi)$, where σ is the substitution mapping v to φ and mapping all the remaining propositional variables to itself.

The term *consecution* in the following definition is from the Restall's book [8]. However, we use it in very simplified version.

Definition 1.4 (Consecution) *A* consecution *in propositional language $\mathcal{L}$ is a pair $< X, \varphi >$, where $X \subseteq \mathbf{FOR}_\mathcal{L}$ and $\varphi \in \mathbf{FOR}_\mathcal{L}$. The set of all consecutions will be denoted as $\mathcal{CON}_\mathcal{L}$.*

Of course we have $\mathcal{P}(\mathbf{FOR}_\mathcal{L}) \times \mathbf{FOR}_\mathcal{L} = \mathcal{CON}_\mathcal{L}$.

Convection 1.5 *Let $X \subseteq \mathbf{FOR}_\mathcal{L}$, $\mathcal{X} \subseteq \mathcal{CON}_\mathcal{L}$ and σ a substitution.*

- *By $\sigma(X)$ we understand the set $\{\sigma(\varphi) \mid \varphi \in X\}$,*
- *By $\sigma(\mathcal{X})$ we understand the set $\{< \sigma(X), \sigma(\varphi) > \mid < X, \varphi > \in \mathcal{X}\}$.*
- *By $\mathcal{SUB}_\mathcal{L}(\mathcal{X})$ we denote the set $\bigcup_{\sigma \in \mathbf{SUB}_\mathcal{L}} \sigma(\mathcal{X})$*

Definition 1.6 (Axiomatic system) *Let $\mathcal{L}$ be a propositional language. The* axiomatic system $\mathcal{AS}$ *in language $\mathcal{L}$ is a non-empty set $\mathcal{AS} \subseteq \mathcal{CON}_\mathcal{L}$, which is closed under arbitrary substitution (i.e., $\mathcal{SUB}_\mathcal{L}(\mathcal{AS}) = \mathcal{AS}$).*

The elements of $\mathcal{AS}$ of the form $< X, \varphi > \in \mathcal{AS}$ are called axioms *for $X = \emptyset$,* n-ary deduction rules *for $|X| = n$, and* κ-infinitary deduction rules *for X being of infinite cardinality κ.*

The axiomatic system is said to be finite *if there is a finite set $\mathcal{X} \subseteq \mathcal{AS}$ such that $\mathcal{SUB}_\mathcal{L}(\mathcal{X}) = \mathcal{AS}$. Furthermore, the axiomatic system is said to be* finitary *if all its deduction rules are finite. Finally, the axiomatic system is said to be* strongly finite *if it is finite and finitary.*

The usual way of presenting an axiomatic system is in form of schemata.

Definition 1.7 (Consequence) *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$.* Theory *T in $\mathcal{L}$ is a subset of $\mathbf{FOR}_\mathcal{L}$. The set $CNS_\mathcal{AS}(T)$ of all* provable formulas *in T is the smallest set of formulas, which contains T, axioms of $\mathcal{AS}$ and is closed under all deduction rules of $\mathcal{AS}$ (i.e., if $X \subseteq CNS_\mathcal{AS}(T)$ and $< X, \varphi > \in \mathcal{AS}$ then $\varphi \in CNS_\mathcal{AS}(T)$. We shall write $T \vdash_\mathcal{AS} \varphi$ to denote $\varphi \in CNS_\mathcal{AS}(T)$ and $\vdash_\mathcal{AS} \varphi$ to denote $\varphi \in CNS_\mathcal{AS}(\emptyset)$.*

Notice that the relation $\vdash_\mathcal{AS}$ can be understood as a subset of $\mathcal{CON}_\mathcal{L}$ and $\mathcal{AS} \subseteq \vdash_\mathcal{AS}$.

Definition 1.8 (Logic) *Let $\mathcal{L}$ be a propositional language. A non-empty set $\mathbf{L} \subseteq \mathcal{CON}_\mathcal{L}$ is called a* logic *in language $\mathcal{L}$ if it is closed under arbitrary substitution and $\vdash_\mathbf{L} = \mathbf{L}$*

Logic is a consequence relation in the usual sense. The elements of a logic are consecutions and we write $X \vdash_\mathbf{L} \varphi$ instead of $< X, \varphi > \in \mathbf{L}$. Sometimes when the logic $\mathbf{L}$ is clear from the context, we write just $\vdash$ instead of $\vdash_\mathbf{L}$. Observe that $\vdash_\mathcal{AS}$ is the smallest logic containing $\mathcal{AS}$

Definition 1.9 (Presentation) *Let $\mathcal{L}$ be a propositional language, $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$, and $\mathbf{L}$ a logic in $\mathcal{L}$. We say that $\mathcal{AS}$ is an* axiomatic system *for (a* presentation *of) the logic $\mathbf{L}$ iff $\mathbf{L} = \vdash_\mathcal{AS}$. We denote the set $CNS_\mathcal{AS}(\emptyset)$ as $\mathcal{THM}(\mathbf{L})$. Logic is said to be finite (finitary, strongly finite) if it has some finite (finitary, strongly finite) presentation.*

Later on we show that our notion of finitary logic coincides with the usual one. Observe that each logic has at least one presentation.

Definition 1.10 (Proof) *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$. A* proof *of the formula φ in theory T is a founded tree labelled by formulas; the root is labelled by φ and leaves by either axioms or elements of T; and if a node is labelled by ψ and its preceding nodes are labelled by $\psi_1, \psi_2, \ldots$ then $<\{\psi_1, \psi_2, \ldots\}, \psi> \in \mathcal{AS}$. We shall write $T \vdash^p_{\mathcal{AS}} \varphi$ if there is a proof of φ in T.*

We understand the tree in an top-to-bottom fashion: the leaves are at the top and the root is at the bottom of the tree, so the fact that tree is founded just means that there is no infinitely long branch.

Theorem 1.11 *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$. Then $T \vdash_{\mathcal{AS}} \varphi$ iff $T \vdash^p_{\mathcal{AS}} \varphi$.*

Proof: Let us define the set $CNS_p(T) = \{\varphi \mid T \vdash^p_{\mathcal{AS}} \varphi\}$. If we show that $CNS_{\mathcal{AS}}(T) = CNS^p_{\mathcal{AS}}(T)$ for each T the proof is done. Obviously $CNS^p_{\mathcal{AS}}(T)$ contains T, axioms of $\mathcal{AS}$ and is closed under all deduction rules of $\mathcal{AS}$, thus $CNS_{\mathcal{AS}}(T) \subseteq CNS^p_{\mathcal{AS}}(T)$. Reverse direction is trivial using the induction over well-founded relation. QED

Lemma 1.12 (Finitary logic) *Let* **L** *be a logic. Then* **L** *is finitary iff for each theory T and formula φ we have: if $T \vdash \varphi$ then there is finite $T' \subseteq T$ such that $T' \vdash \varphi$.*

Proof: Then **L** is finitary then there is its finitary presentation $\mathcal{AS}$. Observe that for each finitary $\mathcal{AS}$ the proofs are always finite (because the tree has no infinite branch and because $\mathcal{AS}$ is finitary each node has finitely many preceding nodes and so we can use König's Lemma to get that the tree is finite). The reverse direction is almost straightforward. QED

Observe that in finitary case we can linearize the tree, i.e., define the notion of the proof in the usual way. The notion of proof allows us to illustrate one important (and usually overlooked) feature of our way of introducing logic. The deduction rule $<\{\psi_1, \psi_2, \ldots\}, \varphi>$, or better written as $\psi_1, \psi_2, \ldots \vdash \varphi$ gives us a way how to construct proof of φ using proofs of $\psi_1, \psi_2, \ldots$. However, the *meta-rule*: from $\vdash \psi_1$, $\vdash \psi_2, \ldots$ get $\vdash \varphi$ only tells us that if there are proof of $\psi_1, \psi_2, \ldots$, then there is proof of φ, without any hint how to construct it. When we introduce semantics, we will se that the former corresponds to the so-called *local* and the latter to the *global* consequence. This distinction deserves more treatment! Also rules are rules of inference between *formulas*, whereas meta-rules are rules of inference between *consecutions*. Both our definitions hide (in some sense) the *default* rules of consequence (thinning, permutation, contraction, and cut).

Definition 1.13 *A logic* **L** *is an* extension of a logic **L**$'$ *iff $L' \subseteq L$. The extension is* axiomatic *if the logic* **L** *is axiomatized by logic* **L**$'$ *(understood as an axiomatic system) with some additional* axioms. *The extension is* conservative *if for each theory T and formula φ in the language of* **L**$'$ *we have: $T \vdash_{\mathbf{L}} \varphi$ entails $T \vdash_{\mathbf{L}'} \varphi$.*

Notice, that the above definition does not mention the language of the logic in question. However, it is obvious that if **L** is an extension of the logic **L**$'$ then the language of **L** has to be larger than the language of **L**$'$. If the language is strictly larger we speak about *expansion* rather than about *extension*.

Now we define the crucial concept of this paper: the notion of weakly implicative logic. We assume that there is a binary connective $\to$ in the propositional language. There is an obvious generalization of this concept, if we drop the condition of the presence of the connective $\to$ in language, and following AAL tradition we would understand the term $\varphi \to \psi$ as a set of formulas of two propositional variables, where the formula φ is substituted for the first one and the formula ψ for the second one. Observe that some of theorems proven in this paper remain theorems under this interpretation. It is easy to see, that in this more general approach (which we will not pursuit in this paper) the defined class of logics would be a subclass of so-called *equivalential logics*, and in the more specific approach (we use in this paper) the class of logics defined by the following definition is a subclass of so-called *finitely equivalential logics*. The name *weakly implicative logics* is inspired by the notion of *implicative logics* by Helena Rasiowa (see [7]). As you will see our notion is really a generalization of her notion.

Definition 1.14 (Weakly implicative logics) *Let $\mathcal{L}$ be a propositional language, such that $\rightarrow\in\mathcal{L}$ and let $\mathbf{L}$ be a logic in $\mathcal{L}$. We say that $\mathbf{L}$ is a* weakly implicative *logic iff the following consecutions are elements of $\mathbf{L}$:*

(Ref) $\vdash_{\mathbf{L}} \varphi \rightarrow \varphi$

(MP) $\varphi, \varphi \rightarrow \psi \vdash_{\mathbf{L}} \psi$

(WT) $\varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_{\mathbf{L}} \varphi \rightarrow \chi$

(CON) $\varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_{\mathbf{L}} c(\chi_1, \dots, \chi_{i-1}, \varphi, \dots, \chi_n) \rightarrow c(\chi_1, \dots, \chi_{i-1}, \psi, \dots, \chi_n)$ *for each n-ary connective c in $\mathcal{L}$ and each $i \leq n$.*

The (*Ref*) is for *reflexivity*, (MP) is for *modus ponens*, (WT) is for *weak transitivity*, and (CON) is for *congruence*. Let $\varphi \leftrightarrow \psi$ be a shortcut for $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi\}$. Observe that we assume neither Exchange nor Weakening nor Contraction as a rules for implication. However, we have all of them as meta-rules, i.e., the connective $\rightarrow$ is by no means an internalization of $\vdash$. This approach is usually called Hilbert's style calculus, and it is defined in the same fashion as the Hilbert calculi for particular substructural logics. This paper can be seen as a contribution to the general (universal, abstract) theory of these calculi. For another general treatment of this topic see Restall's book [8]. Our approaches are rather incomparable in generality, I will comment more on this approach in the subsequent sections.

Lemma 1.15 (Alternative definition) *The condition* (CON) *from the definition of weakly implicative logic can be equivalently replaced by one of the conditions:*

- $\varphi_1 \leftrightarrow \psi_1, \dots \varphi_n \leftrightarrow \psi_n \vdash_{\mathbf{L}} c(\varphi_1, \dots, \varphi_n) \rightarrow c(\psi_1, \dots, \psi_n)$ *for each n-ary connective c in $\mathcal{L}$.*
- $\varphi \leftrightarrow \psi \vdash_{\mathbf{L}} \chi \rightarrow \chi'$ *for each formula χ, where χ' is a result of replacing arbitrary occurrence of subformula φ with formula ψ in formula χ.*

Proof: We show only the first part, the second one is analogous. Assume (for simplicity) that c is a binary connective,

1. $\varphi_1 \leftrightarrow \psi_1 \vdash_{\mathbf{L}} c(\varphi_1, \varphi_2) \rightarrow c(\psi_1, \varphi_2)$ ((CON) for χ_2 being φ_2)
2. $\varphi_2 \leftrightarrow \psi_2 \vdash_{\mathbf{L}} c(\psi_1, \varphi_2) \rightarrow c(\psi_1, \psi_2)$ ((CON) for χ_1 being ψ_1)
3. $\varphi_1 \leftrightarrow \psi_1, \varphi_1 \leftrightarrow \psi_2 \vdash_{\mathbf{L}} c(\varphi_1, \varphi_2) \rightarrow c(\psi_1, \psi_2)$ (1., 2., and WT)

The other direction is trivial (just take $\psi_2 = \varphi_2$ and use (Ref)) QED

Observe that the second part of this lemma can be understood as a *substitution* rule and thus we will use it heavily in the formal proof in this paper. Now we list several lemmata with rather trivial proofs.

Lemma 1.16 *Let $\mathbf{L}$ be a weakly implicative logic in language $\mathcal{L}$ and $\mathbf{L}'$ a logic in language $\mathcal{L}'$, which is an extension of $\mathbf{L}$. Then $\mathbf{L}'$ is weakly implicative logic iff for each n-ary connective c in $\mathcal{L}' \setminus \mathcal{L}$ and each $i \leq n$ we have $\varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_{\mathbf{L}'} c(\chi_1, \dots, \chi_{i-1}, \varphi, \dots, \chi_n) \rightarrow c(\chi_1, \dots, \chi_{i-1}, \psi, \dots, \chi_n)$*

Lemma 1.17 (Intersection) *The intersection of an arbitrary system of weakly implicative logics is a weakly implicative logic.*

Lemma 1.18 (Extension) *Arbitrary axiomatic extension of an arbitrary of weakly implicative logic is a weakly implicative logic.*

Definition 1.19 (Consistency) *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$*,* T *a theory in* $\mathcal{L}$*. A theory* T *is* consistent *if there is formula* φ *such that* $T \nvdash \varphi$*. A logic* $\mathbf{L}$ *is consistent iff the theory* $\emptyset$ *is consistent.*

Definition 1.20 (Linear theory) *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$*,* T *a theory in* $\mathcal{L}$*. A theory* T *is linear if* T *is consistent and* $T \vdash \varphi \to \psi$ *or* $T \vdash \psi \to \varphi$ *for each formulae* φ, ψ*.*

Definition 1.21 *Let* m *be a natural number and* φ *and* ψ *formulas. Then the formula* $\varphi^m \to \psi$ *is defined inductively as:* $\varphi^0 \to \psi = \psi$ *and* $\varphi^{i+1} \to \psi = \varphi \to (\varphi^i \to \psi)$

Observe that $\varphi^3 \to \psi = \varphi \to (\varphi \to (\varphi \to \psi))$

1.2 Semantics

We start be recalling some well-known definitions. The completeness theorem for weakly implicative logics (which we prove in this section) is a consequence of some more general theorem known in AAL. However, our concern is not to reprove known facts, we concentrate on the notion of linearity of a logical matrix, which (as far as I know) was not so deeply studied.

Definition 1.22 (Algebra and matrix) *Let* $\mathcal{L}$ *be a propositional language. An algebra* $\mathrm{A} = (A, \mathbf{C})$ *with signature* $(\mathbf{C}, \mathbf{a})$ *is called* $\mathcal{L}$*-algebra. Let us denote the realization of* c *in* A *as* c_{A}*.*

A pair $\mathbf{B} = (\mathrm{A}_{\mathbf{B}}, D_{\mathbf{B}})$*, where* $\mathrm{A}_{\mathbf{B}}$ *is* $\mathcal{L}$*-algebra and* $D_{\mathbf{B}}$ *is a subset of* A *is called* $\mathcal{L}$-matrix.

The elements of the set D are called designated elements. Notice that substitution can be understand as a endomorphism of the absolutely free $\mathcal{L}$-algebra. We shall write $c_{\mathbf{B}}$ instead of $c_{\mathrm{A}_{\mathbf{B}}}$

Definition 1.23 (Evaluation) *Let* $\mathcal{L}$ *be a propositional language and* A *an* $\mathcal{L}$*-algebra. Then the* A-evaluation *is a mapping* e*:* $\mathbf{FOR}_{\mathcal{L}} \to A$*, such that* $e(c(\varphi_1, \dots, \varphi_n)) = c_{\mathrm{A}}(e(\varphi_1), \dots e(\varphi_n))$*.*

Of course, each A-evaluation is fully determined by its values on propositional variables. We can understand the A-evaluation as a homomorphism from the absolutely free $\mathcal{L}$-algebra to A. Again, we speak about $\mathbf{B}$-evaluation instead of $\mathrm{A}_{\mathbf{B}}$-evaluation.

