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2003

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Technical report No. 890

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Algebraic structures related to Combinations of Belief Functions. The consensus operator and Jøsang's semigroup.[1]

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Abstract:

To overcome frequent criticism of Dempster's rule for combination of belief functions several alternatives was defined, among them the consensus operator. Algebraic analysis of the consensus operator is presented using the methodology introduced by Hájek-Valdés for Dempster's semigroup. The methodology and Dempster's semigroup is recalled. Jøsang's semigroup and related structures are introduced, analysed, and compared with those related to the Dempster's case.

Keywords:
Belief functions, Dempster-Shafer theory, Combination of belief functions, Dempster's rule, Dempster's semigroup, Consensus operator, Jøsang's semigroup, Expert systems.

[1]Partial support by the COST action 274 TARSKI is acknowledged.
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1 Introduction

Ever since the publication of Shafer's book *A Mathematical Theory of Evidence* [15] there has been continuous controversy around the so-called *Dempster's rule*. The purpose of Dempster's rule is to combine two beliefs into a single belief that reflects the two beliefs in a fair and equal way.

Dempster's rule has been criticised mainly because highly conflicting beliefs tend to produce counterintuitive results. This has been formulated in the form of examples by Zadeh [19], Cohen [1], and Daniel [4] among others. The problem with Dempster's rule is due to its normalisation which redistributes conflicting belief masses to non-conflicting ones, and thereby tends to eliminate any conflicting characteristics in the resulting belief mass distribution. An alternative called the non-normalised Dempster's rule proposed by Smets [16] avoids this particular problem by allocating all conflicting belief masses to the empty set. The idea is that conflicting belief masses should be allocated to this missing (empty) event.

Unfortunately nor the non-normalised version does not solve all the disadvantages of Dempster's rule. Thus several other alternatives were suggested later. Among them the *consensus operator* [13], [14], which is developed with an intension to combine better highly conflicting beliefs. The consensus operator forms part of subjective logic described by Jøsang in [13].

An algebraic structure of binary belief functions with Demspter's rule $\oplus$, called *Dempster's semigroup*, was in detail studied in a series of publications, e.g. [2], [3], [11], [12], [18]. The appearing of the consensus operator © is the motivation for a study of algebraic structures of belief functions with © to obtain a better theoretical comparison of both approaches.

The next section briefly recalls the basic definitions. An algebraic analysis of Dempster's semigroup which is used as a methodology for the presented investigation is overviewed in the third section.

Section 4 brings basic ideas and facts about the opinion space to prepare us for introduction of the consensus operator in the consecutive section.

In Section 6, a new algebraic structure — the algebraic structure of binary belief functions with the consensus operator © — is defined. The new structure called *Jøsang's semigroup* is analysed there. The results are discussed and compared with those of Dempster's semigroup in Section 7.

In the end, some ideas for future research are outlined as well.

2 Preliminaries

Let us recall some basic algebraic notions and some basic notions from the Dempster-Shafer theory before we begin a description of its algebra.

A *commutative semigroup* (called also an *Abelian semigroup*) is a structure $\mathbf{X} = (X, \oplus)$ formed by the set X and a binary operation $\oplus$ on X which is commutative and associative ($x \oplus y = y \oplus x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ holds for all $x, y, z \in X$). A *commutative group* is a structure $\mathbf{X} = (X, \oplus, -, o)$ such that $(Y, \oplus)$ is a commutative semigroup, o is a neutral element ($x \oplus o = x$) and $-$ is a unary operation of the inverse ($x \oplus -x = o$). An *ordered Abelian (semi)group* consists of a commutative (semi)group $\mathbf{X}$ as above and a linear ordering $\leq$ of its elements satisfying monotonicity ($x \leq y$ implies $x \oplus z \leq y \oplus z$ for all $x, y, z \in X$). A subset of X which is a (semi)group itself is called a *sub(semi)group*. A subsemigroup $(\{x | x \geq o, x \in X\}, \oplus, o)$ is called *a positive cone* of ordered Abelian group (OAG) X, similarly *a negative cone* of OAG Y for $x \leq o$.

For uncertainty processing, we extend OAG with *extremal elements* $\top$ and $\bot$ representing *True* and *False*, $\top \oplus x = \top$, $\bot \oplus x = \bot$, $\top \oplus \bot$ not defined. [3]

A *homomorphism* $p : (X, \oplus_1) \longrightarrow (Y, \oplus_2)$ is a mapping which preserves structure, i.e. $p(x \oplus_1 y) = p(x) \oplus_2 p(y)$ for each $x, y \in X$. The special cases are *automorphisms*, which are bijective morphisms from a structure onto itself. Morphisms which also preserve ordering of elements are called *ordered morphisms*, see [9].

[3] Some examples are OAG$^+$ $\mathbf{PP} = ([0,1], \oplus_{PP}, 1-x, \frac{1}{2}, \leq)$ and $\mathbf{MC} = ([-1,1], \oplus_{MC}, -, 0, \leq)$ corresponding to combining structures of the classical expert systems PROSPECTOR and EMYCIN, see [11], where $x \oplus_{PP} y = \frac{xy}{xy+(1-x)(1-y)}$ and $x \oplus_{MC} y = x + y - xy$ for $x, y \geq 0$, $x + y + xy$ for $x, y \leq 0$ and $\frac{x+y}{1-min(|x|,|y|)}$ for $xy \leq 0$.

Ordered structures and ordered morphisms are very important for a comparative approach to uncertainty management and decision making.

Let us consider a two-element frame of discernment $\Theta = \{0,1\}$. A basic belief assignment is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$. A belief function is a mapping $bel : \mathcal{P}(\Theta) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. In our special case $bel(1) = m(1)$, $bel(0) = m(0)$, $bel(\{0,1\}) = m(1) + m(0) + m(\{0,1\}) = 1$. Each basic belief assignment determines a d-pair $(m(1), m(0))$ and conversely, each d-pair determines a basic belief assignment.

The Dempster's conjunctive rule of combination is given as $(bel_1 \circledcirc bel_2)(A) = \sum_{X\cap Y=A} \frac{1}{K} m_1(X)m_2(Y)$, where $K = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)$, while *the disjunctive rule of combination* is given by the formula $(bel_1 \circledcirc bel_2)(A) = \sum_{X\cup Y=A} m_1(X)m_2(Y)$, see [15]. Specially for $(m_1(1), m_1(0)) = (a,b)$, $(m_2(1), m_2(0)) = (c,d)$ we have $(a,b)\oplus(c,d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ and $(a,b)\circledcirc(c,d) = (ac, bd)$.

If all the focal elements are singletons (i.e. one-element subsets of Ω) then we speak about *Bayesian belief functions*. *A dogmatic belief function* is defined by Smets as a belief function for which $m(\Omega = 0)$. Let us note, that trivially, every Bayesian belief function is dogmatic.

Bayesian transformation is a mapping $t : Bel_\Omega \longrightarrow Prob_\Omega$, such that $bel(x) \leq t(bel)(x) \leq 1 - bel(\overline{x})$. Thus Bayesian transformation assigns a Bayesian belief function (i.e. probability fuction) to every general one. The fundamental example of Bayesian transformation is pignistic transformation justified by Smets: $BetP(A) = \sum_{A\in X\subseteq\Omega} \frac{1}{|X|}\frac{m(X)}{1-m(\emptyset)}$, i.e. $BetP(0) = m(0) + \frac{1}{2}m(0,1)$ and $BetP(1) = m(1) + \frac{1}{2}m(0,1)$.

3 On the Dempster's semigroup

Now we introduce some principal notions according to [11].

Definition 1 *A Dempster's pair (or d-pair) is a pair of reals such that $a, b \geq 0$ and $a + b \leq 1$. A d-pair (a,b) is Bayesian if $a + b = 1$, (a,b) is simple if $a = 0$ or $b = 0$, in particular, extremal d-pairs are pairs (1,0) and (0,1). (Definitions of Bayesian and simple d-pairs correspond evidently to the usual definitions of Bayesian and simple belief assignments [11], [15]).*

Definition 2 (Standard/conjunctive) Dempster's semigroup[4] $\mathbf{D_0} = (D_0, \oplus)$ *is the set of all non extremal Dempster's pairs, endowed with the operation $\oplus$ and two distinguished elements $0 = (0,0)$ and $0' = (\frac{1}{2}, \frac{1}{2})$, where the operation $\oplus$ is defined by*

$$(a,b)\oplus(c,d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)}).$$

Definition 3 *For $(a,b) \in \mathbf{D}_0$ we define*
$-(a,b) = (b,a)$,
$h(a,b) = (a,b)\oplus 0' = (\frac{1-b}{2-a-b},\ \frac{1-a}{2-a-b})$,
$h_1(a,b) = \frac{1-b}{2-a-b}$,
$f(a,b) = (a,b)\oplus(b,a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2},\ \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$.

For $(a,b), (c,d) \in \mathbf{D_0}$ we further define
$(a,b) \leq_\oplus (c,d)$ *iff* $h_1(a,b) < h_1(c,d)$ *or if* $h_1(a,b) = h_1(c,d)$ *and* $a \leq c$.

Let G denote the set of all Bayesian non-extremal d-pairs. Let us denote the set of all simple d-pairs such that $b = 0$ ($a = 0$) as S_1 (S_2). Furthermore, put $S = \{(a,a) : 0 \leq a \leq 0.5\}$.
(Note: $h(a,b)$ is an abbreviation for $h((a,b))$, etc.)

Note that homomophism h is a homomorphic Bayesian transformation in fact. $h(x)$ expresses *certainty / uncertainy* of belief x, while $f^{-1}(f(x)) \cap S$ expresses *vagueness / preciseness* of x.

[4]A generalization of a notion of the Dempster's semigroup is described in [12], see also [11]. The resulting algebraic structure is called *a dempsteroid*. It has a similar relation to the Dempster's semigroup as it has OAG to **PP** or **MC**. The special case — the standard dempsteroid $\mathbf{D_0} = (D_0, \oplus, -, 0, 0', \leq)$ is defined by the Dempster's semigroup.

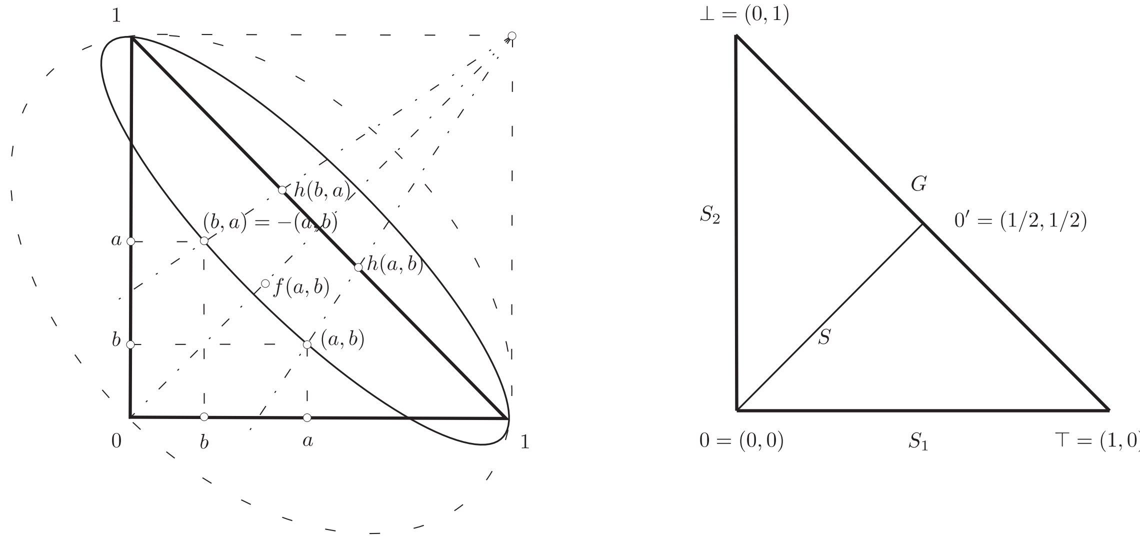

Figure 3.1: ***Dempster's semigroup.*** *Homomorphism h is in this representation a projection to group G along the straight lines running through the point $(1,1)$. All the Dempster's pairs laying on the same ellipse are mapped by homomorphism f to the same d-pair in semigroup S.*

Theorem 1

(i) The Dempster's semigroup with the relation $\leq_\oplus$ is an ordered commutative semigroup with the neutral element 0; $0'$ is the only nonzero idempotent of it.

(ii) The set G with the ordering $\leq_\oplus$ is an ordered Abelian group $(G, \oplus, -, 0', \leq_\oplus)$ which is isomorphic to the PROSPECTOR group **PP** *(cf. [11]) and consequently isomorphic to the additive group of reals with usual ordering.*

(iii) The sets S, S_1 and S_2 with the operation $\oplus$ and the ordering $\leq$ form ordered commutative semigroups with neutral element 0, and are all isomorphic to the semigroup of nonnegative elements (positive cone) of the MYCIN group **MC**.

(iv) The mapping h is an ordered homomorphism of the ordered Dempster's semigroup onto its subgroup G (i.e. onto **PP***).*

(v) The mapping f is a homomorphism of the Dempster's semigroup onto its subsemigroup S (but it is not an ordered homomorphism).

For proofs see [11], [12], [18]. Using the theorem, see (iv) and (v), we can express

$$(a \oplus b) = h^{-1}(h(a) \oplus h(b)) \cap f^{-1}(f(a) \oplus f(b)). \tag{3.1}$$

4 The Opinion Space

Let us briefly recall some notions from [13], [14] before the definition of the consensus operator. Let us consider a binary frame of discernment Θ again. Let $\Theta = \{x, \overline{x}\}$, where x (resp. $\overline{x}$) could be a simple element from an application domain or it could be a subset of an original multidimensional frame of discernment Θ_0 and $\overline{x} = \Theta_0 - x$. In the later case, let belief function on Θ be constructed by the method of focusing, see [13], [14]. Let us assume a basic belief assignment m such that $m(x) = b, m(\overline{x}) = d, m(\Theta) = u$. Hence $bel(x) = b, bel(\overline{x}) = d$, and we can consider b as a belief about the truth of x, d as a disbelief about x (a belief about the complement of x), and $u = 1 - b - d$ as an uncertainty[5] about x. Let us further recall a 3-dimensional metric[6] called *opinion*.