Definition 1.24 (Models) *Let* $\mathcal{L}$ *be a propositional language,* T *a theory in* $\mathcal{L}$*, and* $\mathbf{B}$ *an* $\mathcal{L}$*-matrix. We say that* $\mathbf{B}$*-evaluation is an* $\mathbf{B}$-model *of* T *if for each* $\varphi \in T$ *holds* $e(\varphi) \in D_{\mathbf{B}}$*. We denote the class of* $\mathbf{B}$*-models of* T *by* $\mathbf{MOD}(T, \mathbf{B})$*.*

Definition 1.25 (Semantical consequence) *Let* $\mathbf{L}$ *be a logic in* $\mathcal{L}$*,* T *a theory in* $\mathcal{L}$*, and* $\mathcal{K}$ *a class of* $\mathcal{L}$*-matrices. We say that* φ *is a semantical consequence of the* T *w.r.t. class* $\mathcal{K}$ *if* $\mathbf{MOD}(T, \mathbf{B}) = \mathbf{MOD}(T \cup \{\varphi\}, \mathbf{B})$ *for each* $\mathbf{B} \in \mathcal{K}$*; we denote it by* $T \models_{\mathcal{K}} \varphi$*. By* $\mathcal{TAUT}(\mathcal{K})$ *we understand the set* $\{\varphi \mid \emptyset \models_{\mathcal{K}} \varphi\}$

Observe that $\models_{\mathcal{K}} \subseteq \mathcal{CON}_{\mathcal{L}}$ and that it is a logic in language $\mathcal{L}$.

Definition 1.26 (Soundness and completeness) *We say that the logic* $\mathbf{L}$ *is* sound *w.r.t. class* $\mathcal{K}$ *iff* $\models_{\mathcal{K}} \supseteq \mathbf{L}$*. We say that the logic* $\mathbf{L}$ *is* complete *w.r.t. class* $\mathcal{K}$ *iff* $\models_{\mathcal{K}} \subseteq \mathbf{L}$*.*

Definition 1.27 (L-matrices) *Let* $\mathbf{L}$ *be a logic in* $\mathcal{L}$*,* T *a theory in* $\mathcal{L}$*, and* $\mathbf{B}$ *an* $\mathcal{L}$*-matrix. We say that* $\mathbf{B}$ *is an* $\mathbf{L}$-matrix *if* $\mathbf{L} \subseteq \models_{\{\mathbf{B}\}}$*. We denote the class of* $\mathbf{L}$*-matrices by* $\mathbf{MAT}(\mathbf{L})$*. Finally, we write* $T \models_{\mathbf{L}} \varphi$ *instead of* $T \models_{\mathbf{MAT}(\mathbf{L})} \varphi$ *and* $\mathcal{TAUT}(\mathbf{L})$ *instead of* $\mathcal{TAUT}(\mathbf{MAT}(\mathbf{L}))$

Observe that for each presentation $\mathcal{AS}$ of $\mathbf{L}$ holds: $\mathbf{L} \subseteq \models_{\{\mathbf{B}\}}$ iff $\mathcal{AS} \subseteq \models_{\{\mathbf{B}\}}$. The proof of the following lemma is almost straightforward.

Lemma 1.28 *Let* $\mathbf{L}$ *be a weakly implicative logic and* $\mathbf{B}$ *an* $\mathbf{L}$*-matrix. Then relation* $\leq_{\mathbf{B}}$ *defined as* $x \leq_{\mathbf{B}} y$ *iff* $x \to_{\mathbf{B}} y \in D_{\mathbf{B}}$ *is a preorder. Furthermore, its symmetrization* $x \sim_{\mathbf{B}} y$ *iff* $x \leq_{\mathbf{B}} y$ *and* $y \leq_{\mathbf{B}} x$ *is a congruence on* A*. Finally, the set* $D_{\mathbf{B}}$ *is a cone w.r.t.* $\leq_{\mathbf{B}}$*, i.e., if* $x \in D_{\mathbf{B}}$ *and* $x \leq_{\mathbf{B}} y$ *then* $y \in D_{\mathbf{B}}$*.*

Definition 1.29 (Matrix preorder) *Let* $\mathbf{L}$ *be a weakly implicative logic and* $\mathbf{B}$ *an* $\mathbf{L}$*-matrix. The relation* $\leq_{\mathbf{B}}$ *defined in the previous lemma is called* matrix preorder *of* $\mathbf{B}$*.*

The matrix is said to be ordered *iff the relation* $\leq_{\mathbf{B}}$ *is order. We denote the class of ordered* $\mathbf{L}$*-matrixes by* $o{-}\mathbf{MAT}(\mathbf{L})$*.*

The matrix is said to be linearly ordered *(or just* linear*) iff the relation* $\leq_{\mathbf{B}}$ *is linear order. We denote the class of linearly ordered* $\mathbf{L}$*-matrixes by* $l{-}\mathbf{MAT}(\mathbf{L})$*.*

Matrices for weakly implicative logics coincide with the class of so-called *prestandard matrices* (see Dunn) [2], whereas the ordered matrices coincides with so-called *standard matrices*. Obviously, $l{-}\mathbf{MAT}(\mathbf{L}) \subseteq o{-}\mathbf{MAT}(\mathbf{L}) \subseteq \mathbf{MAT}(\mathbf{L})$. The proofs of the following lemma is trivial.

Lemma 1.30 (Soundness) *The logic* $\mathbf{L}$ *is sound w.r.t. class* $\mathbf{MAT}(\mathbf{L})$ *(i.e.,* $T \vdash_{\mathbf{L}} \varphi$*, then* $T \models_{\mathbf{L}} \varphi$*). Furthermore,* $\mathbf{MAT}(\mathbf{L})$ *is the greatest class w.r.t. which is the logic* $\mathbf{L}$ *sound.*

An interesting question is where the opposite hold, i.e., when the logic $\mathbf{L}$ is complete w.r.t. class $\mathbf{MAT}(\mathbf{L})$. To answer this question we recall the well-known concept of a Lindenbaum matrix (sometimes also called Lindenbaum-Tarski matrix).

Definition 1.31 (Lindenbaum matrix) *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$*,* T *be a theory in* $\mathcal{L}$*. We define* $[\varphi]_T = \{\psi \mid T \vdash \varphi \leftrightarrow \psi\}$ *and* $L_T = \{[\varphi]_T \mid \varphi \in \mathbf{FOR}_{\mathcal{L}}\}$*. We define* $\mathcal{L}$*-matrix* $\mathbf{Lin}_T$*, where the* $\mathcal{L}$*-algebra has the domain* L_T*, operations* $c_{\mathbf{Lin}_T}([\varphi_1]_T, \dots [\varphi_n]_T) = [c(\varphi_1, \dots \varphi_n)]_T$ *and the designated set* $D = \{[\varphi]_T \mid T \vdash_{\mathbf{L}} \varphi\}$*.*

It is obvious that the definition is sound.

Lemma 1.32 *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$*,* T *a theory in* $\mathcal{L}$*, and* e *an* $\mathbf{Lin}_T$*-evaluation defined as* $e(\varphi) = [\varphi]_T$*. Then:*

(1) $\mathbf{Lin}_T \in \mathbf{MAT}(\mathbf{L})$,

(2) $e \in \mathbf{MOD}(T, \mathbf{Lin_T})$,

(3) $[\varphi] \leq_{\mathbf{Lin}_T} [\psi]_T$ *iff* $T \vdash \varphi \to \psi$,

(4) $\mathbf{Lin}_T \in o{-}\mathbf{MAT}(\mathbf{L})$,

(5) $\mathbf{Lin}_T \in l{-}\mathbf{MAT}(\mathbf{L})$ *iff* T *is a linear theory.*

Proof:

(1) We show that if $X \vdash_{\mathbf{L}} \varphi$ then $X \models_{\mathbf{Lin}_T} \varphi$, i.e., if $X \vdash_{\mathbf{L}} \varphi$ and $f \in \mathbf{MOD}(X, \mathbf{Lin_T})$ then $f(\varphi) \in D_{\mathbf{Lin}_T}$. Recall that $f(\varphi) = [\psi]_T$ for some formula ψ.

Let us define substitution σ by setting $\sigma(v)$ be some $\psi \in f(v)$. Next we show that for each φ we get $\sigma(\varphi) \in f(\varphi)$: let $\varphi = c(\varphi_1, \dots, \varphi_n)$, from the induction property $\sigma(\varphi_i) \in f(\varphi_i)$ we get $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots \sigma(\varphi_n)) \in c_{\mathbf{Lin}_T}(f(\varphi_1), \dots, f(\varphi_n)) = f(c(\varphi_1, \dots, \varphi_n))$. Thus we know that for each φ we get $f(\varphi) = [\sigma(\varphi)]_T$

From $f \in \mathbf{MOD}(X, \mathbf{Lin_T})$ we get that $f(\psi) \in D_{\mathbf{Lin}_T}$ for each $\psi \in X$ and thus $T \vdash \sigma(\psi)$. From $X \vdash \varphi$ we get $\sigma(X) \vdash \sigma\varphi$. Thus together we have $T \vdash \sigma(\varphi)$ and so $f(\varphi) = [\sigma(\varphi)]_T \in D_{\mathbf{Lin}_T}$.

(2) Trivial.

(3) $[\varphi] \leq_{\mathbf{Lin}_T} [\psi]_T$ iff $[\varphi] \to_{\mathbf{Lin}_T} [\psi]_T \in D_{\mathbf{Lin}_T}$ iff $[\varphi \to \psi]_T \in D_{\mathbf{Lin}_T}$ iff $T \vdash \varphi \to \psi$.

(4) Since $[\varphi] \leq_{\mathbf{Lin}_T} [\psi]_T$ and $[\psi] \leq_{\mathbf{Lin}_T} [\varphi]_T$ entails $T \vdash \varphi \leftrightarrow \psi$ the proof is obvious.

(5) Straightforward.

QED

Theorem 1.33 (Completeness) *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$*. Then for each theory* T *and formula* φ *holds:* $T \vdash \varphi$ *iff* $T \models_{\mathbf{L}} \varphi$*.*

Proof: One direction is Lemma 1.30. Reverse direction: from Lemma 1.32 we get $\mathbf{Lin}_T \in \mathbf{MAT}(\mathbf{L})$ and for the $\mathbf{Lin}_T$-evaluation e defined as $e(\psi) = [\psi]_T$ holds $e \in \mathbf{MOD}(T, \mathbf{Lin_T})$. Thus from $T \models_{\mathbf{L}} \varphi$ we get that $[\varphi]_T = e(\varphi) \in D_{\mathbf{Lin}_T}$ and so $T \vdash \varphi$. QED

The proofs of the following two corollaries are trivial.

Corollary 1.34 *Let* $\mathbf{L}$ *be a weakly implicative logic. Then* $\mathcal{THM}(\mathbf{L}) = \mathcal{TAUT}(\mathbf{L})$*.*

Corollary 1.35 *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$*,* T *a theory in* $\mathcal{L}$*. Then for each formula* φ *holds* $T \vdash \varphi$ *iff* $T \models_{\mathbf{L}} \varphi$ *iff* $T \models_{o-\mathbf{MAT(L)}} \varphi$*.*

Now we recall the definition of a direct and subdirect product of matrices. This definitions are obvious, if we understand matrix as a first-order structure with functions corresponding to the operations and one unary predicate, whose realization is the set of designated elements.

Definition 1.36 *Let* $\mathcal{L} = (\mathbf{VAR}, \mathbf{C}, \mathbf{a})$ *be a propositional language and* $\mathcal{I}$ *a class of* $\mathcal{L}$*-matrices. The* direct product *of matrices from* $\mathcal{I}$ *is a matrix* $\mathbf{X} = \prod_{\mathbf{B}\in\mathcal{I}} \mathbf{B} = (X, (c_{\mathbf{X}})_{c\in\mathbf{C}}, D_{\mathbf{X}})$*, where* X *is a cartesian product od domain of matrices from* $\mathcal{I}$*, operations are defined pointwise, and* $(x_{\mathbf{B}})_{\mathbf{B}\in\mathcal{I}} \in D_{\mathbf{X}}$ *iff* $x_{\mathbf{B}} \in D_{\mathbf{B}}$ *for each* $\mathbf{B} \in \mathcal{I}$*.*

Furthermore, we say that the matrix $\mathbf{X}$ *is a* subdirect product *of matrices from* $\mathcal{I}$ *if there is an embedding* $f : \mathbf{X} \to \prod_{\mathbf{B}\in\mathcal{I}} \mathbf{B}$*, such that for each* $\mathbf{B} \in \mathcal{I}$ *holds* $\pi_{\mathbf{B}}(f(\mathbf{X})) = B$*.*

By $\pi_{\mathbf{B}}$ we mean the projection to the component $\mathbf{B}$.

Lemma 1.37 *Let* $\mathbf{L}$ *be a logic and* $\mathcal{I}$ *a class of* $\mathbf{L}$*-matrices. Then each matrix* $\mathbf{X}$ *which is a subdirect product of matrices from* $\mathcal{I}$ *is an* $\mathbf{L}$*-matrix.*

Proof: Trivial. QED

1.3 Fuzzy logic

In the previous section we have seen that each weakly implicative logic is sound and complete w.r.t. class of its *ordered* matrices. There is an obvious question, which (if not all) of them are complete w.r.t. class of its *linearly ordered* matrices. This will lead us to the second central definition of this paper: the notion of weakly implicative fuzzy logics.

Definition 1.38 (Fuzzy logics) *Weakly implicative logic* $\mathbf{L}$ *in language* $\mathcal{L}$ *is called* fuzzy *logic if* $\mathbf{L} = \models_{l-\mathbf{MAT(L)}}$ *(i.e., if the logic* $\mathbf{L}$ *is sound and complete w.r.t. linearly ordered* $\mathbf{L}$*-matrices.)*

The full proper name of the above defined class of logics if *weakly implicative fuzzy logics*, however since all the logics we encounter from now on are weakly implicative, we just say that a logic is *fuzzy*. There is a joint paper by the author and Libor Běhounek [1] given *philosophical, methodological, and pragmatical* reasons for using the term fuzzy, and for *formal* delimitation of the existing *informal* class of fuzzy logic.

Corollary 1.39 (Completeness) *Let* $\mathbf{L}$ *be a fuzzy logic in* $\mathcal{L}$*. Then* $\mathcal{THM}(\mathbf{L}) = \mathcal{TAUT}(\mathbf{L}) = \mathcal{TAUT}(l-\mathbf{MAT}(\mathbf{L}))$*.*

We are going to show the equivalent definitions the class of weakly implicative fuzzy logics.

Definition 1.40 (Linear extension) *A weakly implicative logic* $\mathbf{L}$ *has the* Linear Extension Property (LEP) *if for each theory* T *formula* φ *such that* $T \nvdash \varphi$ *there is a linear theory* T'*, such that* $T \subseteq T'$ *and* $T' \nvdash \varphi$*.*

Definition 1.41 (Prelinearity) *A weakly implicative logic* **L** *has the* Prelinearity Property (PP) *if for each theory* T *we get* $T \vdash \chi$ *whenever* $T, \varphi \to \psi \vdash \chi$ *and* $T, \psi \to \varphi \vdash \chi$*.*

Definition 1.42 (Subdirect Decomposition) *A weakly implicative logic* **L** *has the* Subdirect Decomposition Property (SDP) *if each ordered* **L***-matrix is a subdirect product of linear* **L***-matrices.*

Theorem 1.43 (Characterization of fuzzy logics) *A weakly implicative logic is fuzzy iff it has* LEP*.*

Proof: Assume that **L** is fuzzy logic. If **L** and $T \nvdash \varphi$ then there is linear **L**-matrix **B** and **B**-evaluation e such that $e(\varphi) \notin D_{\mathbf{B}}$. Let us define $T' = T \cup \{\psi \mid e(\psi) \in D_{\mathbf{B}}\}$. Obviously $T \subseteq T'$ and $T' \nvdash \varphi$. Since $\leq_{\mathbf{B}}$ is linear order we get $e(\chi) \leq_{\mathbf{B}} e(\delta)$ or $e(\delta) \leq_{\mathbf{B}} e(\chi)$ for each χ and δ. Thus either $e(\chi \to \delta) \in D_{\mathbf{B}}$ or $e(\delta \to \chi) \in D_{\mathbf{B}}$.