Definition 4 *Let Θ be a binary frame of discernment containing x and $\overline{x}$ as its elements, let m be a basic belief assignment which defines observer's belief b about x, disbelief d about x (a belief of the complement of x), and uncertainty u about x. Let a represent the relative atomicity of x in Θ. Then the observer's opinion about x is the tuple:*

$$\omega = (b, d, u, a).$$

Thus an opinion ω_x represents an observer's belief, disbelief and uncertainty about the truth of x and *a relative atomicity* a_x of x in the original frame of discernment Θ_0 in the case of focusing. The opinion contains a redundant parametr $u = 1 - b - d$ which allows a simple definition of the consensus operator, see the next section. Because we consider the only x, we can omit indexing of b, d, u, a by x, which is used in the case, where focusing given by different subsets of Θ is considered.

The opinion space can be graphically represented by a triangle as shown in Fig. 4.1.

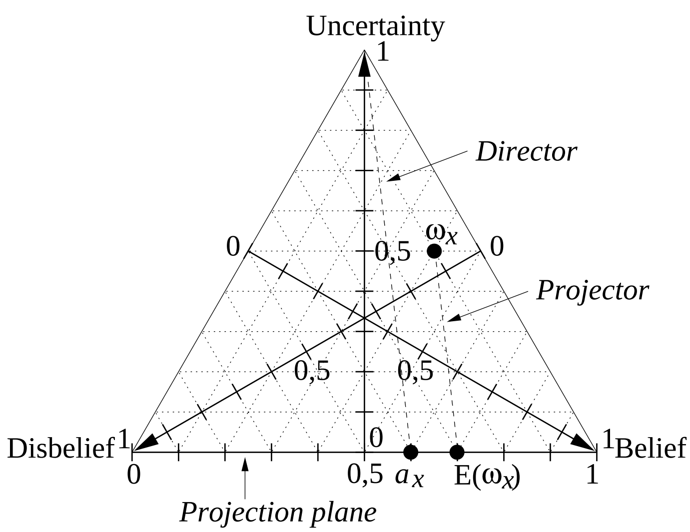

Figure 4.1: ***Opinion triangle.*** *ω_x is an example of an opinion about $x \in \Theta$.*

As an example the position of the opinion $\omega_x = (0.4, 0.1, 0.5, 0.6)$ is indicated as a point in the triangle. The horizontal base line between the Belief and Disbelief corners is called *the probability axis*. As shown in the figure, the probability expectation value $E(x) = 0.7$ and the relative atomicity $a(x) = 0.6$ can be graphically represented as points on the probability axis. The line joining the top corner of the triangle and the relative atomicity point is called *the director*. *The projector* is parallel to the director and passes through the opinion point ω_x. Opinions situate on the probability axis

[5] We use Jøsangs terminology here. Note, that $bel(\overline{x}) = dou(x)$ is called (degree of) *doubt of x* by Shafer in [15]. $u = 1 - b - d$ corresponds rather to vagueness than to uncertainty in Hájek-Valdés.

[6] From the mathematical point of view, it is not any metric. It is just an extended representation of binary belief function (belief).

are called *dogmatic opinions*, representing traditional probability without uncertainty. The distance between an opinion point and the probability axis can be interpreted as a degree of uncertainty. Opinions situate in the left of right corner, i.e. with either $b = 1$ or $d = 1$ are called *the absolute opinions*, corresponding to TRUE or FALSE values in two-valued logic.

Because the relative atomicity does not play any role in consensus operator (it is used for computing of the probability expectation and by another operator of Jøsang's subjective logic), we can omit it as redundant from our point of interest[7].

Several opinions can be indexed by $A, B, C, \ldots$ for observers $A, B, C, \ldots$ as it is used by Jøsang or simply by $1, 2, 3, \ldots$ as we have used in the case of the Dempster's semigroup[8].

4.1 Analogy of opinions and d-pairs

Trivially, any opinion (b, d, u) gives the unique d-pair (b, d), and analogically any d-pair (v, w) gives the opinion $(v, w, 1 - v - w)$ which is unique if relative atomicity is omitted or fixed. We can observe, that the absolute opinion $(1, 0, 0)$ in the right corner of the opinion triangle (Belief) corresponds to $\top = (1, 0)$ in the notation of the Dempster's semigroup, while Disbelief $(0, 1, 0)$ in the left corner corresponds to $\perp = (0, 1)$, and Uncertainty $(0, 0, 1)$ in the top corner corresponds to $0 = (0, 0)$ which is interpreted as *total ignorance* in the Dempster's semigroup. Analogically the probability axis corresponds to the set G of all the Bayesian d-pairs, and right (or left) arm of the opinion space triangle corresponds to S_1, i.e. to the set of all simple d-pairs $(b, 0)$ (or to S_2 respectivelly). And vertical median of the opinion triangle connecting $(0, 0, 1)$ and $(\frac{1}{2}, \frac{1}{2}, 0)$ corresponds to the set S. Using the analogies we will use denotations G, S, S_1, and S_2 also in the context of the opinion space.

5 The Consensus Operator

The consensus of two opinions is an opinion that reflects both argument opinions in a fair and equal way, i.e. when two observers have beliefs about the truth of x resulting from distinct pieces of evidence about x, the consensus operator produces a consensus belief that combines the two separate beliefs into one.

Definition 5 *Let $\omega_A = (b_A, d_A, u_A)$ and $\omega_B = (b_B, d_B, u_B)$ be opinions*[9] *respectively held by agents A and B about the same element x of $\Theta = \{x, \overline{x}\}$, and let $\kappa = u_A + u_B - u_A u_B$. When $u_A,\ u_B \ \rightarrow 0$, the relative dogmatism between ω_A and ω_B is defined by γ so that $\gamma = u_A/u_B$. Let $\omega_{AB} = (b_{AB}, d_{AB}, u_{AB})$ be the opinion such that:*

	for $\kappa \neq 0$:	*for* $\kappa = 0$:
1.	$b_{AB} = (b_A u_B + b_B u_A)/\kappa$	$b_{AB} = \frac{\gamma b_A + b_B}{\gamma + 1}$
2.	$d_{AB} = (d_A u_B + d_B u_A)/\kappa$	$d_{AB} = \frac{\gamma d_A + d_B}{\gamma + 1}$
3.	$u_{AB} = (u_A u_B)/\kappa$	$u_{AB} = 0$.