Reverse direction: we need to prove that $\vdash_{\mathbf{L}} = \models_{l-\mathbf{MAT}(\mathbf{L})}$. One inclusion is trivial consequence of Theorem 1.33, the reverse one we prove by contradiction: assume that $T \nvdash \varphi$. Let us take a linear supertheory $T' \nvdash \varphi$. From Lemma 1.32 we know that $\mathbf{Lin}_{T'} \in l-\mathbf{MAT}(\mathbf{L})$, $\mathbf{Lin}_{T'}$ is linearly ordered and for the $\mathbf{Lin}_{T'}$-evaluation e defined as $e(\psi) = [\psi]_{T'}$ holds $e \in \mathbf{MOD}(T', \mathbf{Lin_{T'}})$. This entails that also $e \in \mathbf{MOD}(T, \mathbf{Lin_{T'}})$. The rest of the proof is analogous to the one for weakly implicative logics. QED

Lemma 1.44 *Let* **L** *be a weakly implicative logic with* LEP*. Then* **L** *has* PP*.*

Proof: We argument contrapositively: let $T \nvdash \chi$, then (using LEP) there is linear theory T', such that $T' \nvdash \chi$. Assume that $T' \vdash \varphi \to \psi$, then obviously $T, \varphi \to \psi \nvdash \chi$ QED

To reverse this lemma it seems that we need one additional assumption: the logic has to be finitary. However, it is obvious that for some infinitary rules the equivalence will hold as well, the question exactly for which subclass of weakly implicative logics the equivalence holds seems to be interesting open problem.

Lemma 1.45 *Let* **L** *be a finitary weakly implicative logic with* PP*,* T *a theory, and* φ *a formula, such that* $T \nvdash \varphi$*. Then there is a linear theory* T'*, such that* $T \subseteq T'$ *and* $T' \nvdash \varphi$*.*

Proof: Let $||\mathcal{L}|| = \kappa$. Let us enumerate all tuples of formula by ordinals $< \kappa$. Let $T_0 = T$. We construct theories T_μ using transfinite induction. Let $\hat{T}_\mu = \bigcup\limits_{\nu<\mu} T_\nu$.

Observe that if $T_\nu \nvdash \varphi$ for each $\nu < \mu$ then $\hat{T}_\mu \nvdash \varphi$ as well (the logic is finitary). We know that either $\hat{T}_\mu \cup \{\varphi_\mu \to \psi_\mu\} \nvdash \varphi$ or $\hat{T}_\mu \cup \{\psi_\mu \to \varphi_\mu\} \nvdash \varphi$ (otherwise using PP we get contradiction with $\hat{T}_\mu \nvdash \varphi$), define T_μ accordingly. Finally, define $T' = \hat{T}_\kappa$; obviously $T' \nvdash \varphi$ and T' is linear. QED

Theorem 1.46 (Equivalent characterization) *Let* **L** *be a finitary weakly implicative logic. Then the following are equivalent:*

(1) **L** *is a fuzzy logic,*

(2) **L** *has* LEP*,*

(3) **L** *has* PP*,*

(4) **L** *has* SDP*.*

Proof: (1) $\to$ (2) : C.f. Theorem 1.43.

(2) $\to$ (3) : C.f. Lemma 1.44.

(3) $\to$ (4) : Let us denote the language of **L** as $\mathcal{L} = (\mathbf{VAR}, \mathbf{C}, \mathbf{a})$. Let $\mathbf{B} = (\mathrm{A}, D)$ be an ordered **L**-matrix and $\leq$ its matrix order. Let us take $\mathbf{VAR}' = A$, for clearness we will use $v_a \in \mathbf{VAR}'$ and $a \in A$. We define the propositional language $\mathcal{L}' = (\mathbf{VAR}', \mathbf{C}, \mathbf{a})$ and the logic $\mathbf{L}'$ as $\vdash_{\mathcal{SUB}_{\mathcal{L}'}(\mathbf{L})}$. The logic $\mathbf{L}'$ has obviously PP as well and thus it has LEP (using Lemma 1.45). Notice, that here we use the assumption that **L** is finitary.

Let $T = \{c(v_{a_1}, \dots v_{a_n}) \leftrightarrow v_{c_{\mathbf{B}}(a_1,\dots a_n)} \mid c \in \mathbf{C}, \mathbf{a}(c) = n, \text{ and } a_1, \dots, a_n \in A\} \cup \{v_a \mid a \in D\}$. Observe that $\mathbf{B}$ is an $\mathbf{L}'$-matrix. Let us define $\mathbf{B}$-evaluation $e(v_a) = a$ and observe that $\{e\} = \mathbf{MOD}(T, B)$. Now we show $T \vdash v_a \to v_b$ iff $a \leq b$: one direction is simple (from $e \in \mathbf{MOD}(T, B)$ we get that $e(v_a) \to_{\mathbf{B}} e(v_b) \in D$ and so $a \leq b$); the other direction is similar (if $a \leq b$ then $a \to b \in D$ thus $T \vdash v_{a \to b}$ and so $T \vdash v_a \to v_b$). Finally, we observe that for each formula φ there is $a \in A$, such that $T \vdash \varphi \leftrightarrow v_a$

Let us define the set $\mathcal{I}$ of all linear theories extending T. Next we define $\mathbf{L}$-matrix $\mathbf{X} = \prod_{S \in \mathcal{I}} \mathbf{Lin}_S$ (direct product of Lindenbaum matrices). Finally, we define $f(a) = ([v_a]_S)_{S \in \mathcal{I}}$.

Now we show that f is an embedding of $\mathbf{B}$ into $\mathbf{X}$: since for each $S \in \mathcal{I}$ we have $[v_{c_{\mathbf{B}}(a_1,\dots,a_n)}]_S = [c(v_{a_1}, \dots v_{a_n})]_S = c_{\mathbf{Lin}_S}([v_{a_1}]_S, \dots [v_{a_n}]_S)$ we get $f(c_{\mathbf{B}}(a_1, \dots, a_n)) = (c_{\mathbf{Lin}_S}([v_{a_1}]_S, \dots [v_{a_n}]_S))_{S \in \mathcal{I}} = c_{\mathbf{X}}(f(a_1), \dots f(a_n))$. Since obviously $a \in D$ entails $f(a) \in D_{\mathbf{X}}$ (from $a \in D$ we have $T \vdash v_a$ and so $S \vdash v_a$ and thus $[v_a]_S \in D_{\mathbf{Lin}_S}$ for each $S \in \mathcal{I}$) we know that f is a morphism. It remains to be shown that f is one-one: from $a \neq b$ we get that either $a \not\leq b$ or $b \not\leq a$. Let us assume that $a \not\leq b$ then $T \not\vdash v_a \to v_b$, using Lemma 1.45 we know that there is a linear theory $S \in \mathcal{I}$ such that $S \not\vdash v_a \to v_b$ thus $[v_a]_S \not\leq_{\mathbf{Lin}_S} [v_b]_S$ and so $f(a) \not\leq_{\mathbf{X}} f(b)$.

Finally, we observe that for each S we have $\pi_S(f(A)) = L_S$ (just recall that for each φ there is $a \in A$, such that $T \vdash \varphi \leftrightarrow v_a$).

Since $\mathbf{Lin}_S$ is linearly ordered $\mathbf{L}$-matrix for each $S \in \mathcal{I}$ and $\mathbf{B}$ can be embedded into direct product of $(\mathbf{Lin}_S)_{S \in \mathcal{I}}$ in the way that $\pi_S(f(A)) = L_S$. We conclude that $\mathbf{B}$ is a subdirect product of linearly ordered $\mathbf{L}$-matrices.

$(4) \to (1)$: Trivial.

QED

Lemma 1.47 *Let $\mathbf{L}$ be a fuzzy logic. Then $(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash_{\mathbf{L}} \chi$ for each naturals i and j.*

Proof: Let $T = \{(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi\}$. Obviously $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ (using (MP)). Since each fuzzy logic has PP the proof is done. QED

The proofs of the following lemmata are obvious.

Lemma 1.48 (Intersection) *The intersection of an arbitrary system of fuzzy logics is a fuzzy logic.*

Lemma 1.49 (Conservative expansion) *Let $\mathbf{L}'$ be a conservative expansion of a finitary fuzzy logic $\mathbf{L}$. Then $\mathbf{L}$ is fuzzy logic as well.*

Lemma 1.50 (Axiomatic extension) *An axiomatic extension of arbitrary fuzzy logic in the same language is a fuzzy logic.*

The assumption of being in the same language can be omitted if we assume some additional properties of the logic in question.

Proof: We know that $\mathbf{L}'$ has PP, we show that $\mathbf{L}$ has PP as well and because $\mathbf{L}$ is finitary we get that $\mathbf{L}$ is fuzzy. Let us take theory T and formulae φ, ψ, χ in language of $\mathbf{L}$. Assume that $T, \varphi \to \psi \vdash_{\mathbf{L}} \chi$ and $T, \psi \to \varphi \vdash_{\mathbf{L}} \chi$, then also $T, \varphi \to \psi \vdash_{\mathbf{L}'} \chi$ and $T, \psi \to \varphi \vdash_{\mathbf{L}'} \chi$. Using PP for $\mathbf{L}'$ we get $T \vdash_{\mathbf{L}'} \chi$. Conservativeness completes the proof. QED

2 Special propositional logics

In this section we introduce some additional rules (see Tables 2.1) and 2.2) and some additional connectives (see table 2.3) and prove some facts about these extensions. It is just a sketch of a huge work to be done. The ultimate goal is to characterize known logics and put them into our context with modularly designed Hilbert's style calculi. Having this done, we identify which of them are fuzzy, or we find minimal fuzzy logics extending some known logics (eg. minimal fuzzy logic over intuitionistic

Table 2.1: Structural rules

consecution	symbol	name
$\varphi \vdash \psi \to \varphi$	W	Weakening
$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$	E	Exchange
$\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$	C	Contraction

Table 2.2: Addition rules

consecution	symbol	name
$\vdash \varphi \to ((\varphi \to \psi) \to \psi)$	As	assertion
$\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$	Sf	suffixing
$\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$	Pf	prefixing
$\vdash \varphi \to (\varphi \to \varphi)$	M	mingle

logic is Gödel logic, minimal fuzzy logic over Full Lambek calculus with exchange and weakening is MTL logic, etc.)

Having stronger logic/language can lead to simplified semantics, eg. having $\mathbf{1}$ we can replace matrices with ordered structures (the designated set will be the upper cone of $\mathbf{1_B}$), having one of the lattice connectives we can even work with algebras, thus being able to use powerful methods of Abstract Algebraic Logics (which, of course, we can do anyway, but some theorems of ALL hold in algebraizable logics only). There are other possible simplification of the semantics allowed be adding some structural rules (weakening leads to algebraic semantics, exchange to ordered structures, with φ being valid iff $e(\varphi \to \varphi) \leq e(\varphi)$, etc.).

2.1 Adding rules

In this section we restrict ourselves to the propositional languages with implication only. The basic rules correspond to the structural rules are exchange, contraction and weakening (see Table 2.1), extended by some additional important rules summarized in Table 2.2. We formulate them as rules, however in some situation we can use their stronger forms—we formulate then as axioms. To do this in a general way we present the following definition.

Definition 2.1 *Let R be a unary deduction rule of the form $\varphi \vdash \psi$. By the corresponding axiom we understand axiom $\vdash \varphi \to \psi$, we will denote it as $ax(R)$.*

Table 2.3: Propositional connectives

Symbol	Arity	Name	Alternative name
$\top$	0	verum	additive truth
$\mathbf{1}$	0	one	multiplicative truth
$\mathbf{0}$	0	zero	multiplicative falsum
$\bot$	0	falsum	additive falsum
$\triangle$	1	Baaz delta	globalization
$\wedge$	2	min-conjunction	additive conjunction
$\vee$	2	max-disjunction	additive disjunction
&	2	strong conjunction	fusion, multiplicative conjunction
$\rightsquigarrow$	2	c-implication	reverse implication

Of course if $\mathbf{L}$ is weakly implicative logic then $\vdash_{\mathbf{L}} \varphi \to \psi$ entails $\varphi \vdash_{\mathbf{L}} \psi$, i.e., if $ax(R) \in \mathbf{L}$ then $R \in \mathbf{L}$. Recall, that in the literature some of the axioms are known under different names.

Definition 2.2 (Adding rules) *Let $\mathbf{L}$ be a weakly implicative logic in language $\{\to\}$ and Q be a subset of $\{\mathrm{W}, \mathrm{E}, \mathrm{C}, \mathrm{Pf}, \mathrm{Sf}, ax(\mathrm{W}), ax(E), ax(C), ax(\mathrm{Pf}), ax(\mathrm{Sf}), \mathrm{As}\}$. We say that $\mathbf{L}$ is an Q-implication fragment if the consecutions from Q are elements of $\mathbf{L}$.*

We say that $\mathbf{L}$ is an fuzzy Q-implication fragment if $\mathbf{L}$ is fuzzy logic and $\mathbf{L}$ is Q-implication fragment.

Definition 2.3 *The weakest Q-implication fragment is denoted as $\mathcal{MIN}(Q)$. Furthermore, the weakest fuzzy Q-implication fragment is denoted as $\mathcal{FUZZ}(Q)$*

Both definition are sound thanks to the Lemma 1.17 and 1.48. The following lemma show interplay between transitivity and exchange.

Lemma 2.4 *The following logics are equivalent:*

(1) BCI *logic (implicational fragment of linear logic)*

(2) $\mathcal{MIN}(\{ax(\mathrm{Sf}), \mathrm{As}\})$

(3) $\mathcal{MIN}(\{ax(\mathrm{Sf}), ax(\mathrm{E})\})$

(4) $\mathcal{MIN}(\{ax(\mathrm{Pf}), ax(\mathrm{E})\})$

(5) $\mathcal{MIN}(\{\mathrm{Pf}, ax(\mathrm{E})\})$

(6) $\mathcal{MIN}(\{ax(\mathrm{Sf}), \mathrm{E}\})$

(7) $\mathcal{MIN}(\{ax(\mathrm{Pf}), \mathrm{E}\})$

Proof: We show that each logic is stronger than the next one (cyclicly):

(1) $\supseteq$ (2): recall that BCI logic is axiomatized by $ax(\mathrm{Sf})$, As, (Ref), and (MP). So all we have to do is to show (CON) for $\to$ but this is almost straightforward.

(2) $\supseteq$ (3): All we need to show is $\vdash_{\mathcal{MIN}(\{ax(\mathrm{Sf}),\mathrm{As}\})} ax(\mathrm{E})$

(i) $\psi \to ((\psi \to \chi) \to \chi)$	As
(ii) $(\psi \to ((\psi \to \chi) \to \chi)) \to ((((\psi \to \chi) \to \chi) \to (\varphi \to \chi)) \to (\psi \to (\varphi \to \chi)))$	$ax(\mathrm{Sf})$
(iii) $(((\psi \to \chi) \to \chi) \to (\varphi \to \chi)) \to (\psi \to (\varphi \to \chi))$	$(i), (ii)$, (MP)
(iv) $(\varphi \to (\psi \to \chi)) \to (((\psi \to \chi) \to \chi) \to (\varphi \to \chi))$	$ax(\mathrm{Sf})$
(v) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	$(iv), (iii)$, (WT)

(3) $\supseteq$ (4): trivial

(4) $\supseteq$ (5): trivial

(5) $\supseteq$ (6): All we need to show is $\vdash_{\mathcal{MIN}(\{\mathrm{Pf},ax(\mathrm{E})\})} ax(\mathrm{Sf})$

(i) $(\psi \to \chi) \to (\psi \to \chi)$	(Ref)
(ii) $\psi \to ((\psi \to \chi) \to \chi)$	$(i), ax\mathrm{E}$, (MP)
(iii) $(\varphi \to \psi) \to (\varphi \to ((\psi \to \chi) \to \chi))$	(ii), Pf
(iv) $(\varphi \to ((\psi \to \chi) \to \chi)) \to ((\psi \to \chi) \to (\varphi \to \chi))$	$ax(\mathrm{E})$
(v) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	$(iii), (iv)$, Pf

(6) $\supseteq$ (7): trivial

(7) $\supseteq$ (1): trivial

QED

Notice two open problems: what about logics $\mathcal{MIN}(\{\mathrm{Sf}, ax(\mathrm{E})\})$ and $\mathcal{MIN}(\{ax(\mathrm{Pf}), \mathrm{As}\})$?