Then ω_{AB} is called the consensus opinion between ω_A and ω_B, representing an imaginary agent [A,B]'s opinion about x, as if that agent represented both A and B. By using the symbol © to designate this operator[10] *we define $\omega_{AB} = \omega_A © \omega_B$.*

Note that $\kappa = u_A + u_B - u_A u_B = 0$ *iff* $u_A u_B = 0$, i. e. *iff* both the opinions $\omega_a,\ \omega_B$ are Bayesian (laying on the probability axis), i. e. being dogmatic in the case of 2-element frame of discernment.

[7] Especially, in the case of two simple elements x and $\overline{x}$ of a domain ($\Theta = \Theta_0$, i.e. $|\Theta_0| = 2$), or in the case where $|x| = |\overline{x}| \in \Theta_0$ for $|\Theta_0| > 2$, there is the fix relative atomicity $a_x = \frac{1}{2}$, and all the projectors are perpendicular to the probability axis, and the probability expectaction is equal to the pignistic probability defined in the Transferable Belief Model [16], [17].

[8] Note that these indices are upper indices in [13], [14], while they are lower indices in the context of the Dempster's semigroup and here.

[9] Let us note that (from our point of view) redundant relative atomicity and indexing by x is omitted in this definition, originally from [14], further upper indices $A,\ B$ are substituted by the lower ones.

[10] $\oplus$ is used in [13], [14]. Let us use © here to distinguish the consensus operator © from the Dempster's rule $\oplus$.

6 Jøsang's semigroup

Let us turn our attention to an algebra of belief functions on a binary frame of discernment (i.e. to an algebra of d-pairs — opinions) with the binary consensus operator $\circledcirc$. As it is already stated in [13] the set of all the opinions is closed with respect to the consensu operator $\circledcirc$. Further, the consensus operator $\circledcirc$ is a commutative and associative operation on the set of all non-dogmatic binary belief functions (opinions), hence we can speak about an Abelian semigroup again. Associativity of consensus of several dogmatic beliefs is more complicated, thus we will postpone its discussion and a formal definition of Jøsang's semigroup for a later time.

Proof of closeness of D_0 with respect to $\circledcirc$: We have $0 \leq b_i, d_i, u_i \leq 1$ such that $b_i + d_i + u_i = 1$. Let $(b_1, d_1, u_1) \circledcirc (b_2, d_2, u_2) = (b_{12}, d_{12}, u_{12}) = (\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{u_{1u}2}{u_1+u_2-u_1u_2})$. We have $b_1u_2 + b_2u_2 \geq 0$, $0 \leq u_1u_2 \leq u_i$, and $u_1 + u_2 - u_1u_2 \geq 0$, thus $b_{12} \geq 0$, and similarly $d_{12}, u_{12} \geq 0$.
$b_{12} + d_{12} + u_{12} = \frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2} + \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2} + \frac{u_1u_2}{u_1+u_2-u_1u_2} =$
$\frac{b_1(1-b_2-d_2)+b_2(1-b_1-d_1)+d_1(1-b_2-d_2)+d_2(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}{(1-b_1-d_1)+(1-b_2-d_2)-(1-b_1-d_1)(1-b_2-d_2)} =$
$\frac{(b_1+d_1+1-b_1-d_1)(1-b_2-d_2)+(b_2+d_2)(1-b_1-d_1)}{(1-b_1-d_1)+(1-b_2-d_2)-(1-b_1-d_1)(1-b_2-d_2)} =$
$\frac{(1-b_2-d_2)+(b_2+d_2+1-b_2-d_2)(1-b_1-d_1)-(1-b_1-d_1)(1-b_2-d_2)}{(1-b_1-d_1)+(1-b_2-d_2)-(1-b_1-d_1)(1-b_2-d_2)} =$
$\frac{(1-b_2-d_2)+(1-b_1-d_1)-(1-b_1-d_1)(1-b_2-d_2)}{(1-b_1-d_1)+(1-b_2-d_2)-(1-b_1-d_1)(1-b_2-d_2)} = 1$. Hence $0 \leq b_{12}, d_{12}, u_{12} \leq 1$ and $b_{12} + d_{12} + u_{12} = 1$, thus $(b_{12}, d_{12}, u_{12}) \in D_0$. $(b_{12}, d_{12}, u_{12}) \in G$ *iff* $u_1u_2 = 0$.
If $u_1 = u_2 = 0$ then we obtain $(b_1, d_1, u_1) \circledcirc (b_2, d_2, u_2) = (b_1, 1 - b_1, 0) \circledcirc (b_2, 1 - b_2, 0) = (\frac{b_1+b+2}{2}, 1 - \frac{b_1+b+2}{2}, 0)$, thus again $(b_{12}, d_{12}, u_{12}) \in G$.
Hence we have proved that all the sets $D_0, G, D_0 - G$ are closed with respect to the consensus operator $\circledcirc$.