Table 2.4 puts some known logics into our context (we list axioms which has to be added to $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E})$). We can add all of them as rules or axioms—in the presence of exchange these two options are equivalent—as shown by the following observation:

Table 2.4: Known implicational fragments

consecutions	implicational fragment of
$\emptyset$	linear logic
C	relevance logic
C, M	relevance logic with mingle
W	affine linear logic
W, C	intuitionistic logic

1. $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}) = \mathcal{MIN}(ax(\mathrm{Sf}), ax(\mathrm{E}))$
2. $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}) = \mathcal{MIN}(ax(\mathrm{Sf}), ax(\mathrm{E}), ax(\mathrm{W}))$
3. $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{C}) = \mathcal{MIN}(ax(\mathrm{Sf}), ax(\mathrm{E}), ax(\mathrm{C}))$

Proof: Part 1. was shown in Lemma 2.4. Part 2. is trivial. To prove part 3. we only show that $\vdash_{\mathcal{MIN}(ax(\mathrm{Sf}),\mathrm{E},\mathrm{C})} ax(\mathrm{C})$

(i)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to (\varphi \to \psi))$	(Ref)
(ii)	$\varphi \to ((\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi))$	(i) and E.
(iii)	$\varphi \to (\varphi \to ((\varphi \to (\varphi \to \psi)) \to \psi))$	(ii), $ax(E), ax(\mathrm{Sf})$
(iv)	$\varphi \to ((\varphi \to (\varphi \to \psi)) \to \psi)$	(iii) and C
(v)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	(iv) and E

QED

Before we proceed further we observe some properties of BCI.

Lemma 2.5 *It holds:*

1. $\vdash_{\mathrm{BCI}} (\varphi^n \to (\psi^m \to \chi)) \to (\psi^m \to (\varphi^n \to \chi))$
2. $\vdash_{\mathrm{BCI}} (\varphi^n \to \psi) \to ((\psi \to \chi) \to (\varphi^n \to \chi))$

Now we present an important definition—the deduction theorem—for rather wide class of weakly implicative logic. We present it in a rather strange strong form. Our formulation allows us to show exactly which logics have this deduction theorem.

Definition 2.6 *Let $\mathcal{L}$ be a language and* **L** *a logic in $\mathcal{L}$, such that* **L** *has a presentation $\mathcal{AX}$, where* (MP) *is the only deduction rule. We say that* **L** *has* Implicational Deduction Theorem *(*$\mathrm{DT}_\to$*) if for each, theory T, formulas φ, ψ, and proof $\mathcal{P}$ of ψ in theory T, φ (in presentation $\mathcal{AX}$) we have: $T, \varphi \vdash \psi$ iff $T \vdash \varphi^n \to \psi$, where n is a number of occurrences of φ in the leaves of the proof $\mathcal{P}$ and there is a proof $\mathcal{P}'$ of $\varphi^n \to \psi$ in T, such that each $\psi \in T$ occurs in the leaves of $\mathcal{P}$ same number of times as in the leaves of $\mathcal{P}'$.*

Observe that each logic with $\mathrm{DT}_\to$ has also the "standard" form of the deduction theorem.

Corollary 2.7 *Each logic with* $\mathrm{DT}_\to$ *has so called* Local Deduction Theorem *(*LDT*): for each theory T and formulas φ, ψ: $T, \varphi \vdash \psi$ iff there is n such that $T \vdash \varphi^n \to \psi$.*

Now we present sufficient and necessary condition for **L** to have $\mathrm{DT}_\to$.

Theorem 2.8 (Deduction theorem) *Let $\mathcal{L}$ be a language and* **L** *a logic in $\mathcal{L}$, such that* **L** *has a presentation, where* (MP) *is the only deduction rule. Then* **L** *has* $\mathrm{DT}_\to$ *iff implicational fragment of* **L** *is an extension of* BCI.

Proof: First direction: All we have to do is to show that $\vdash_{\mathbf{L}} ax(\mathrm{Sf})$ and $\vdash_{\mathbf{L}} ax(\mathrm{E})$. Observe that $\varphi, \psi, \varphi \to (\psi \to \chi) \vdash_{\mathbf{L}} \chi$ and each of the premises is used exactly once, applying $\mathrm{DT}_{\to}$ three times we get $\vdash_{\mathbf{L}} ax(\mathrm{E})$. Observe that $\varphi, \varphi \to \psi, \psi \to \chi \vdash_{\mathbf{L}} \chi$ and each of the premises is used exactly once, applying $\mathrm{DT}_{\to}$ three times we get $\vdash_{\mathbf{L}} ax(\mathrm{Sf})$.

Reverse direction: we need to prove two directions. One is obvious. To prove the other one we use the induction over the proof of ψ in T, φ (in $\mathcal{AX}$). We show that it holds for each χ in the proof $\mathcal{P}'$:

- χ is a leaf of $\mathcal{P}$, i.e., $\chi \in T$, χ is an axiom, or $\chi = \varphi$: trivial
- χ has predecessors $\psi_2 = \psi_1 \to \chi$ and ψ_1, using the induction property we get $T \vdash \varphi^n \to (\psi_1 \to \chi)$ and $T \vdash \varphi^m \to \psi_1$ (and the number of occurrences of formulas from T is the same), we distinguish two cases
  - $\psi_2 = \varphi$, using Lemma 2.5 (2) we know that $\vdash_{\mathrm{BCI}} (\varphi^m \to \psi_1) \to ((\psi_1 \to \chi) \to (\varphi^m \to \chi))$, thus we get $T \vdash \varphi \to (\varphi^m \to \chi)$ and so $T \vdash \varphi^{m+1} \to \chi$.
  - $\psi_2 \neq \varphi$, using Lemma 2.5 (1) we get $T \vdash \psi_1 \to (\varphi^n \to \chi)$. From Lemma 2.5 (2) we obtain $T \vdash \varphi^m \to (\varphi^n \to \chi)$. Thus $T \vdash \varphi^{m+n} \to \chi$.

  In both cases the number of occurrences of formulas from T is not changed.

QED

We have proved even more: given proof of ψ in T, φ, we construct the proof of $\varphi^n \to \psi$ in T.

Corollary 2.9 *Let $\mathcal{L}$ be a language and* **L** *a logic in $\mathcal{L}$, such that* **L** *has a presentation, where* (MP) *is the only deduction rule and implicational fragment of* **L** *is an extension of* BCI*. Then* **L** *has* LDT*.*

In the presence of LDT we can prove the "converse" of Lemma 1.47. Thus having an equivalent definition of fuzzy logics in some class of logics.

Lemma 2.10 *Let* **L** *be a finitary logic with* LDT*. Then* $\vdash_{\mathbf{L}} (\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash_{\mathbf{L}} \chi$ *iff* **L** *is fuzzy.*

Proof: One direction is just Lemma 1.47. To prove the other direction we only show that **L** has PP. Assume that $T, \varphi \to \psi \vdash \chi$ and $T, \varphi \to \psi \vdash \chi$. By LDT we get $T \vdash (\varphi \to \psi)^i \to \chi$ and $T \vdash (\psi \to \varphi)^j \to \chi$ and so we have $T \vdash \chi$. QED

At the end of this section we present an axiomatic system for the minimal fuzzy $\{ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}\}$-implication fragment. We decided to use classical names for axioms $ax(\mathrm{Sf})$, $ax(\mathrm{E})$, and $ax(\mathrm{W})$.

Definition 2.11 *The* fuzzy BCK logic *(*FBCK*) has the following presentation:*

$\mathcal{B}$	$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$\mathcal{C}$	$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
$\mathcal{K}$	$\vdash \varphi \to (\psi \to \varphi)$
F_n	$\vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$

Observe that FBCK $= \mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{F})$ (observe that having exchange we can easily get $ax(\mathrm{W})$ from W). Using Lemma 2.4 we could write several different equivalent axiomatic systems. Also observe that *ordered* FBCK-matrices are exactly BCK-algebras satisfying axioms F_n.

Theorem 2.12 FBCK $= \mathcal{FUZZ}(\{ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}\})$

Proof: First, we show that FBCK, is fuzzy logic. Using Theorem 1.46 it is enough to show that FBCK has PP. Let $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ then using $\mathrm{DT}_{\to}$ we get that $T \vdash (\varphi \to \psi)^m \to \chi$ and $T \vdash (\psi \to \varphi)^n \to \chi$ for some n and m. Let us take $t = \max(m, n)$, using Weakening we get the $T \vdash (\varphi \to \psi)^t \to \chi$ and $T \vdash (\psi \to \varphi)^t \to \chi$, axiom F_t completes the proof.

Next, we have to show that each fuzzy logic extending BCK proves F_n. We recall that each fuzzy logic has PP. Now observe that

Table 2.5: Rules

Consecution	symbol	Name	match for
$\psi\&\varphi \to \chi \dashv\vdash \varphi \to (\psi \rightsquigarrow \chi)$	$\rightsquigarrow$R	$\rightsquigarrow$-Residuation	$\rightsquigarrow$
$\varphi \to (\psi \to \chi) \dashv\vdash \varphi\&\psi \to \chi$	R	Residuation	&
$\varphi \to \psi \dashv\vdash \varphi \rightsquigarrow \psi$	Imp	Implications	$\rightsquigarrow$
$\vdash \varphi \to \top$	Tr	veritas ex quolibet	$\top$
$\vdash \bot \to \varphi$	Fa	ex-falso quodlibet	$\bot$
$\varphi \dashv\vdash \mathbf{1} \to \varphi$	?	?	**1**
$\varphi \to \chi, \psi \to \chi \vdash \varphi \vee \psi \to \chi$	$\vee$1	supremum	$\vee$
$\vdash \varphi \to \varphi \vee \psi$	$\vee$2	idempotency	$\vee$
$\vdash \varphi \vee \psi \to \psi \vee \varphi$	$\vee$3	commutativity	$\vee$
$\chi \to \varphi, \chi \to \psi \vdash \chi \to \varphi \wedge \psi$	$\wedge$1	infimum	$\wedge$
$\vdash \varphi \wedge \psi \to \varphi$	$\wedge$2	idempotency	$\wedge$
$\vdash \varphi \wedge \psi \to \psi \wedge \varphi$	$\wedge$3	commutativity	$\wedge$

Table 2.6: Matching rules for $\triangle$

Consecution	symbol	Name
$\vdash \triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$	$\triangle$1	$\triangle$-monotonicity
$\vdash \triangle\varphi \to \varphi$	$\triangle$2	$\triangle$-reflexivity
$\vdash \triangle\varphi \to \triangle\triangle\varphi$	$\triangle$3	$\triangle$-transitivity
$\vdash \triangle\varphi \to (\triangle\psi \to \varphi)$	$\triangle$W	$\triangle$-weakening
$\vdash \triangle(\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to \psi)$	$\triangle$C	$\triangle$-contraction
$\vdash \triangle(\triangle\varphi \to (\triangle\psi \to \chi)) \to (\triangle\psi \to (\triangle\varphi \to \chi))$	$\triangle$E	$\triangle$-exchange
$\varphi \vdash \triangle\varphi$	(NEC)	necessitation

(i) $((\varphi \to \psi)^n \to \chi) \to ((\varphi \to \psi)^n \to \chi)$ (Ref)
(ii) $(\varphi \to \psi)^n \to (((\varphi \to \psi)^n \to \chi) \to \chi)$ (i) and Lemma 2.5 1.
(iii) $\varphi \to \psi \vdash (((\varphi \to \psi)^n \to \chi) \to \chi)$ (ii)
(iv) $\varphi \to \psi \vdash ((\psi \to \varphi)^n \to \chi) \to (((\varphi \to \psi)^n \to \chi) \to \chi)$ (iii) and $\mathcal{K}$
(v) $\varphi \to \psi \vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$ (iv) and $\mathcal{C}$

By replacing φ and ψ in (iv) we get:

(vi) $\psi \to \varphi \vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$

Since we **L** has PP we get $\vdash \mathrm{F}_n$ QED

Since obviously BCK $= \mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W})$ we get the following corollary.

Corollary 2.13 FBCK *is the weakest fuzzy logic stronger than* BCK*.*

We can alter the axioms F_n by using two different natural number m, n as "exponents", we get axioms $\mathrm{F}_{m,n}$. If we add axioms $\mathrm{F}_{m,n}$ to the BCI logic, we get fuzzy logic (by Lemma 2.10). However, we are not able to the prove the converse statement, i.e., that this logic is minimal fuzzy logic over BCI.

2.2 Adding connectives

As mentioned before, we consider the connectives from Table 2.3. For the matching rules for the particular connective see Tables 2.5 and 2.6. There are many interesting interplays between them and

Table 2.7: Known logics

$\mathcal{L}$	Q	$\mathcal{MIN}_{\mathcal{L}}(Q)$
$\bot$, & $\vee$	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{C}$	Intuitionistic logic
$\rightsquigarrow$, $\bot$, $\mathbf{1}$, $\mathbf{0}$, &, $\wedge$, $\vee$	Sf, Pf	Full Lambek
$\bot$, &	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{C}, \mathrm{PL}$	Gödel logic
$\bot$, &, $\wedge$	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{PL}$	MTL logic
&, $\wedge$	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{PL}$	MTLH logic

the matching consecution (eg. in the presence of residuation *rule* we can prove that the residuation *axiom* is equivalent to the *associativity axiom* for &). Again, we present only the basic definitions, a lot of work is to be done yet.

Definition 2.14 (Adding connectives) *A logic* **L** *is* Q-weakly implicative logic in $\mathcal{L}$ *if its implication fragment is* Q*-implication fragment and if some of the connectives* $\{\top, \mathbf{0}, \mathbf{1}, \bot, \wedge, \vee, \&, \rightsquigarrow, \triangle\}$ *are in* $\mathcal{L}$*, then their matching rules and instances of* (CON) *rules for the connectives in question are elements of* **L**. *We say that* **L** *is an* Q*-fuzzy logic if* **L** *is fuzzy logic and* **L** *is* Q*-weakly implicative logic.*

To simplify things we will write that **L** is Q-logic instead of **L** is Q-weakly implicative logic.

Definition 2.15 *Let* $\mathcal{L}$ *be a propositional language. We denote the weakest* Q*-logic in* $\mathcal{L}$ *as* $\mathcal{MIN}_{\mathcal{L}}(Q)$ *and the weakest fuzzy* Q*-logic is denoted as* $\mathcal{FUZZ}_{\mathcal{L}}(Q)$

Table 2.7 puts some known logics into our context. The only thing we need to observe is the fact that in presence of $ax(\mathrm{Sf})$ and E we get from the residuation rules the residuation axioms (having this it is easy to show that & is associative. We show one direction:

(i)	$(\varphi \to (\psi \to \chi)) \to (\varphi \to (\psi \to \chi))$	(Ref)
(ii)	$\varphi \to ((\varphi \to (\psi \to \chi)) \to (\psi \to \chi))$	(i) and E
(iii)	$\varphi \to (\psi \to ((\varphi \to (\psi \to \chi)) \to \chi))$	$(ii), ax(\mathrm{E}), (\mathrm{WT})$
(iv)	$\varphi \& \psi \to ((\varphi \to (\psi \to \chi)) \to \chi)$	(iii) and residuation rule
(v)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$	$(iv), \mathrm{E}.$

The logic $\mathcal{FUZZ}_{\{\&\}}(ax(dT), \mathrm{E}, \mathrm{W})$ is the newest logic of Petr Hájek—the quasihoop logic (for details see [5]).

Definition 2.16 *The* quasihoop logic *(*QH*) has the following presentation:*

(1)	$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(2)	$\vdash \varphi \& \psi \to \psi \& \varphi$
(3)	$\vdash \varphi \& \psi \to \varphi$
(4)	$\vdash (\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(5)	$\vdash (\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
F_n	$\vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$

Observe that axiom (2) corresponds to E axiom (3) to W. Axioms (4) and (5) are axiomatic version of residuation rules R. The ordered QH-matrices are just BCK(RP)-algebras.

2.3 The connective $\vee$

Having disjunction in the language, we can express several concepts of this paper in more common ways. In this subsection we assume that $\vee \in \mathcal{L}$ for each logics we encounter here.

Lemma 2.17 *Let* **L** *be a weakly implicative logic. Then* $\varphi \vee \psi, \varphi \to \psi \vdash \psi$.

Proof: We give a formal proof:

(i) $\varphi \to \psi, \psi \to \psi \vdash \varphi \vee \psi \to \psi$ — $\vee 1$
(ii) $\varphi \vee \psi, \varphi \to \psi \vdash \varphi \vee \psi \to \psi$ — (i)
(iii) $\varphi \vee \psi, \varphi \to \psi \vdash \psi$ — (ii) and (MP)

QED

Now we define the notion of prime theory. This is more known concept than the concept of linear theory. However, we will see that in fuzzy logics both notions coincide.

Definition 2.18 (Prime theory) *Let* **L** *be a weakly implicative logic. A theory* T *is* prime *if from* $T \vdash \varphi \vee \psi$ *we get* $T \vdash \varphi$ *or* $T \vdash \psi$.