Proof of commutativity and associativity of $\circledcirc$: Let it be $\omega_1 = (b_1, d_1, u_1)$, $\omega_2 = (b_2, d_2, u_2)$, $\omega_3 = (b_3, d_3, u_3)$, $\omega_{ij} = \omega_i \circledcirc \omega_j$ for $i, j = 1, 2, 3$.
Let it hold $\kappa \neq 0$. There is $b_{12} = \frac{b_1u_2+b_2u_1}{\kappa_{12}} = \frac{b_1u_2+b_2u_1}{u_1+u_2-u_{12}} = \frac{b_2u_1+b_1u_2}{u_2+u_1-u_{21}} = \frac{b_2u_1+b_1u_2}{\kappa_{21}} = b_{21}$, analogically $d_{12} = d_{21}$, and $u_{12} = \frac{u_1u_2}{\kappa_{12}} = \frac{u_2u_1}{\kappa_{21}} = u_{21}$. Hence $\omega_1 \circledcirc \omega_2 = \omega_{12} = \omega_{21} = \omega_2 \circledcirc \omega_1$, i.e. $\circledcirc$ is commutative for $\kappa \neq 0$.
Let it hold $\kappa = 0$ now. $b_{12} = \frac{\frac{u_1}{u_2}b_1+b_2}{\frac{u_1}{u_2}+1} = \frac{\frac{u_1}{u_2}b_1+b_2}{\frac{u_1}{u_2}+1}\frac{\frac{u_2}{u_1}}{\frac{u_2}{u_1}} = \frac{b_1+\frac{u_2}{u_1}b_2}{1+\frac{u_2}{u_1}} = \frac{\frac{u_2}{u_1}b_2+b_1}{\frac{u_2}{u_1}+1} = b_{21}$, analogically $d_{12} = d_{21}$, and trivially $u_{12} = 0 = u_{21}$. Hence the consensus operator $\circledcirc$ is commutative also for combination of two Bayesian opinions.
Let us suppose that all three opinions ω_1, ω_2, and ω_3 are non-Bayesian, i.e. $u_1 \neq 0$, $u_2 \neq 0, u_3 \neq 0$, thus $\kappa_{ij} \neq 0$ for $i, j \in \{1, 2, 3\}$. $b_{(12)3} = \frac{b_{12}u_3+b_3u_{12}}{\kappa_{(12)3}} = \frac{\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}u_3 + b_3\frac{u_1u_2}{u_1+u_2-u_1u_2}}{\frac{u_1u_2}{u_1+u_2-u_1u_2} + u_3 - \frac{u_1u_2}{u_1+u_2-u_1u_2}u_3} =$
$\frac{\frac{b_1u_2u_3+b_2u_1u_3}{u_1+u_2-u_1u_2} + \frac{b_3u_1u_2}{u_1+u_2-u_1u_2}}{\frac{u_1u_2}{u_1+u_2-u_1u_2} + \frac{u_1u_3+u_2u_3-u_1u_2u_3}{u_1+u_2-u_1u_2} - \frac{u_1u_2u_3}{u_1+u_2-u_1u_2}} = \frac{b_1u_2u_3+b_2u_1u_3 + b_3u_1u_2}{u_1u_2 + u_1u_3+u_2u_3-u_1u_2u_3 - u_1u_2u_3} =$
$\frac{b_1u_2u_3 + b_2u_1u_3+b_3u_1u_2}{u_1u_2+u_1u_3-u_1u_2u_3 + u_2u_3 - u_1u_2u_3} = \frac{\frac{b_1u_2u_3}{u_2+u_3-u_2u_3} + \frac{b_2u_1u_3+b_3u_1u_2}{u_2+u_3-u_2u_3}}{\frac{u_1u_2+u_1u_3-u_1u_2u_3}{u_2+u_3-u_2u_3} + \frac{u_2u_3}{u_2+u_3-u_2u_3} - \frac{u_1u_2u_3}{u_2+u_3-u_2u_3}} =$
$\frac{b_1\frac{u_2u_3}{u_2+u_3-u_2u_3} + \frac{b_2u_3+b_3u_2}{u_2+u_3-u_2u_3}u_1}{u_1 + \frac{u_2u_3}{u_2+u_3-u_2u_3} - u_1\frac{u_2u_3}{u_2+u_3-u_2u_3}} = \frac{b_1u_{23}+b_{23}u_1}{u_1+u_{23}-u_1u_{23}} = \frac{b_1u_{23}+b_{23}u_1}{\kappa_{1(23)}} = b_{1(23)}$, analogically $d_{(12)3} = d_{1(23)}$, $u_{(12)3} = u_{1(23)}$, and moreover $u_{(12)3} = 1 - b_{(12)3} - d_{(12)3} = 1 - b_{1(23)} - d_{1(23)} = u_{1(23)}$. Hence the consensus operator $\circledcirc$ is associative for non-Bayesian opinions. As it was already mentioned above, we post pone the question of associativity of Bayesian opinions for later time.

Because we have no more information than beliefs, i.e. opinion and we do not expect any additional one, we have to concider the same approximation of u to 0 for all dogmatic opinions. Hence $\gamma = 1$ and we can express the consensus operator as follows:
$(b_A, d_A, u_A) \circledcirc (b_B, d_B, u_B) = (\frac{b_Au_B+b_Bu_A}{u_A+u_B-u_Au_B}, \frac{d_Au_B+d_Bu_A}{u_A+u_B-u_Au_B}, \frac{u_Au_B}{u_A+u_B-u_Au_B})$ for $u_Au_B \neq 0$
$(b_A, d_A, 0) \circledcirc (b_B, d_B, 0) = (\frac{b_A+b_B}{2}, \frac{d_A+d_B}{2}, 0)$ for two dogmatic opinions, a consensus of several dogmatic opinions we will discus later.

We can simply derive expressions for the consensus operator in special cases: for simple d-pairs (opinions) with the same and different focal elements, and for cases where $b = d$.
$(b_1, 0, 1 - b_1) \circledcirc (b_2, 0, 1 - b_2) = (\frac{b_1(1-b_2)+b_2(1-b_1)}{(1-b_1)+(1-b_2)-(1-b_1)(1-b_2)}, 0, \frac{(1-b_1)(1-b_2)}{2-b_1-b_2-(1-b_1-b_2+b_1b_2))}) =$

$= (\frac{b_1+b_2-2b_1b_2}{1-b_1b_2}, 0, \frac{1-b_1-b_2+b_1b_2}{1-b_1b_2})$,
$(0, d_1, 1-d_1) \circledcirc (0, d_2, 1-d_2) = (0, \frac{d_1+d_2-2d_1d_2}{1-d_1d_2}, \frac{1-d_1-d_2+d_1d_2}{1-d_1d_2})$,
$(b, 0, 1-b) \circledcirc (0, d, 1-d) = (\frac{b(1-d)}{(1-b)+(1-d)-(1-b)(1-d)}, \frac{d(1-b)}{(2-b-d-(1-b-d+bd)}, \frac{(1-b)(1-d)}{1-bd}) = (\frac{b-bd}{1-bd}, \frac{d-bd}{1-bd}, \frac{1-b-d+bd}{1-bd})$,
$(b_1, b_1, 1-2b_1) \circledcirc (b_2, b_2, 1-2b_2) =$
$(\frac{b_1(1-2b_2)+b_2(1-2b_1)}{(1-2b_1)+(1-2b_2)-(1-2b_1)(1-2b_2)}, \frac{b_1+b_2-4b_1b_2}{2-2b_1-2b_2-(1-2b_1-2b_2+4b_1b_2)}, \frac{(1-2b_1)(1-2b_2)}{1-4b_1b_2}) =$
$(\frac{b_1+b_2-4b_1b_2}{1-4b_1b_2}, \frac{b_1+b_2-4b_1b_2}{1-4b_1b_2}, \frac{(1-2b_1)(1-2b_2)}{1-4b_1b_2})$.

Lemma 1 *(i) Both the* $0 = (0, 0, 1)$ *and* $0' = (\frac{1}{2}, \frac{1}{2}, 0)$ *are idempotents of the consensus operator.*
(ii) All the Bayesian d-pairs (dogmatic opinions)[11] *are idempotents with respect to the consensus operator.*
(iii) All the Bayesian d-pairs (dogmatic opinions)[12] *are absorbing elements with respect to the consensus with non-bayesian ones.*
(iv) $0 = (0, 0, 1)$ *is the only non-Bayesian idempotent.*
(v) $0 = (0, 0, 1)$ *is the neutral element for non-Bayesian d-pairs (opinions).*