Definition 2.19 (Prime extension) *A weakly implicative logic* **L** *has the* Prime Extension Property (PEP) *if for each theory* T *formula* φ *such that* $T \nvdash \varphi$ *there is a prime theory* T', *such that* $T \subseteq T'$ *and* $T' \nvdash \varphi$.

Definition 2.20 (Proof by cases) *A weakly implicative logic* **L** *has the* Proof by Cases Property (PCP) *if for each theory* T *we get* $T, \varphi \vee \psi \vdash \chi$ *whenever* $T, \varphi \vdash \chi$ *and* $T, \psi \vdash \chi$.

Observe that above defined principles PCP and PEP differ from seemingly analogous principles PP and LEP. For example Intuitionistic logic has both PCP and PEP but doesn't have the other two. Let us examine this in more details:

Lemma 2.21 *Let* **L** *be a weakly implicative logic. Then:*

1. *each linear theory is prime;*
2. *if* **L** *has* PP *we have* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$;
3. *if* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$ *then each prime theory is linear;*
4. *if* **L** *has* PP *then* **L** *has* PCP;
5. *if* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$ *and* **L** *has* PCP *then* **L** *has* PP.

Proof:

1. Let $T \vdash \varphi \vee \psi$. Since T is linear we know that $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$. Thus (using Lemma 2.17) we get $T \vdash \psi$ or $T \vdash \varphi$.
2. Trivial.
3. Trivial.
4. From PP and $\vee 2$ we easily get $\vdash_{\mathbf{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$. Now let T be a theory such that $T, \varphi \vdash \chi$ and $T, \psi \vdash \chi$. Using Lemma 2.17 we know that $T, \varphi \vee \psi, \varphi \to \psi \vdash \psi$ and $T, \varphi \vee \psi, \psi \to \varphi \vdash \varphi$. Thus $T, \varphi \vee \psi, \varphi \to \psi \vdash \chi$ and $T, \varphi \vee \psi, \psi \to \varphi \vdash \chi$. PP completes the proof.
5. just observe that from $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ we get $T, (\varphi \to \psi) \vee (\psi \to \varphi) \vdash \chi$. Knowing that $\vdash_{\mathbf{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ we get $T \vdash \chi$.

QED

This lemma has three interesting corollaries.

Corollary 2.22 *In fuzzy logic the theory* T *is prime iff* T *is linear.*

Corollary 2.23 *A logic* **L** *is fuzzy iff* **L** *has* PEP *and* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$.

Corollary 2.24 *A finitary logic* **L** *is fuzzy iff* **L** *has* PCP *and* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$.

2.4 The connective $\triangle$

The connective $\triangle$ is a special one, in intuitionistic logic it is known as *globalization* (with some additional assumptions); in linear logic it is a kind of *exponential*; and in fuzzy logics it is known as Baaz delta. Roughly speaking, this connective allows us *controlled* use of structural rules. In this subsection we assume that $\triangle$ is an element of all the propositional languages $\mathcal{L}$ (unless the opposite is explicitly mentioned)

Now we present the analog of Definition 2.6 and prove the analog of Theorem 2.8. However, the $\triangle$ connective allows us much simpler formulations. In the following definition and theorem we understand **L** as arbitrary weakly implicative logic (not necessarily with matching rules $\triangle$1, $\triangle$2, $\triangle$3,$\triangle$W, $\triangle$C, $\triangle$E, and (NEC)).

Definition 2.25 *Let* **L** *a weakly implicative logic. We say that* **L** *has* Delta Deduction Theorem *(*$\mathrm{DT}_\triangle$*) if for each theory* T *and formulas* φ, ψ *we have:* $T, \varphi \vdash \psi$ *iff* $T \vdash \triangle\varphi \to \psi$*.*

Now we present sufficient and necessary condition for **L** to have $\mathrm{DT}_\triangle$. And we also show that in each weakly implicative logic with $\mathrm{DT}_\triangle$ the matching rules for $\triangle$ hold.

Theorem 2.26 (Deduction theorem) *Let* **L***be a finitary logic. Then* **L** *has* $\mathrm{DT}_\triangle$ *iff* **L** *has some presentation* $\mathcal{AX}$*, where* (MP) *and* (NEC) *are the only deduction rules and all the matching axioms for* $\triangle$ *hold.*

Proof: First direction: Assume that **L** has $\mathrm{DT}_\triangle$, then obviously **L** has presentation where (MP) and (NEC) are the only deduction rules (just replace each rule $\varphi_1, \ldots \varphi_n \vdash \psi$ with the following axiom $\vdash \triangle\varphi_1 \to (\ldots \to (\triangle\varphi_n \to \psi)\ldots)$). All we have to do is to show that the matching rules for $\triangle$ hold:

- (NEC): From $\vdash \triangle\varphi \to \triangle\varphi$ we get $\varphi \vdash \triangle\varphi$ (using $\mathrm{DT}_\triangle$)
- $\triangle$1: From $\varphi, \varphi \to \psi \vdash \psi$ we get $\varphi, \varphi \to \psi \vdash \triangle\psi$ (using (NEC)). Applying $\mathrm{DT}_\triangle$ twice completes the proof.
- $\triangle$2: From $\varphi \vdash \varphi$ we get $\vdash \triangle\varphi \to \varphi$ (using $\mathrm{DT}_\triangle$)
- $\triangle$3: From $\varphi \vdash \triangle\varphi$ we get $\vdash \triangle\varphi \to \triangle\triangle\varphi$ (using $\mathrm{DT}_\triangle$)
- $\triangle$W: From $\varphi, \psi \vdash \varphi$ we get $\varphi \vdash \triangle\psi \to \varphi$ (using $\mathrm{DT}_\triangle$ twice)
- $\triangle$C: We know that $\varphi, \triangle\varphi \to (\triangle\varphi \to \psi) \vdash \psi$ ((NEC) and (MP) twice). Applying $\mathrm{DT}_\triangle$ twice completes the proof.
- $\triangle$E: We know that $\varphi, \psi, \triangle\varphi \to (\triangle\psi \to \chi) \vdash \chi$ ((NEC) twice and (MP) twice). Applying $\mathrm{DT}_\triangle$ three times completes the proof.

Reverse direction: we need to prove two directions. One is obvious. To prove the other one we use the induction over the proof of ψ in T, φ (in $\mathcal{AX}$). We show that it holds for each χ in the proof of ψ in T, φ

- $\chi \in T$, χ is an axiom - trivial using $\triangle$W
- $\chi = \varphi$ - trivial using $\triangle$2
- $\chi = \triangle\psi_1$ is obtained from its predecessor ψ_1 by (NEC). From the induction property for ψ_2 we know that $T \vdash \triangle\varphi \to \psi_1$. We apply (NEC), $\triangle$1, and (MP) to get $T \vdash \triangle\triangle\varphi \to \triangle\psi_1$. Using $\triangle$3 and (WT) we get $T \vdash \triangle\varphi \to \triangle\psi_1$.
- χ is obtained from its predecessors $\psi_2 = \psi_1 \to \chi$ and ψ_1 by (MP). From the induction property for ψ_2 we know that $T \vdash \triangle\varphi \to (\psi_1 \to \chi)$. We apply $\triangle$1 twice to get $T \vdash \triangle\triangle\varphi \to (\triangle\psi_1 \to \triangle\chi)$, using $\triangle$3 and $\triangle$E we get $T \vdash \triangle\psi_1 \to (\triangle\varphi \to \triangle\chi)$.

  From the induction property for ψ_1 we know that $T \vdash \triangle\varphi \to \psi_1$. We apply $\triangle$1 and $\triangle$3 to get $T \vdash \triangle\varphi \to \triangle\psi_1$. Now using (WT) we obtain $T \vdash \triangle\varphi \to (\triangle\varphi \to \triangle\chi)$. Axiom $\triangle$C gets us $T \vdash \triangle\varphi \to \triangle\chi$ and $\triangle$2 completes the proof.

QED

We can easily prove analogy of Lemmata 1.47 and 2.10. Thus having an equivalent definition of fuzzy logics in some class of logics.

Lemma 2.27 *Let* $\mathbf{L}$ *be a fuzzy logic. Then* $\triangle(\varphi \to \psi) \to \chi, \triangle(\psi \to \varphi) \to \chi \vdash_{\mathbf{L}} \chi$

Lemma 2.28 *Let* $\mathbf{L}$ *be a finitary fuzzy logic with* $\mathrm{DT}_{\triangle}$*. Then* $\triangle(\varphi \to \psi) \to \chi, \triangle(\psi \to \varphi) \to \chi \vdash_{\mathbf{L}} \chi$ *iff* $\mathbf{L}$ *is fuzzy.*

Unlike in Lemma 2.10, in this case the rule appearing in the previous lemma is equivalent to the following axiom: $\vdash_{\mathbf{L}} \triangle(\triangle(\varphi \to \psi) \to \chi) \to (\triangle(\triangle(\psi \to \varphi) \to \chi) \to \chi)$. Having $\vee$ in the language we can formulate even stronger claim.

Corollary 2.29 *Let* $\mathcal{L}$ *be a language,* $\vee \in \mathcal{L}$ *and* $\mathbf{L}$ *a finitary logic in* $\mathcal{L}$ *with* $\mathrm{DT}_{\triangle}$*. Then* $\mathbf{L}$ *is fuzzy iff* $\vdash_{\mathbf{L}} \triangle(\varphi \to \psi) \vee \triangle(\psi \to \varphi)$

Now we observe that $\triangle(\varphi \to \varphi)$ can be used as definition of $\mathbf{1}$. Thus we may assume that whenever $\triangle \in \mathcal{L}$, then $\mathbf{1} \in \mathcal{L}$.

Lemma 2.30 *It holds:*

- $\vdash \triangle(\varphi \to \varphi) \to \triangle(\psi \to \psi)$
- $\psi \vdash \triangle(\varphi \to \varphi) \to \psi$
- $\triangle(\varphi \to \varphi) \to \psi \vdash \psi$
- $\vdash \triangle(\varphi \to \varphi)$

Recall, the in presence of $\perp$ we can define the derived connective negation as $\neg\varphi = \varphi \to \perp$. Now we show some rather trivial properties of logics with $\triangle$ and $\perp$ in the language.

Lemma 2.31 *Let* $\mathcal{L}$ *be a language such that* $\perp \in \mathcal{L}$ *and* $\mathbf{L}$ *a logic in* $\mathcal{L}$*. Then*

1. $\vdash_{\mathbf{L}} \triangle(\mathbf{1} \to \perp) \to \perp$
2. $\vdash_{\mathbf{L}} \perp \leftrightarrow \triangle\perp$
3. $\vdash_{\mathbf{L}} \mathbf{1} \leftrightarrow \triangle\mathbf{1}$
4. $\vdash_{\mathbf{L}} (\triangle\varphi \to \neg\triangle\varphi) \to \neg\triangle\varphi$

Proof: The only non-trivial is Part 1. We give a formal proof.

(i)	$\triangle(\mathbf{1} \to \perp) \to \triangle(\mathbf{1} \to \perp)$	(Ref)
(ii)	$\triangle(\mathbf{1} \to \perp) \to (\triangle\mathbf{1} \to \triangle\perp)$	(i), $\triangle 1$, and (WT)
(iii)	$\triangle\mathbf{1} \to (\triangle(\mathbf{1} \to \perp) \to \triangle\perp)$	(ii), (NEC), $\triangle$E, (WT)
(iv)	$\triangle(\mathbf{1} \to \perp) \to \triangle\perp$	(iii) and (MP)
(v)	$\triangle(\mathbf{1} \to \perp) \to \perp$	(iv), $\triangle 2$ and (WT)

QED

The reader familiar with fuzzy logic with $\triangle$ can notice that matching rules are somehow "weak", $\triangle$ has more properties in this case, which do not hold in general. In the rest of this section we add some additional rules for $\triangle$ to get known fuzzy logics with $\triangle$. We start by showing that fuzzy logic $\mathbf{L}$, where $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ has some interesting properties.

Lemma 2.32 *Let* $\mathcal{L}$ *be a language,* $\perp \in \mathcal{L}$*,* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$*, and* $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$*. Then:*

1. *For each linearly ordered* $\mathbf{L}$*-matrix* $\mathbf{B}$ *holds:* $\triangle_{\mathbf{B}} x = \mathbf{1}_{\mathbf{B}}$ *if* $\mathbf{1}_{\mathbf{B}} \leq_{\mathbf{B}} x$ *and* $\triangle_{\mathbf{B}} x = \perp_{\mathbf{B}}$ *otherwise.*

2. $\mathbf{L}$ *has* $\mathrm{DT}_\triangle$*.*

3. $T, \triangle\varphi \vdash \chi$ *and* $T, \neg\triangle\varphi \vdash \chi$ *entails* $T \vdash \chi$*.*

4. $\vdash (\mathbf{1} \to \bot) \to \bot$ *iff* $\vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$*.*

5. $\vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$ *iff* $\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$*.*

Proof: 1. We omit the subscripts $\mathbf{B}$ in this proof. Let $\mathbf{1} \leq x$, then obviously $\mathbf{1} \leq \triangle x$ (using (NEC)). Using $\triangle$W we get $\mathbf{1} \leq \triangle\mathbf{1} \leq \triangle x \to \mathbf{1}$, thus $\triangle x \leq \mathbf{1}$.

If $x < \mathbf{1}$ then $\triangle x < \mathbf{1}$ as well (because ($\triangle$2) we know $\triangle x \leq x$). Since $\neg\triangle x \to \triangle x \leq \triangle x$ we have to have $\neg\triangle x > \triangle x$ (otherwise $\mathbf{1} \leq \neg\triangle x \to \triangle x$ and so $\mathbf{1} \leq \triangle x$—a contradiction). Observe that from $\triangle$C we get $\vdash_{\mathbf{L}} \triangle(\triangle\varphi \to (\triangle\varphi \to \bot)) \to (\triangle\varphi \to \bot)$, i.e., $\triangle(\triangle x \to \neg\triangle x) \leq \neg\triangle x$. We know that $\triangle x \leq \neg\triangle x$ so $\triangle(\triangle x \to \neg\triangle x) = \mathbf{1}$ and so $\mathbf{1} \leq \triangle x \to \bot$. Finally, $\triangle x \leq \bot$.

2. One direction is obvious. We show the reverse direction contrapositively: if $T \not\vdash \triangle\varphi \to \psi$, then there is linearly ordered $\mathbf{L}$-matrix $\mathbf{B}$ and $\mathbf{B}$-model e of T and $e(\triangle\varphi \to \psi) < \mathbf{1}$ (again we omit the subscripts $\mathbf{B}$). Then obviously $\mathbf{1} \leq e(\varphi)$ (otherwise $e(\triangle\varphi) = \bot$ and so $\mathbf{1} \leq e(\triangle\varphi \to \psi)$—a contradiction), thus $e(\triangle\varphi) = \mathbf{1}$ and so $e(\triangle\varphi \to \psi) = \mathbf{1} \to e(\psi) = e(\psi)$. So we know that e is $\mathbf{B}$-model e of T, φ and $e(\psi) < \mathbf{1}$. Thus $T, \varphi \not\vdash \psi$.

3. We observe that since $\vdash \triangle(\neg\triangle\psi \to \triangle\psi) \to \triangle\psi$ and $\vdash \triangle(\triangle\psi \to \neg\triangle\psi) \to \triangle\neg\triangle\psi$ (from Lemma 2.31), whenever we prove $T \vdash \triangle\psi \to \chi$ and $T \vdash \triangle\neg\triangle\psi \to \chi$ then $T \vdash \chi$ (using Lemma 2.28 and the fact that $\mathbf{L}$ has $\mathrm{DT}_\triangle$). Using $\mathrm{DT}_\triangle$ once more completes the proof.

4. Assume that $\vdash (\mathbf{1} \to \bot) \to \bot$, then also $\neg\mathbf{1} \to \chi$. We use part 3 of this lemma: first notice that $\triangle\varphi \vdash \mathbf{1} \leftrightarrow \triangle\varphi$ and so $\triangle\varphi \vdash \neg\triangle\varphi \to \neg\mathbf{1}$. Thus $\triangle\varphi \vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$. Second, we also know that $\neg\triangle\varphi \vdash \mathbf{1} \leftrightarrow \neg\triangle\varphi$ and since $\vdash \mathbf{1} \to \triangle\mathbf{1}$ we get $\neg\triangle\varphi \vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$.

To prove the reverse direction just set $\varphi = \bot$ and get $\neg\triangle\bot \to \triangle\neg\triangle\bot$. Lemma 2.31 completes the proof.

5. Observe that $\psi \to \varphi \vdash \triangle\neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ (using $\triangle$W, $\triangle$3, and $\mathrm{DT}_\triangle$) and so $\psi \to \varphi \vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ (using $\vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$). Now $\varphi \to \psi \vdash \triangle(\varphi \to \psi) \leftrightarrow \mathbf{1}$ and since we know $\vdash (\mathbf{1} \to \bot) \to \chi$ (from part 4. of this lemma) we get $\varphi \to \psi \vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$.

To prove the reverse direction just set $\varphi = \bot$ and $\psi = \mathbf{1}$ and get $\vdash \neg\triangle(\bot \to \mathbf{1}) \to \triangle(\mathbf{1} \to \bot)$. Observe that $\vdash \triangle(\bot \to \mathbf{1}) \leftrightarrow \mathbf{1}$ (using $\mathrm{DT}_\triangle$) and so we have $\vdash (\mathbf{1} \to \bot) \to \triangle(\mathbf{1} \to \bot)$. Lemma 2.31 and Part 4. of this lemma complete the proof. QED

Observe that the part 1. holds even without assumption that $\mathbf{L}$ is fuzzy and that part 2. has an interesting corollary:

Corollary 2.33 *Let $\mathcal{L}$ be a language such that $\bot \in \mathcal{L}$, $\mathbf{L}$ a finitary fuzzy logic in $\mathcal{L}$, such that $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$. Then $\mathbf{L}$ has a presentation, where* (MP) *and* (NEC) *are the only deduction rules.*

Of course, we even know this presentation: just replace each rule $\varphi_1, \ldots \varphi_n \vdash \psi$ with the axiom $\vdash \triangle\varphi_1 \to (\ldots \to (\triangle\varphi_n \to \psi) \ldots)$. Now we try to formulate the "essence" of the connective $\triangle$, when used in fuzzy logics.

Definition 2.34 *Let $\mathcal{L}$ be a language such that $\bot, \triangle \in \mathcal{L}$ and $\mathbf{L}$ a fuzzy logic in $\mathcal{L}$. We say that $\mathbf{L}$ is logic with* Baaz delta *iff for each formula in language $\{\to, \bot, \mathbf{1}\}$, if we define substitution $\sigma(v) = \triangle v$ then $\vdash_{\mathbf{L}} \sigma\varphi$ iff φ is a theorem of classical logic.*

Obviously, not all logics with $\triangle$ connective are logic with Baaz delta (take Intuitionistic logic, and in all Heyting algebras interpret $\triangle$ as identity). Observe that if $\mathbf{L}$ is logic with Baaz delta it is consistent. This definition can be viewed as rather peculiar, but we are going to present more convenient alternative definition. The following lemma works for fuzzy logic only: even with Intuitionistic logic fulfills the properties 1.–5. from the upcoming definition it is not a logic with Baaz delta, analogously Łukasiewicz logic with globalization is not fuzzy logic and so it is not a logic with Baaz delta.

Lemma 2.35 *Let $\mathbf{L}$ be a consistent fuzzy logic. Then $\mathbf{L}$ is a fuzzy logic with Baaz delta iff the following hold:*

1. $\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$
2. $\vdash (\mathbf{1} \to \mathbf{1}) \to \mathbf{1}$
3. $\vdash (\bot \to \mathbf{1}) \to \mathbf{1}$
4. $\vdash (\bot \to \bot) \to \mathbf{1}$
5. $\vdash (\mathbf{1} \to \bot) \to \bot$

Proof: One direction is obvious (only non-trivial part is to show $\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$—to do this just notice that $(\neg p \to p) \to p$ is a theorem of classical logic).

To prove converse direction we need to show two things: first, if φ is a theorem of the classical logic, then $\vdash \sigma\varphi$. We prove this contrapositively: assume that $\not\vdash \sigma\varphi$, then there is linear **L**-matrix **B** and **B**-evaluation e such that $e(\sigma\varphi) < \mathbf{1}$. Observe that form Lemma 2.32 part 1. we know that $e(\triangle v)$ is either $\mathbf{1}$ or $\bot$, from the form of the formula $\sigma\varphi$ and from theorems 2.–5. we know that $e(\sigma\varphi) = \bot$ and if we define evaluation $f(v) = e(\triangle v)$ then φ is a classical evaluation not satisfying φ.

Second, we need to show that if $\vdash \sigma\varphi$ then φ is a theorem of the classical logic. We prove this by a contradiction: assume that there is a classical evaluation e, such that $e(\varphi) = \bot$. Then there is substitution $\rho(v) = e(v)$ (we identify constants $\mathbf{1}$ and $\bot$ with two truth values of classical logic) and $\rho\varphi \to \bot$ is a theorem of the classical logic. Thus $\vdash \sigma(\rho\varphi \to \bot)$. Because $\sigma(\rho\varphi \to \bot) = \rho\varphi \to \bot$ (there are no variables in $\rho\varphi \to \bot$) and theorems 2.–5. we get $\vdash \rho\varphi \to \bot$ (using the previous direction). Since we assume that $\vdash \sigma\varphi$ we also have $\vdash \rho\sigma\varphi$ and so finally $\vdash \rho\varphi$ (because $\triangle\bot \leftrightarrow \bot$ and $\triangle\mathbf{1} \leftrightarrow \mathbf{1}$). Thus together we have $\vdash \bot$—a contradiction with consistency of **L**. QED

In the literature, it is common that for fuzzy logic **L** there is defined its conservative expansion by the connective $\triangle$ (usually denoted as $\mathbf{L}_\triangle$), which is fuzzy as well. It is usual that $\mathbf{L}_\triangle$ is a logic with Baaz delta and has $\mathrm{DT}_\triangle$ (even if **L** has not some variant of deduction theorem). Now we introduce general way of expanding the logic **L** into the logic $\mathbf{L}_\triangle$. First we give an indirect definition and then we show how to find a presentation of $\mathbf{L}_\triangle$ based on the presentation of $\mathbf{L}_\triangle$.

Definition 2.36 *Let **L** be a consistent fuzzy logic in $\mathcal{L}$, such that $\bot \in \mathcal{L}$ and $\triangle \notin \mathcal{L}$ By $\mathbf{L}_\triangle$ we denote the weakest fuzzy logic with Baaz delta in language $\mathcal{L} \cup \{\triangle\}$ expanding **L**.*

Theorem 2.37 *Let **L** be a consistent finitary fuzzy logic in $\mathcal{L}$, such that $\bot \in \mathcal{L}$ and $\triangle \notin \mathcal{L}$ and let $\mathcal{AX}$ be some finitary presentation of **L**. Then the following is a presentation of $\mathbf{L}_\triangle$:*

A	*axioms of $\mathcal{AX}$,*
B	$\vdash \triangle\varphi_1 \to (\ldots(\triangle\varphi_n \to \psi)$ *for each n-ary deduction rule* $<\varphi_1,\ldots,\varphi_n,\psi> \in \mathcal{AX}$,
C	*matching rules for $\triangle$,*
(MP)	$\varphi, \varphi \to \psi \vdash \psi$
$\triangle 4$	$\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi,$
$\triangle 5$	$\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi),$
$\triangle 6$	$\vdash (\mathbf{1} \to \mathbf{1}) \to \mathbf{1}.$

Proof: Let $\mathbf{L}'$ be the logic with the above presentation presentation. We have to show that $\mathbf{L}'$ is a fuzzy logic with Baaz delta (it obviously extends **L**).

Since **L** is fuzzy we get $(\varphi \to \psi) \to \chi, (\psi \to \varphi) \to \chi \vdash_{\mathbf{L}} \chi$ and so $(\varphi \to \psi) \to \chi, (\psi \to \varphi) \to \chi \vdash_{\mathbf{L}'} \chi$. From Lemma 2.31 and $\triangle 4$ we know that $\vdash_{\mathbf{L}'} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ and $\vdash_{\mathbf{L}'} (\triangle\varphi \to \neg\triangle\varphi) \to \neg\triangle\varphi$. Thus if we prove $T \vdash_{\mathbf{L}'} \triangle\varphi \to \chi$ and $T \vdash_{\mathbf{L}'} \neg\triangle\varphi \to \chi$ we get $T \vdash_{\mathbf{L}'} \chi$.

Let us denote $T = \{\triangle(\varphi \to \psi) \to \chi, \triangle(\psi \to \varphi) \to \chi\}$. Observe that $T, \triangle(\varphi \to \psi) \vdash_{\mathbf{L}'} \chi$ and $T, \neg\triangle(\varphi \to \psi) \vdash_{\mathbf{L}'} \chi$ (the first is obvious, to prove the second use $\triangle 4$). Thus $T \vdash_{\mathbf{L}'} \chi$. From Theorem 2.26 we know that $\mathbf{L}'$ has $\mathrm{DT}_\triangle$ and from the fact $T \vdash_{\mathbf{L}'} \chi$ we know that $\mathbf{L}'$ is fuzzy (using Lemma 2.28).

Now we observe that $\mathbf{L}'$ is a conservative expansion of **L** (we can extend any linear **L**-matrix **B**, which is counterexample to $T \vdash \varphi$, into the linear $\mathbf{L}'$-matrix $\mathbf{B}_\triangle$ and is is a counterexample as well—since $\mathbf{L}'$ is fuzzy). So we know that $\mathbf{L}'$ is consistent (because **L** is consistent) .

Thus we can use Lemma 2.35 to prove that $\mathbf{L}'$ is a logic with Baaz delta. Notice that all we need to show is the following:

1. $\vdash (\bot \to \mathbf{1}) \to \mathbf{1}$

2. $\vdash (\bot \to \bot) \to \mathbf{1}$

3. $\vdash (\mathbf{1} \to \bot) \to \bot$

To prove 1. use theorem $\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ for $\varphi = \mathbf{1}$, you get $\vdash (\neg\triangle\mathbf{1} \to \triangle\mathbf{1}) \to \triangle\mathbf{1}$. Lemma 2.31 completes the proof. To prove 2. use theorem $\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ for $\varphi = \mathbf{1}$ and $\psi = \bot$, you get $\vdash \neg\triangle(\mathbf{1} \to \bot) \to \triangle(\bot \to \mathbf{1})$. Lemma 2.31 completes the proof. To prove 3. use theorem $\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ for $\varphi = \bot$ and $\psi = \mathbf{1}$, you get $\vdash \neg\triangle(\bot \to \mathbf{1}) \to \triangle(\mathbf{1} \to \bot)$. Lemma 2.31 completes the proof.

To complete the proof of this theorem we need to show that each fuzzy logic $\mathbf{L}'$ with Baaz delta proofs all formulas from our presentation. From Lemma 2.35 we know that $\vdash_{\mathbf{L}'} \triangle 6$, $\vdash_{\mathbf{L}'} \triangle 4$, and $\vdash_{\mathbf{L}'} (\mathbf{1} \to \bot) \to \bot$. Observe that in this case we can use Lemma 2.32 Parts 4. and 5. to get $\vdash_{\mathbf{L}'} \triangle 5$. Logic $\mathbf{L}'$ obviously proves all consecution from the group A and B. So all we have to show are the axioms from the group B: to do this just observe that $\mathbf{L}'$ has $\mathrm{DT}_\triangle$ (because $\mathbf{L}'$ is fuzzy and $\vdash_{\mathbf{L}'} \triangle 4$ we can use Lemma 2.32) and since $\mathbf{L} \subseteq \mathbf{L}'$ the proof is done. QED

The presence of the axiom $\vdash (\mathbf{1} \to \mathbf{1}) \to \mathbf{1}$ seems to be unavoidable, however under some rather weak additional assumptions we can omit it. One of them is of course the presence of weakening, the other one is the presence of Sf (then we get $\vdash (\mathbf{1} \to \mathbf{1}) \to (\bot \to \mathbf{1})$ and using the know fact that $\vdash (\bot \to \mathbf{1}) \to \mathbf{1}$ we would get our axiom). Now we show that $\mathbf{L}_\triangle$ has some promised nice properties:

Lemma 2.38 *Let* $\mathbf{L}$ *be a consistent fuzzy logic in* $\mathcal{L}$*, such that* $\triangle \notin \mathcal{L}$ *and* $\bot \in \mathcal{L}$*. Then* $\mathbf{L}_\triangle$ *is a fuzzy logic with* $\mathrm{DT}_\triangle$ *and Baaz delta, which is a conservative expansion of* $\mathbf{L}$*.*

3 First-order logic

In this second part of this paper, we move to the first-order logics. We present the very basic theorems only. The broader treatment of this topic will be the content of the subsequent papers. Our approach is inspired by the classical first order logic and by its modification (the axiomatic system, the notion of Henkin theory) for non-classical logics, the main source is Hájek's treatment of basic predicate fuzzy logic (for details see [4]).

3.1 Basic definitions

In the following let $\mathbf{L}$ be a fixed weakly implicative logic in propositional language $\mathcal{L}$.

Definition 3.1 (Predicate language) *By* multi-sorted predicate language $\mathbf{\Gamma}$ *we understand a quintuple* $(\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$*, where* $\mathbf{S}$ *is a non-empty set of sorts,* $\preceq$ *is an ordering on* $\mathbf{S}$ *(indicating the subsumption of sorts),* $\mathbf{P}$ *is a non-empty set of predicate symbols,* $\mathbf{F}$ *is a set of function symbols, and* $\mathbf{A}$ *is a function assigning to each predicate and function symbol a finite sequences of elements of* $\mathbf{S}$*.*

Let $|\mathbf{A}(P)|$ *denote the length of the sequence* $\mathbf{A}(P)$*. The number* $|\mathbf{A}(P)|$ *is called the* arity of the predicate symbol P*. The number* $|\mathbf{A}(f)|-1$ *is called the* arity of the function symbol f*. The functions* f *for which* $\mathbf{A}(f) = <s>$ *are called the* individual constants *of sort* s*. If* $s_1 \preceq s_2$ *holds for sorts* s_1, s_2 *we say that* s_2 subsumes s_1*.*

The $\mathcal{L}$-logical symbols *are individual variables* $x^s, y^s, \ldots$ *for each sort* s*, the logical connectives of* $\mathcal{L}$*, and the quantifiers* $\forall$ *and* $\exists$*.*

Let us denote by $\mathbf{C}_s$ the set of constants of the sort s. In the following let $\mathbf{\Gamma}$ be a fixed multi-sorted predicate language for logic $\mathbf{L}\forall$.

Definition 3.2 (Terms) *Each individual variable of sort* s *is a* $\mathbf{\Gamma}$-term *of sort* s*. Let* $t_1, \ldots, t_n$ *be terms of sorts* $s_1, \ldots, s_n$*, and* f *be a function symbol,* $\mathbf{A}(f) = <w_1, \ldots, w_n, w_{n+1}>$ *such that* $s_i \preceq w_i$ *for* $i \leq n$*. Then* $f(t_1, \ldots, t_n)$ *is a* $\mathbf{\Gamma}$*-term of sort* w_{n+1}*.*

Notice that the set of terms depends on $\mathbf{\Gamma}$ only, whereas the set of formulas depends of the propositional language as well. So we should speak about $\mathbf{\Gamma}$-terms and $(\mathbf{\Gamma}, \mathcal{L})$-formulas (however, we speak about $\mathbf{\Gamma}$-formulas if the propositional language is clear from the context and we speak about terms and formulas if both propositional and the predicate language are clear from the context).

Definition 3.3 (Formulas) *Let $t_1, \dots t_n$ be terms of sorts $s_1, \dots s_n$, and P be a predicate symbol, $\mathbf{A}(P) = <w_1, \dots w_n>$, such that $s_i \preceq w_i$ for $i \leq n$. Then $P(t_1, \dots t_n)$ is an* atomic $\mathbf{\Gamma}$-formula. *The nullary logical connectives of $\mathcal{L}$ are atomic $\mathbf{\Gamma}$-formulas as well.*

Let φ be $\mathbf{\Gamma}$-formula and x^s an object variable of the sort s. Then $(\forall x^s)\varphi$ and $(\exists x^s)\varphi$ are $\mathbf{\Gamma}$-formulas. Furthermore, the class of $\mathbf{\Gamma}$-formulas is closed under logical connectives of $\mathcal{L}$.

Bounded *and* free variables *in a formula are defined as usual. A formula is called a* sentence *iff it contains no free variables. A set of $\mathbf{\Gamma}$-sentences is called a* $\mathbf{\Gamma}$-theory.

Instead of $\xi_1, \dots, \xi_n$ (where ξ_i's are terms or formulae and n is arbitrary or fixed by the context) we shall sometimes write just $\vec{\xi}$.

Unless stated otherwise, the expression $\phi(x_1, \dots, x_n)$ means that all free variables of ϕ are among $x_1, \dots, x_n$.

If $\phi(x_1, \dots, x_n, \vec{z})$ is a formula and we substitute terms t_i for all x_i's in ϕ, we denote the resulting formula in the context simply by $\phi(t_1, \dots, t_n, \vec{z})$.