Proof: (i) $(0, 0, 1) \circledcirc (0, 0, 1) = (\frac{0\cdot 1+0\cdot 1}{1+1-1}, \frac{0+0}{1}, \frac{1\cdot 1}{1}) = (0, 0, 1)$.
$(\frac{1}{2}, \frac{1}{2}, 0) \circledcirc (\frac{1}{2}, \frac{1}{2}, 0) = (\frac{\frac{1}{2}+\frac{1}{2}}{2}, \frac{\frac{1}{2}+\frac{1}{2}}{2}, 0) = (\frac{1}{2}, \frac{1}{2}, 0)$.
(ii) $(b, 1-b, 0) \circledcirc (b, 1-b, 0) = (\frac{b+b}{2}, \frac{(1-b)+(1-b)}{2}, 0) = (b, 1-b, 0)$,
spec. $(1, 0, 0) \circledcirc (1, 0, 0) = (\frac{1+1}{2}, \frac{0+0}{2}, 0) = (1, 0, 0)$, $(0, 1, 0)$ analogically.
(iii) $(b, 1-bd, 0) \circledcirc (b_B, d_B, 0) = (\frac{0+b_Bu_A}{u_A+0-0}, \frac{d_Bu_A}{u_A}, \frac{0}{u_A}) = (b_B, d_B, 0)$,
spec. $(b, 1-bd, 0) \circledcirc (1, 0, 0) = (\frac{0+u_A}{u_A+0-0}, \frac{0u_A}{u_A}, \frac{0}{u_A}) = (1, 1, 0)$, $(0, 1, 0)$ analogically.
(iv) $(0, 0, 1) \circledcirc (0, 0, 1) = (\frac{0\cdot 1+0\cdot 1}{1+1-1}, \frac{0}{1}, \frac{1}{1}) = (0, 0, 1)$, i.e. it is an idempotent.
$(b, d, u) \circledcirc (b, d, u) = (\frac{bu+bu}{u+u-uu}, \frac{du+du}{2u-u^2}, \frac{u^2}{2u-u^2}) = (b, d, u)$, let (b, d, u) be idempotent, i.e. it holds $b = \frac{2bu}{2u-u^2}$, $d = \frac{2du}{2u-u^2}$ it holds if either $b = 0$ & $d = 0$ or $2u - u^2 = 2u$, i.e. $u = 0$, thus (b, d, u) is Bayesian. Hence $(0, 0, 1)$ is the only non-Bayesian idempotent.
(v) $(b, d, u) \circledcirc (0, 0, 1) = (\frac{b+0}{1+u-u}, \frac{d+0}{1+u-u}, \frac{u}{1}) = (b, d, u)$. □

Lemma 2 *(i) All the subsets* G, S, S_1, *and* S_2 *of the opinion space are closed with respect to the consensus operator.*
(ii) Consensus of two opinions is Bayesian iff at least one of the opinions consensed is Bayesian.
(iii) All the subsets $S_{(k)} = \{(b, kb, 1-(1+k)b) | (b, kb, 1-(1+k)b) \text{ is opinion } \}$ *of the opinion space are closed with respect to the consensus operator.*

Proof: (i) The subset G is closed from the definition because $u_{12} = \frac{u_1u_2}{\kappa} = \frac{0\cdot 0}{\kappa} = 0$ for Bayesian opinions. The other statements follow the above introduced formulas for the special cases: $(b_1, b_1, 1-2b_1) \circledcirc (b_2, b_2, 1-2b_2) =$
$(\frac{b_1+b_2-4b_1b_2}{1-4b_1b_2}, \frac{b_1+b_2-4b_1b_2}{1-4b_1b_2}, \frac{(1-2b_1)(1-2b_2)}{1-4b_1b_2})$, hence the subset S is closed with respect to the consensus operator; $(b_1, 0, 1-b_1) \circledcirc (b_2, 0, 1-b_2) = (\frac{b_1+b_2-2b_1b_2}{1-b_1b_2}, 0, \frac{1-b_1-b_2+b_1b_2}{1-b_1b_2})$, hence S_1 is also closed; closeness of S_2 with respect to the consensus operator is analogical.
(ii) $\frac{u_1u_2}{u_1+u_2-u_1u_2} = 0$ iff $u_1 = 0 \lor u_2 = 0$.
(iii) $S_{(k)}$: $(b_1, kb_1, 1-(1+k)b_1) \circledcirc (b_2, kb_2, 1-(1+k)b_2) =$
$(\frac{b_1(1-(1+k)b_2)+b_2(1-(1+k)b_1)}{(1-(1+k)b_1)+(1-(1+k)b_2)-(1-(1+k)b_1)(1-(1+k)b_2)},$
$\frac{kb_1(1-(1+k)b_2)+kb_2(1-(1+k)b_1)}{1-b_1-kb_1+1-b_2-kb_2-(1-b_1-kb_1-b_2-kb_2-(1+k)^2b_1b_2)}, \frac{(1-(1+k)b_1)(1-(1+k)b_2)}{(1-(1+k)^2b_1b_2)}) =$
$(\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, \frac{k(b_1+b_2-2(1+k)b_1b_2)}{(1-(1+k)^2b_1b_2)}, \frac{1-(1+k)b_1-(1+k)b_2+(1+k)^2b_1b_2}{(1-(1+k)^2b_1b_2)}) =$
$(\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, k\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, \frac{1-(1+k)(b_1+b_2)+(1+k)^2b_1b_2}{(1-(1+k)^2b_1b_2)}) =$
$(\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, k\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, \frac{1-(1+k)^2b_1b_2+(1+k)^2b_1b_2-(1+k)(b_1+b_2)+(1+k)^2b_1b_2}{(1-(1+k)^2b_1b_2)}) =$
$(\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, k\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, 1+\frac{(1+k)(2(1+k)b_1b_2-(b_1+b_2)}{(1-(1+k)^2b_1b_2)}) =$

[11] Including extremal d-pairs (absolute opinions) TRUE and FALSE.
[12] Including extremal d-pairs (absolute opinions) TRUE and FALSE.

$(\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, k\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}, 1-(1+k)\frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}) =$
$(b_3, kb_3, 1-(1+k)b_3)$, where $b_3 = \frac{b_1+b_2-2(1+k)b_1b_2}{(1-(1+k)^2b_1b_2)}$. □

Definition 6 *Let us define for (b,d,u) from the opinion space the following:* $-(b,d,u) = (d,b,u)$,
$q(b,d,u) = (b,d,u)\circledcirc 0' = 0'$,
$q_0(b,d,u) = q^{-1}(q(b,d,u)) \cap (S_1 \cup S_2)$, *where*
$q_0(b,d,u) = (\frac{d-b}{2d-1}, 0, \frac{b+d-1}{2d-1})$ *for* $b \geq d$, $q_0(b,d,u) = (0, \frac{b-d}{2b-1}, \frac{b+d-1}{2b-1})$ *for* $b \leq d$,
$r(b,d,u) = (b,d,u)\circledcirc - (b,d,u) = (b,d,u)\circledcirc(d,b,u) = (\frac{bu+du}{2u-u^2}, \frac{bu+du}{2u-u^2}, \frac{u^2}{2u-u^2}) = (\frac{b+d}{2-u}, \frac{b+d}{2-u}, \frac{u}{2-u}) = (\frac{1-u}{2-u}, \frac{1-u}{2-u}, \frac{u}{2-u})$ *for* $u \neq 0$,
$r(b,d,0) = (b,d,0)\circledcirc(d,b,0) = (\frac{b+d}{2}, \frac{b+d}{2}, 0) = (\frac{1}{2}, \frac{1}{2}, 0) = 0'$.