Definition 3.4 (Substitutability) *A term t of sort w is* substitutable *for the individual variable x^s in a formula $\varphi(x^s, \vec{z})$ iff $w \preceq s$ and no occurrence of any variable y occurring in t is bounded in $\varphi(t, \vec{z})$.*

Let $\mathbf{B}$ be fixed ordered $\mathbf{L}$-matrix in the following text.

Definition 3.5 (Structure) *An* $\mathbf{B}$-structure $\mathbb{M}$ *for $\mathbf{\Gamma}$ has form: $\mathbb{M} = ((M_s)_{s \in \mathbf{S}}, (P_\mathbb{M})_{P \in \mathbf{P}}, (f_\mathbb{M})_{f \in \mathbf{F}})$, where M_s is a non-empty domain for each $s \in \mathbf{S}$ and $M_s \subseteq M_w$ iff $s \preceq w$; $P_\mathbb{M}$ is an n-ary fuzzy relation $\prod_{i=1}^n M_{s_i} \to \mathbf{L}$ for each predicate symbol $P \in \mathbf{P}$ such that $\mathbf{A}(P) = <s_1, \dots, s_n>$; $f_\mathbb{M}$ is a function $\prod_{i=1}^n M_{s_i} \to M_{s_{n+1}}$ for each function symbol $f \in \mathbf{F}$ such that $\mathbf{A}(f) = <s_1, \dots, s_n, s_{n+1}>$, and an element of M_s if f is a constant of sort s.*

Definition 3.6 (Evaluation) *Let $\mathbb{M}$ be a $\mathbf{B}$-structure for $\mathbf{\Gamma}$. An* $\mathbb{M}$-evaluation *of the object variables is a mapping* e *which assigns to each variable of sort s an element from M_s (for all sorts $s \in \mathbf{S}$).*

Let e *be an $\mathbb{M}$-evaluation, x a variable of sort s, and $a \in M_s$. Then $\mathrm{e}[x \to a]$ is an $\mathbb{M}$-evaluation such that $\mathrm{e}[x \to a](x) = a$ and $\mathrm{e}[x \to a](y) = \mathrm{e}(y)$ for each individual variable y different from x.*

Definition 3.7 (Truth definition) *Let $\mathbb{M}$ be a $\mathbf{B}$-structure for $\mathbf{\Gamma}$, and* v *an $\mathbb{M}$-evaluation. A* values *of the terms and a* truth values *of the formulas in $\mathbb{M}$ for an evaluation* v *are defined as follows:*

$$
\begin{array}{rcl}
||x_s||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & \mathrm{v}(x)\,, \\
||f(t_1, t_2, \dots, t_n)||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & f_\mathbb{M}(||t_1||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ ||t_2||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ \dots,\ ||t_n||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}})\,, \\
||P(t_1, t_2, \dots, t_n)||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & P_\mathbb{M}(||t_1||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ ||t_2||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ \dots,\ ||t_n||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}})\,, \\
||c(\varphi_1, \varphi_2, \dots, \varphi_n)||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & c_{\mathbf{B}}(||\varphi_1||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ ||\varphi_2||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ \dots,\ ||\varphi_n||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}})\,, \\
||(\forall x^s)\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & \inf\{||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}[x^s \to a]} \mid a \in M_s\}\,, \\
||(\exists x^s)\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & \sup\{||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}[x^s \to a]} \mid a \in M_s\}\,,
\end{array}
$$

If the infimum or supremum does not exist, we take its value as undefined. We say that a $\mathbf{B}$-structure $\mathbb{M}$ for $\mathbf{\Gamma}$ is safe *iff $||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}}$ is defined for each $\mathbf{\Gamma}$-formula φ and each $\mathbb{M}$-evaluation* v.

Definition 3.8 (Value of formula) *Let $\mathbb{M}$ be a safe $\mathbf{B}$-structure for $\mathbf{\Gamma}$, and φ a $\mathbf{\Gamma}$ formula.* A truth value of the formula φ *in $\mathbb{M}$ is defined as follows:*

$$||\varphi||^{\mathbf{B}}_{\mathbb{M}} = \inf\{||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} \mid \mathrm{v} \text{ is an } \mathbb{M}\text{-evaluation}\}.$$

We say that φ is an $\mathbf{B}$-tautology *if $||\varphi||^{\mathbf{A}}_{\mathbb{M}} \in D_{\mathbf{B}}$ for each safe* $\mathbf{B}$-structure $\mathbb{M}$.

Definition 3.9 (Model) *Let $\mathbb{M}$ be a $\mathbf{B}$-structure for $\mathbf{\Gamma}$, and T a $\mathbf{\Gamma}$-theory. The $\mathbf{B}$-structure $\mathbb{M}$ for $\mathbf{\Gamma}$ is called $\mathbf{B}$-model of T if $||\varphi||^{\mathbf{A}}_{\mathbb{M}} \in D_{\mathbf{B}}$ for each $\varphi \in T$. We denote the set of $\mathbf{A}$-models od T by $\mathcal{MOD}(T, \mathbf{A})$*

Definition 3.10 (Semantical consequence) *Let $\mathcal{K}$ be a class of $\mathcal{L}$-matrices. We say that φ is a semantical consequence of the T w.r.t. class $\mathcal{K}$ if $\mathcal{MOD}(T, \mathbf{B}) = \mathcal{MOD}(T \cup \{\varphi\}, \mathbf{B})$ for each $\mathbf{B} \in \mathcal{K}$; we denote it by $T \models_{\mathcal{K}} \varphi$. By $\mathcal{TAUT}(\mathcal{K})$ we understand the set $\{\varphi \mid \emptyset \models_{\mathcal{K}} \varphi\}$.*

We write $T \models_{\mathbf{L}} \varphi$ instead of $T \models_{o-\mathbf{MAT(L)}} \varphi$ and we also write $T \models^{l}_{\mathbf{L}} \varphi$ instead of $T \models_{l-\mathbf{MAT(L)}} \varphi$

For a fixed $\mathbf{B}$-model $\mathbb{M}$ and an $\mathbb{M}$-valuation e such that $\mathrm{e}(x_i) = a_i$ (for all i's), instead of $\|\varphi(x_1, \ldots, x_n)\|^{\mathbf{B}}_{\mathbb{M},\mathrm{e}}$ we write simply $\|\varphi(a_1, \ldots, a_n)\|$ and speak of the value of $\varphi(a_1, \ldots, a_n)$.

Now we define the predicate logic to each weakly implicative logic (and stronger predicate logic for each fuzzy logics). As in the propositional case we understand the predicate logic as an asymmetric consequence relation (following Dunn's terminology). For simplicity of this introductory paper we made two extra design choices. First, we assume that $\vee$ is the part of the language (there are ways how ovoid the need for $\vee$ under some additional assumptions eg. having exchange, or having $\rightsquigarrow$ in the language). Second, we formulate consecutions ($\forall$2) and ($\exists$2) as axioms rather than rules, which would result into the weaker definition. However, it is obvious that under some rather weak assumptions these two notions would coincide (eg. under the presence of & in the language).

These two topics will be more elaborated in some subsequent paper.

Definition 3.11 *Let $\mathbf{L}$ be a weakly implicative logic. The logic $\mathbf{L}\forall^-$ is given by the following axioms and the deduction rules:*

(P)	*the formulas and deduction rules resulting from the axioms and deduction rules of $\mathbf{L}$ by the substitution of the propositional variables by the formulas of $\mathbf{\Gamma}$*
($\forall$1)	$\vdash_{\mathbf{L}\forall^-} (\forall x)\varphi(x) \to \varphi(t)$, *where t is substitutable for x in φ*
($\exists$1)	$\vdash_{\mathbf{L}\forall^-} \varphi(t) \to (\exists x)\varphi(x)$, *where t is substitutable for x in φ*
($\forall$2)	$\vdash_{\mathbf{L}\forall^-} \chi(\forall x)(\chi \to \varphi) \to (\forall x)\varphi$, *where x is not free in χ*
($\exists$2)	$\vdash_{\mathbf{L}\forall^-} (\forall x)(\varphi \to \chi) \to (\exists x)\varphi \to \chi$, *where x is not free in χ*
(Gen)	$\varphi \vdash_{\mathbf{L}\forall^-} (\forall x)\varphi$

Furthermore, if $\mathbf{L}$ is a fuzzy logic we define the logic $\mathbf{L}\forall$ as an extension of $\mathbf{L}\forall^-$ by axiom:

($\forall$3) $\vdash_{\mathbf{L}\forall} (\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, *where x is not free in χ*

Logics $\mathbf{L}\forall$ and $\mathbf{L}\forall^-$ are sometimes the same (Łukasiewicz predicate logic) and sometimes they are different (Gödel predicate logic).

Now we recall the concept of the Proof by Cases Property, we will need this property to prove the completeness theorem of $\mathbf{L}\forall$ w.r.t. linearly ordered matrices.

Definition 3.12 (Proof by Cases) *Fuzzy logic $\mathbf{L}\forall$ has the Proof by Cases Property (*PP*) if for each theory T and each sentences φ and ψ we get $T \vdash \chi$ whenever $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$.*

Unluckily, we are not able to prove that each predicate fuzzy logic has PP. However, we can give some simply checkable sufficient conditions. Before we do so we observe that we can easily prove the both deduction theorems (we assume the same definition of Implicational (Delta) Deduction theorem are in propositional case, we only assume that φ is sentence).

Theorem 3.13 (Deduction Theorems) *Let $\mathbf{L}$ be a logic with $\mathrm{DT}_{\to}$ ($\mathrm{DT}_{\triangle}$ respectively). Then both logics $\mathbf{L}\forall^-$ and $\mathbf{L}\forall$ has $\mathrm{DT}_{\to}$ ($\mathrm{DT}_{\triangle}$ resp.).*

Corollary 3.14 *Let $\mathbf{L}$ be logic, such that $\mathbf{L}$ has some presentation where* (MP) *is the only deduction rule and implicational fragment of $\mathbf{L}$ is an extension of* FBCK*. Then $\mathbf{L}\forall$ has* PP*.*

Corollary 3.15 *Let $\mathcal{L}$ be a propositional language, $\triangle \in \mathcal{L}$ and $\mathbf{L}$ a fuzzy logic in $\mathcal{L}$, such that $\mathbf{L}$ has some presentation where* (MP) *and* (NEC) *are the only deduction rules. Then $\mathbf{L}\forall$ has* PP*.*

3.2 Henkin and witnessed theories

In this subsection we prepare some technical means to proof the completeness.

Definition 3.16 (Henkin and φ-witnessed theories) *The theory is called* Henkin *theory if for each sentence $\varphi = (\forall x)\psi$ and $T \not\vdash \varphi$, there is a constant c such that $T \not\vdash \psi(c)$.*

*Furthermore, let $\varphi(x_1, \dots x_n, y)$ be a formula. Henkin theory is called φ-*witnessed *theory if for each formula $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$, where t_i are closed terms holds: if $T \vdash (\exists y)\psi(y)$, then there is a constant c such that $T \vdash \psi(c)$.*

Definition 3.17 (Henkin and φ-witnessed logics) *Let **L** be a weakly implicative logic and φ a formula. We say that the logic $\mathbf{L}\forall^-$ is Henkin (φ-witnessed) for each theory T and each sentence α, $T \not\vdash \alpha$ there is a Henkin (φ-witnessed) theory T' such that $T \subseteq T'$ and $T' \not\vdash \alpha$.*

*Let **L** be a fuzzy logic and φ a formula. We say that the logic $\mathbf{L}\forall$ is Henkin (φ-witnessed) for each theory T and each sentence α, $T \not\vdash \alpha$ there is a linear Henkin (φ-witnessed) theory T' such that $T \subseteq T'$ and $T' \not\vdash \alpha$.*

Definition 3.18 (proto-φ-witnessed logics) *Let **L** be a weakly implicative logic and φ a formula. We say that the logic $\mathbf{L}\forall^-$ (or the logic $\mathbf{L}\forall$) is proto-φ-witnessed if for each theory T and for each formula $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$, where t_i are closed terms holds: $T \cup \{\psi(c)\}$ is a conservative extension of $T \cup \{(\exists y)\psi(y)\}$.*

In fact, the logic is proto-φ-witnessed iff it supports introduction of Skolem constants. We use this property to show, that in that case we can introduce Skolem function of arbitrary arity.

Lemma 3.19 *Let **L** be a weakly implicative logic and φ a formula. If the logic $\mathbf{L}\forall^-$ is φ-witnessed then it is proto-φ-witnessed .*

*Furthermore, let **L** be a fuzzy logic and φ a formula. If the logic $\mathbf{L}\forall$ is φ-witnessed then it is proto-φ-witnessed .*

Proof: Let T be a theory and $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$, where t_i are closed terms, such that theory $T \cup \{\psi(c)\}$ is not a conservative extension of $T \cup \{(\exists y)\psi(y)\}$, i.e., there is a formula α such that $T \cup \{\psi(c)\} \vdash \alpha$ and $T \cup \{(\exists y)\psi(y)\} \not\vdash \alpha$. Let us take φ-witnessed theory T', such that $T \cup \{(\exists y)\psi(y)\} \subseteq T'$ and $T' \not\vdash \alpha$. Since T' is φ-witnessed and $T' \vdash (\exists y)\psi(y)$ there is a constant d such that $T' \vdash \psi(d)$. Since $T \cup \{\psi(c)\} \vdash \alpha$ we get $T \cup \{\psi(d)\} \vdash \alpha$ and so $T' \vdash \alpha$—a contradiction

The proof of the second part as the same. QED

Definition 3.20 (Directed set of formulas) *Let Ψ be a set of formulas. We say that Ψ is a directed set if for each $\psi, \varphi \in \Psi$ there is $\delta \in \Psi$ such that $\vdash \psi \to \delta$ and $\vdash \varphi \to \delta$ (we call δ the upper bound of φ and ψ)*

This is the crucial lemma of this paper:

Lemma 3.21 (φ-witnessed extension) *Let **L** be a finitary fuzzy logic with* PP *and φ a formula. If the logic $\mathbf{L}\forall$ is proto-φ-witnessed then it is φ-witnessed .*

Proof: We construct our extension by a transfinite induction. Let T be a theory and α a formula $T \not\vdash \alpha$. If Ψ is a set of formulas by $T \not\vdash \Psi$ we mean $T \not\vdash \psi$ for each $\psi \in \Psi$.

Before we start we extend our predicate language by new constants $\{c^s_\nu \mid \nu \leq ||\Gamma||\}$ for each sort s. Let $T_0 = T$ and $\Psi_0 = \{\alpha\}$. We enumerate all formulas with one free variable x by ordinal numbers as χ_μ and all formulas with one free variable x of the form $\varphi(x_1 : t_1, \dots x_n : t_n, x)$ by ordinal numbers as σ_μ.

We construct directed sets Ψ_μ and theories T_μ so $T_\mu \not\vdash \Psi_\mu$ and $T_\mu \subseteq T_\nu$ and $\Psi_\mu \subseteq \Psi_\nu$ for $\mu \leq \nu$. Thus we get $T_\mu \not\vdash \alpha$. Observe that theory T_0 and set Ψ_0 fulfill these conditions. The induction step:

Let us define the sets: $\hat{T}_\mu = \bigcup_{\nu<\mu} T_\nu$ and $\hat{\Psi}_\mu = \bigcup_{\nu<\mu} \Psi_\nu$. Notice that from the induction property we get that $\hat{T}_\mu \not\vdash \hat{\Psi}_\mu$ and $\hat{\Psi}_\mu$ is directed set. Now we construct theory T'_μ and set Ψ_μ. We distinguish two cases:

(H1) There is $\psi \in \hat{\Psi}_\mu$ such that $\hat{T}_\mu \vdash \psi \vee \chi_\mu(c)$. Let $T'_\mu = \hat{T}_\mu \cup \{\psi \rightarrow (\forall x)\chi_\mu(x)\}$ and $\Psi_\mu = \hat{\Psi}_\mu$

(H2) Otherwise, let $T'_\mu = \hat{T}_\mu$ and $\Psi_\mu = \hat{\Psi}_\mu \cup \{\psi \vee \chi_\mu(c) \mid \psi \in \hat{\Psi}_\mu\}$

We show that $T'_\mu \not\vdash \Psi_\mu$ and Ψ_μ are directed. Let c be the first unused constant of the proper sort.

(H1) Let $\varphi \in \hat{\Psi}_\mu$ and δ is the upper bound of φ and ψ. We know that $\hat{T}_\mu \vdash \psi \vee \chi_\mu(x)$ (just replace c by x everywhere in the proof of $\psi \vee \chi_\mu(c)$). Thus $\hat{T}_\mu \vdash \psi \vee (\forall x)\chi_\mu(x)$ (by the generalization and axiom ($\forall$3)). Thus $\hat{T}_\mu \cup \{(\forall x)\chi_\mu(x) \rightarrow \psi\} \vdash \psi$ and so we get $\hat{T}_\mu \cup \{(\forall x)\chi_\mu(x) \rightarrow \psi\} \vdash \delta$. Thus $\hat{T}_\mu \cup \{\psi \rightarrow (\forall x)\chi_\mu(x)\} \not\vdash \delta$ (otherwise $\hat{T}_\mu \vdash \delta$—a contradiction). Finally, if we have $\hat{T}_\mu \cup \{\psi \rightarrow (\forall x)\chi_\mu(x)\} \vdash \varphi$ then $\hat{T}_\mu \cup \{\psi \rightarrow (\forall x)\chi_\mu(x)\} \vdash \delta$—a contradiction. We have show that $T_\mu \not\vdash \Psi_\mu$ the other conditions are in this case obvious.

(H2) The proof of $T_\mu \not\vdash \Psi_\mu$ is trivial. We only have to show that Ψ_μ is directed. We should distinguish three cases, however we show only one (the other are analogous) $\varphi, \alpha \in \hat{\Psi}_\mu$ and $\psi = \alpha \vee \chi_\mu(c)$. Let δ be the upper bound of φ and α then obviously $\delta \vee \chi_\mu(c) \in \Psi_\mu$ is the upper bound φ and ψ.

Next, we construct theory T_μ. Let c be the first unused constant of the proper sort. Again, we distinguish two cases:

(W1) There is $\psi \in \Psi_\mu$ such that $T'_\mu \cup \{(\exists x)\sigma_\mu(x)\} \vdash \psi$. Let $T_\mu = T'_\mu$.

(W2) $T'_\mu \cup \{(\exists x)\sigma_\mu(x)\} \not\vdash \Psi_\mu$. Let $T_\mu = T'_\mu \cup \{\sigma_\mu(c)\}$.

We show that $T_\mu \not\vdash \Psi_\mu$:

(W1) Trivial.

(W2) Since the logic $\mathbf{L}\forall$ is proto-φ-witnessed $T'_\mu \cup \{\sigma(c)\}$ is a conservative extension of $T'_\mu \cup \{(\exists x)\sigma(x)\}$. Since $T'_\mu \cup \{(\exists x)\sigma(x)\} \not\vdash \Psi_\mu$ the proof is done.

Let us define theory $\hat{T} = T_{||\mathbf{\Gamma}||}$ and set $\Psi = \Psi_{||\mathbf{\Gamma}||}$. Now we construct a complete theory T' such that $\hat{T} \subseteq T'$ and $T' \not\vdash \Psi$. We do it again by a transfinite induction. Let us enumerate pair of formulas by ordinals. $T'_0 = \hat{T}$. We construct theories T'_μ such that $T'_\mu \not\vdash \Psi$ and $T'_\mu \subseteq T'_\nu$ for $\mu \leq \nu$. Let us define theory $\hat{T}'_\mu = \bigcup_{\nu<\mu} T'_\nu$. Notice that from the induction property we get that $\hat{T}'_\mu \not\vdash \Psi$. The induction step: we show that $\hat{T}'_\mu \cup \{\varphi_\mu \rightarrow \psi_\mu\} \not\vdash \Psi$ or $\hat{T}'_\mu \cup \{\psi_\mu \rightarrow \varphi_\mu\} \not\vdash \Psi$. By contradiction: let there be formulas $\beta, \gamma \in \Psi$ such that $\hat{T}'_\mu \cup \{\varphi_\mu \rightarrow \psi_\mu\} \vdash \beta$ and $\hat{T}'_\mu \cup \{\psi_\mu \rightarrow \varphi_\mu\} \vdash \gamma$. Let us take upper bound δ of β and γ and we get $\hat{T}'_\mu \cup \{\varphi_\mu \rightarrow \psi_\mu\} \vdash \delta$ and $\hat{T}'_\mu \cup \{\psi_\mu \rightarrow \varphi_\mu\} \vdash \delta$. Thus $\hat{T}'_\mu \vdash \delta$—a contradiction.

Finally, we define $T' = T'_{||\mathbf{\Gamma}||}$. Recall that $T' \not\vdash \Psi$. If we show that T' is φ-witnessed theory the proof is done (because T' is obviously complete and $T' \not\vdash \alpha$).

Is T' Henkin? Let $\varphi(x)$ be processed in the step μ. If $T' \not\vdash (\forall x)\varphi(x)$ then we used the case (H2) (otherwise $\hat{T}_\mu \vdash \psi \vee \varphi(c)$ which leads to $\hat{T}_\mu \vdash \psi \vee (\forall x)\chi_\mu(x)$ and so $\hat{T}_\mu \cup \{\psi \rightarrow (\forall x)\varphi(x)\} \vdash (\forall x)\varphi(x)$ and so $T_\mu \vdash (\forall x)\varphi(x)$—a contradiction). If $T' \vdash \varphi(c)$ then $T' \vdash \varphi(c) \vee \psi$ for all $\psi \in \hat{\Psi}_\mu$. Since we used case (H2) we know that $\varphi(c) \vee \psi \in \Psi_\mu$—a contradiction with $T' \not\vdash \Psi$.

Is T' φ-witnessed ? Let $\psi(x) = \varphi(x_1 : t_1, \ldots x_n : t_n, x)$ be processed in the step μ. If $T' \vdash (\exists x)\varphi(x)$ then we used the case (W2) (since $\hat{T}_\mu \cup \{(\exists x)\varphi(x)\} \vdash \psi$ for some $\psi \in \Psi$ we get $T' \vdash \psi$—a contradiction). Thus $T_\mu \vdash \varphi(c)$ and so $T' \vdash \varphi(c)$. QED

Corollary 3.22 (Henkin extension in fuzzy logics) *Let* $\mathbf{L}$ *be finitary fuzzy logic with* PP*. Then the logic* $\mathbf{L}\forall$ *is Henkin.*

Proof: Just read the proof of the latter lemma without parts (W1) and (W2) and notice that the assumption that $\mathbf{L}\forall$ is proto-φ-witnessed was used only in part (W2). QED

Corollary 3.23 (Henkin extension) *Let* $\mathbf{L}$ *be finitary weakly implicative logic. Then the logic* $\mathbf{L}\forall^-$ *is Henkin.*

Proof: Just read the proof of Theorem 3.4 in [6] QED

3.3 Completeness

We introduce the notion of a Lindenbaum matrix in the same fashion as in the propositional level. We define the *canonical* $\mathbf{Lin}_T$*-structure* $\mathbb{M}_T$ in the usual way—elements are the closed terms, and functions and predicates are defined accordingly. We have the following important lemma.

Lemma 3.24 *Let T be a Henkin theory and φ a formula with only one free variable x of the sort s. Then*

- $[(\forall x^s)\varphi]_{\mathbf{T}} = \inf_{c \in \mathbf{C}_s} [\varphi(c)]_{\mathbf{T}}$.
- $[(\exists x^s)\varphi]_{\mathbf{T}} = \sup_{c \in \mathbf{C}_s} [\varphi(c)]_{\mathbf{T}}$.

Proof: Recall that $[\varphi]_{\mathbf{T}} \leq [\psi]_{\mathbf{T}}$ iff $\mathbf{T} \vdash \varphi \to \psi$ (cf. Lemma 1.32). We prove only the first claim, the proof of the second one is analogous.

We show that $[(\forall x^s)\varphi]_{\mathbf{T}}$ is the greatest lower bound of all $[\varphi(c)]_{\mathbf{T}}$. The proof that $[(\forall x^s)\varphi]_{\mathbf{T}}$ is the lower bound is simple: $[(\forall x^s)\varphi]_{\mathbf{T}} \leq [\varphi(c)]_{\mathbf{T}}$ for all constants $c \in C_s$ (by axiom ($\forall$1)).

Now suppose there is $[\chi]_{\mathbf{T}}$ such that $[\chi]_{\mathbf{T}} \leq [\varphi(c)]_{\mathbf{T}}$ for all $c \in C_s$ and $[\chi]_{\mathbf{T}} \not\leq [(\forall x^s)\varphi]_{\mathbf{T}}$. Thus $T \not\vdash \chi \to (\forall x^s)\varphi$ and so $T \not\vdash (\chi \to \varphi)$ (by rule ($Gen\forall$)). Thus $T \not\vdash (\forall x^s)(\chi \to \varphi)$ (by axiom ($\forall$1)) By a Henkin property we get a constant $d \in C_s$ such that $T \not\vdash \chi \to \varphi(d)$. Finally $[\chi]_{\mathbf{T}} \not\leq [\varphi(d)]_{\mathbf{T}}$ - a contradiction. QED

Obviously, for T being Henkin the canonical $\mathbf{Lin}_T$-structure is safe and we have $[\varphi]_{\mathbf{T}} = ||\varphi||^{\mathbf{Lin}_T}_{\mathbb{M}_T}$ and thus $\mathbb{M}_T$ is a $\mathbf{Lin}_T$-model of T. Since each theory can be extended into Henkin theory (and in the case of fuzzy logic into the *linear* Henkin theory the proof of the following theorems is straightforward.

Theorem 3.25 *Let* $\mathbf{L}$ *be a finitary fuzzy logic with* PP, $\mathbf{\Gamma}$ *a predicate language, and* φ *a formula. Then* $T \vdash_{\mathbf{L}\forall} \varphi$ *iff* $T \models^l_{\mathbf{L}} \varphi$.

Corollary 3.26 *Let* $\mathbf{L}$ *be logic, such that* $\mathbf{L}$ *has some presentation* $\mathcal{AX}$, *where* (MP) *is the only deduction rule and implicational fragment of* $\mathbf{L}$ *is an extension of* FBCK. *Then the logic* $\mathbf{L}\forall$ *is sound and complete w.r.t. corresponding class of linear matrices.*

Corollary 3.27 *Let* $\mathbf{L}$ *be one of the following logics: MTL, IMTL, SMTL,* Π*MTL, NM, WNM, MTLH, IMTLH,* Π*MTLH, NMH, WNMH, Ł,* Π*, G, BL, ŁH,* Π*H, GH, BLH, CHL, SBL, PŁ. Then the logic* $\mathbf{L}\forall$ *is sound and complete w.r.t. corresponding class of linear matrices.*

Corollary 3.28 *Let* $\mathcal{L}$ *be a propositional language,* $\triangle \in \mathcal{L}$ *and* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$, *such that* $\mathbf{L}$ *has some presentation where* (MP) *and* (NEC) *are the only deduction rules. Then the logic* $\mathbf{L}\forall$ *is sound and complete w.r.t. corresponding class of linear matrices.*

Corollary 3.29 *Let* $\mathbf{L}$ *be one of the following logics:* $MTL_\triangle$, $IMTL_\triangle$, $SMTL_\triangle$, $\Pi MTL_\triangle$, $NM_\triangle$, $WNM_\triangle$, $Ł_\triangle$, $\Pi_\triangle$, $G_\triangle$, $BL_\triangle$, $SBL_\triangle$, $PŁ_\triangle$, $PŁ'_\triangle$, $SBL_\sim$, $\Pi_\sim$, $G_\sim$, $Ł\Pi$, $Ł\Pi\frac{1}{2}$. *Then the logic* $\mathbf{L}\forall$ *is sound and complete w.r.t. corresponding class of linear matrices.*

There are fuzzy logic, described in the literature not covered by this general approach (so far), namely the logics RΠ, $\mathrm{R}\Pi_\sim$ and RŁΠ (because their infinitary rule—however this problem can be easily solved) and the logic PŁ′ (because of the unavoidable rule $\neg(\varphi \odot \varphi) \vdash \neg\varphi$ - how to solve this is unknown to me).

At the end of this section we formulate completeness theorem of for weakly implicative logics. The proof is analogous to the one for fuzzy logics, we only use Lemma 3.23 instead of Lemma 3.22. This gives us some kind of first order calculus for a very wide class of logics. There is an interesting research task to examine existing first order calculi for particular logics (substructural, modal, intuitionistic, etc.) and compare them to our approach.

Theorem 3.30 *Let* $\mathbf{L}$ *be a weakly implicative logic,* $\mathbf{\Gamma}$ *a predicate language,* T *a theory, and* φ *a formula. Then* $T \vdash_{\mathbf{L}\forall^-} \varphi$ *iff* $T \models_{\mathbf{L}} \varphi$.

3.4 Skolem functions

Again, this section is only a short sketch of what "can be done". Observe that the majority of known fuzzy logic are φ-witnessed for each formula φ. Whereas, the logics with $\triangle$ are rather limited in this aspect, as shown by the following lemma:

Lemma 3.31 *Let $\mathcal{L}$ be a propositional language, $\triangle \in \mathcal{L}$ and $\mathbf{L}$ a logic in $\mathcal{L}$ with $\mathrm{DT}_\triangle$. Then the logics $\mathbf{L}\forall^-$ and $\mathbf{L}\forall$ are φ-witnessed iff $\vdash \triangle(\exists y)\varphi(y) \to (\exists y)\triangle\varphi(y)$*

Proof: First, let us suppose that the logic is proto-φ-witnessed then $\{\varphi(c)\}$ is a conservative extension of $\{(\exists y)\varphi(y)\}$. Since $\{\varphi(c)\} \vdash (\exists x)\triangle\varphi(x)$ we get $\{(\exists y)\varphi(y)\} \vdash (\exists x)\triangle\varphi(x)$. The deduction theorem gives us $\vdash \triangle(\exists y)\varphi(y) \to (\exists y)\triangle\varphi(y)$.

Other direction if similar. Let $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$, where t_i are closed terms. We want show that $T \cup \{\psi(c)\}$ is a conservative extension of $T \cup \{(\exists y)\psi(y)\}$. For each φ without c we want to get : $T \cup \{\psi(c)\} \vdash \varphi$ iff $T \cup \{(\exists y)\psi(y)\} \vdash \varphi$. By deduction theorem and some simple steps we get: $T \vdash (\exists y)\triangle\psi(y) \to \varphi$ iff $T \vdash \triangle(\exists y)\psi(y) \to \varphi$. Now just notice that if $\vdash \triangle(\exists y)\varphi(y) \to (\exists y)\triangle\varphi(y)$ then $\vdash \triangle(\exists y)\varphi(y) \equiv (\exists y)\triangle\varphi(y)$. QED

Corollary 3.32 *Let $\mathcal{L}$ be a propositional language, $\triangle \in \mathcal{L}$, $\mathbf{L}$ a logic in $\mathcal{L}$ with $\mathrm{DT}_\triangle$, and φ a formula. Then the logics $\mathbf{L}\forall^-$ and $\mathbf{L}\forall$ are $\triangle\varphi$-witnessed.*

Let us examine the behavior of the φ-witnessed logic w.r.t. Skolem functions introduction. We formulate the theorem for fuzzy logics only, its reformulation for weakly implicative logics needs an analogy of Lemma 3.21. This can be done in a rather straightforward way, but we skip this here. For the sake of simplicity we formulate the theorem for the unsorted language.

Theorem 3.33 *Let $\mathbf{L}$ be a proto-φ-witnessed finitary fuzzy logic with PP, T a theory, and $\varphi(x_1, \dots, x_n, y)$ a formula. If $T \vdash (\forall x_1)\dots(\forall x_n)\,(\exists y)\,\varphi(x_1, \dots x_n, y)$. Then the theory T' in the language of T extended by new function symbol f_φ resulting from the theory T by adding the axiom $\vdash (\forall x_1)\dots(\forall x_n)\,\varphi(x_1, \dots x_n, f_\varphi(x_1, \dots, x_n))$ is a conservative extension of T.*

Proof: Let $\bar{T}$ be a φ-witnessed supertheory of T. Then if $T \not\vdash \chi$ there is a canonical $\mathbf{Lin}_{\bar{T}}$-model $\mathbb{M}_{\bar{T}}$ of T. Since for each vector $t_1, \dots, t_n$ of closed terms if $\bar{T} \vdash (\exists y)\,\varphi(t_1, \dots t_n, y)$ there is a constant $c_{t_1,\dots,t_n}$ such that $T \vdash \varphi(t_1, \dots t_n, c_{t_1,\dots,t_n})$. Since $c_{t_1,\dots,t_n}$ is an element of $\mathbb{M}_{\bar{T}}$ (together with all other closed terms) we define $(f_\varphi)_{\mathbb{M}_{\bar{T}}}(t_1, \dots, t_n) = c_{t_1,\dots,t_n}$. Then obviously $\mathbb{M}_{\bar{T}}$ is a model of T' and since $\mathbb{M}_{\bar{T}} \not\models \chi$ we get that $T' \not\vdash \chi$. QED

To prove the Skolem function elimination we need to extend our language with some sort of equality.

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