Definition 7 *For* $(b,d,u),\ (b',d',u') \in \mathbf{D}_0$ *we further define*
$(b,d,u) \leq_{pr} (b',d',u')$ *iff* $p_1(b,d,u) < p_1(b',d',u')$ *or if* $p_1(b,d,u) = p_1(b',d',u')$ *and* $b \leq b'$, *where* $p(b,d,u) = (b+\frac{u}{2}, d+\frac{u}{2}, 0)$, $p_1(b,d,u) = b+\frac{u}{2}$,
$(b,d,u) \leq_{qr} (b',d',u')$ *iff* $(q_0)_2(b,d,u) > (q_0)_2(b',d',u')$ *or if* $(q_0)_2(b,d,u) = (q_0)_2(b',d',u')$ *and* $(q_0)_1(b,d,u) < (q_0)_1(b',d',u')$ *or if* $q_0(b,d,u) = q_0(b',d',u')$ *and* $b \leq b'$, *where* $q_0(b,d,u) = ((q_0)_1(b,d,u), (q_0)_2(b,d,u), (q_0)_3(b,d,u))$.

Motivation: The mapping q is defined as an analogy of the homomorphism h in the context of the Dempster's semigroup. We have shown that $q(x) = 0'$ for all $x \in J_0$, i.e. $q(J_0) = 0'$, thus we cannot use it for a definition of an ordering analogic to the ordering $\leq$ of the standard Dempster's semigroup. The ordering $\leq_{pr}$ is defined as an analogy of the ordering $\leq$ of the standard Dempster's semigroup, where projection p is used instead of homomorphism h. Note that projection p is pignistic transformation in fact. As we'll see later q is not an ordered homomorphism of D_0 to G with respect to $\circledcirc$ and $\leq_{pr}$.

We still keep the idea of lines analogic to h-lines in our mind, thus we can take q-lines connecting opinion x with its q-image $q(x) = 0'$ and we define $q_0(x)$ as an intersection of q-line with $S_0 = S_1 \cup S_2$. And we use mapping q_0 instead of homomorphism h in a definition of the ordering $\leq_{qr}$.

Lemma 3 *(i)* $-(-x) = x$ *(i.e.* $-(-(b,d,u)) = (b,d,u)$*),*
(ii) $-(x\circledcirc y) = -x\circledcirc - y$
(i.e. $-((b_1,d_1,u_1)\circledcirc(b_2,d_2,u_2)) = -(b_1,d_1,u_1) \circledcirc -(b_2,d_2,u_2)$*),*
(iii) $-x$ *is not an inverse to* x*, i.e. the equation* $(b_1,d_1,u_1)\circledcirc(b_2,d_2,u_2) = (0,0,1)$ *has no solution in opinion space for* $(b_1,d_1,u_1) \neq (0,0,1)$ *(a discussion of the cases).*

Proof: (i) $-(-(b,d,u)) = -(d,b,u) = (b,d,u)$,
(ii) $-((b_1,d_1,u_1)\circledcirc(b_2,d_2,u_2)) = -(\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2}) = (\frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2}) = (d_1,b_1,u_1)\circledcirc(d_2,b_2,u_2) = -(b_1,d_1,u_1) \circledcirc -(b_2,d_2,u_2)$,
$-((b_1,d_1,0)\circledcirc(b_2,d_2,0)) = -(\frac{b_1+b_2}{2}, \frac{d_1+d_2}{2}, 0) = (\frac{d_1+d_2}{2}, \frac{b_1+b_2}{2}, 0) = (d_1,b_1,0)\circledcirc(d_2,b_2,0) = -(b_1,d_1,0) \circledcirc -(b_2,d_2,0)$,
(iii) $(\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2}) = (0,0,1)$ iff $u_1 = u_2 = 1$.

□

Lemma 4 *(i) The mapping* q *is a trivial ordered homomorphism of the set of all non-Bayesian opinions to* $\{0'\}$.
(ii) For the mapping $q_0(x) =$ "$q^{-1}(x) \cap (S_1 \cup S_2)$"*, which is expressible as* $q_0(b,d,u) = (\frac{d-b}{2d-1}, 0, \frac{b+d-1}{2d-1})$ *for* $b \geq d$ *and* $q_0(b,d,u) = (\frac{b-d}{2b-1}, 0, \frac{b+d-1}{2b-1})$ *for* $b \leq d$*, the following holds:* q_0 *is a homomorphism of* $(D_1 - G)$ *onto* S_1 *and of* $(D_2 - G)$ *onto* S_2*, where* $D_1 = \{(b,d,u) \in D_0 | b \geq d\}$, $D_2 = \{(b,d,u) \in D_0 | b \leq d\}$*, but it is not a homomorphism of* $(D_0 - G)$ *onto* $S_1 \cup S_2$. q_0 *is an ordered homomorphism with respect to the ordering* $\leq_{qr}$*. (But it is not an ordered homomorphism with respect to the ordering* $\leq_{pr}$*.)*
(iii) The mapping r *is a homomorphism of all the opinion space onto its subalgebra* $S \cup 0'$ *(but it is not an ordered homomorphism).*
(iv) The sets S, S_1, S_2*, and* $S_{(k)}$ *with the consensus operator and with ordering* $\leq_{pr}$ *(or* $\leq_{qr}$ *respectively)*

form ordered Abelian semigroups with neutral element $(0,0,1)$. They are all isomorphic to the positive cone of the aditive group of reals.
(v) There is no neutral element in G, there is no inverse on G, i.e. there is no relation of G to any group.
(vi) The set $S_0 = S_1 \cup S_2$ with operator $\odot_{S_0} = \odot \circ q_0$, with operator $-$, with distiguished element $0 = (0,0,1)$ and with ordering $\leq_{qr}$ forms ordered Abelian group $S_0 = (S_0, \odot_{S_0}, -, 0, \leq_{qr})$. S_0 isomorphic to the MYCIN group MC. The same holds also for the ordering $\leq_{pr}$.

Proof: (i) Trivially $q(x \odot y) = 0' = 0' \odot 0' = q(x) \odot q(y)$. And if $x \leq y$ then $q(x) = 0' \leq 0' = q(y)$.

(ii) Let $x = (b,d,u) \in (D_1 - G)$, then $b \geq d$ and $q(x) = (\frac{d-b}{2d-1}, 0, \frac{b+d-1}{2d-1}) = (\frac{b-d}{1-2d}, 0, \frac{1-b-d}{1-2d})$, further $b - d \geq 0$ and $1 - 2d \geq 0$, hence $\frac{b-d}{1-2d} \geq 0$ and $q(x) \in S_1$. Analogically $q(x) \in S_2$ for $x \in (D_2 - G)$. $q_0(b,0,1-b) = (\frac{-b}{-1}, 0, \frac{b-1}{-1}) = (b,0,1-b)$ and $q_0(0,d,1-d) = (0,d,1-d)$, hence q_0 is onto $S_1 \cup S_2$. Specially $q_0(0,0,1) = (0,0,1)$ thus it remains to prove that $q_0(x_1) \odot q_0(x_2) = q_0(x_1 \odot x_2)$.
Let $x_1, x_2 \in D_1 - G$:

$$q_0(x_1) \odot q_0(x_2) \;=\; q_0(b_1,d_1,u_1) \odot q_0(b_2,d_2,u_2) \;=$$

$$(\tfrac{d_1-b_1}{2d_1-1},\ 0,\ \tfrac{b_1+d_1-1}{2d_1-1}) \odot (\tfrac{d_2-b_2}{2d_2-1},\ 0,\ \tfrac{b_2+d_2-1}{2d_2-1}) \;=$$

$$\left(\frac{\frac{d_1-b_1}{2d_1-1}\cdot\frac{b_2+d_2-1}{2d_2-1}+\frac{d_2-b_2}{2d_2-1}\cdot\frac{b_1+d_1-1}{2d_1-1}}{\frac{b_1+d_1-1}{2d_1-1}+\frac{b_2+d_2-1}{2d_2-1}-\frac{b_1+d_1-1}{2d_1-1}\cdot\frac{b_2+d_2-1}{2d_2-1}},\ 0,\ \frac{\frac{b_1+d_1-1}{2d_1-1}\cdot\frac{b_2+d_2-1}{2d_2-1}}{\frac{b_1+d_1-1}{2d_1-1}+\frac{b_2+d_2-1}{2d_2-1}-\frac{b_1+d_1-1}{2d_1-1}\cdot\frac{b_2+d_2-1}{2d_2-1}}\right) \;=$$

$$\left(\frac{(d_1-b_1)(b_2+d_2-1)+(d_2-b_2)(b_1+d_1-1)}{(b_1+d_1-1)(2d_2-1)+(b_2+d_2-1)(2d_1-1)-(b_1+d_1-1)(b_2+d_2-1)},\ 0,\ \frac{(b_1+d_1-1)(b_2+d_2-1)}{(b_1+d_1-1)(2d_2-1)+(b_2+d_2-1)(2d_1-1)-(b_1+d_1-1)(b_2+d_2-1)}\right) \;=$$

$$\left(\frac{-[(d_1-b_1)(1-b_2-d_2)+(d_2-b_2)(1-b_1-d_1)]}{-[(1-b_1-d_1)(2d_2-1)+(1-b_2-d_2)(2d_1-1)+(1-b_1-d_1)(1-b_2+d_2)]},\ 0,\ \frac{(1-b_1-d_1)(1-b_2-d_2)}{-[(1-b_1-d_1)(2d_2-1)+(1-b_2-d_2)(2d_1-1)+(1-b_1-d_1)(1-b_2+d_2)]}\right) \;=$$

$$\left(\frac{(d_1-b_1)(1-b_2-d_2)+(d_2-b_2)(1-b_1-d_1)}{(1-b_1-d_1)(2d_2-1)+(1-b_2-d_2)(2d_1-1)+(1-b_1-d_1)(1-b_2+d_2)},\ 0,\ \frac{-(1-b_1-d_1)(1-b_2-d_2)}{(1-b_1-d_1)(2d_2-1)+(1-b_2-d_2)(2d_1-1)+(1-b_1-d_1)(1-b_2+d_2)}\right);$$

$$q_0(x_1 \odot x_2) \;=\; q_0((b_1,d_1,u_1) \odot (b_2,d_2,u_2)) \;=\; q_0(\tfrac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \tfrac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \tfrac{u_1u_2}{u_1+u_2-u_1u_2}) \;=$$

$$\left(\frac{\frac{(d_1-b_1)u_2+(d_2-b_2)u_1}{u_1+u_2-u_1u_2}}{\frac{2d_1u_2+2d_2u_1-u_1-u_2+u_1u_2}{u_1+u_2-u_1u_2}},\ 0,\ \frac{\frac{(b_1+d_1)u_2+(b_2+d_2)u_1-u_1-u_2+u_1u_2}{u_1+u_2-u_1u_2}}{\frac{2d_1u_2+2d_2u_1-u_1-u_2+u_1u_2}{u_1+u_2-u_1u_2}}\right) \;=$$

$$\left(\frac{(d_1-b_1)u_2+(d_2-b_2)u_1}{2d_1u_2+2d_2u_1-u_1-u_2+u_1u_2},\ 0,\ \frac{(b_1+d_1)u_2+(b_2+d_2)u_1-u_1-u_2+u_1u_2}{2d_1u_2+2d_2u_1-u_1-u_2+u_1u_2}\right) \;=$$

$$\left(\frac{(d_1-b_1)(1-b_2-d_2)+(d_2-b_2)(1-b_1-d_1)}{2d_1(1-b_2-d_2)+2d_2(1-b_1-d_1)-(1-b_1-d_1)-(1-b_2-d_2)+(1-b_1-d_1)(1-b_2-d_2)},\ 0,\ \frac{(b_1+d_1)(1-b_2-d_2)+(b_2+d_2)(1-b_1-d_1)-(1-b_1-d_1)-(1-b_2-d_2)+(1-b_1-d_1)(1-b_2-d_2)}{2d_1(1-b_2-d_2)+2d_2(1-b_1-d_1)-(1-b_1-d_1)-(1-b_2-d_2)+(1-b_1-d_1)(1-b_2-d_2)}\right) \;=$$

$$\left(\frac{(d_1-b_1)(1-b_2-d_2)+(d_2-b_2)(1-b_1-d_1)}{(2d_1-1)(1-b_2-d_2)+(2d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)},\ 0,\ \frac{(b_1+d_1-1)(1-b_2-d_2)+(b_2+d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}{(2d_1-1)(1-b_2-d_2)+(2d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}\right) \;=$$

$$\left(\frac{(d_1-b_1)(1-b_2-d_2)+(d_2-b_2)(1-b_1-d_1)}{(2d_1-1)(1-b_2-d_2)+(2d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)},\ 0,\ \frac{-(1-b_1-d_1)1)(1-b_2-d_2)-(1-b_2-d_2))(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}{(2d_1-1)(1-b_2-d_2)+(2d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}\right) \;=$$

$$\left(\frac{(d_1-b_1)(1-b_2-d_2)+(d_2-b_2)(1-b_1-d_1)}{(2d_1-1)(1-b_2-d_2)+(2d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)},\ 0,\ \frac{-(1-b_1-d_1)(1-b_2-d_2)-(1-b_2-d_2))(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}{(2d_1-1)(1-b_2-d_2)+(2d_2-1)(1-b_1-d_1)+(1-b_1-d_1)(1-b_2-d_2)}\right) \;=\; q_0(b_1,d_1,u_1) \odot q_0(b_2,d_2,u_2)) \;=$$

$q_0(x_1) \odot q_0(x_2))$ and analogically for $x_1, x_2 \in D_2 - G$. Hence q_0 is homomorphism of $D_i - G$ to S_i.
It remains to prove that q_0 satisfies the ordering $\leq_{qr}$ but not $\leq_{pr}$. Let $x_1 = (b_1,d_1,u_1) \leq_{qr} (b_2,d_2,u_2) = x_2$: if $(q_0)_2(b_1,d_1,u_1) < (q_0)_2(b_2,d_2,u_2)$ then $q_0(b_1,d_1,u_1) = (0, (q_0)_2(b_1,d_1,u_1), 1 - (q_0)_2(b_1,d_1,u_1)) <_{qr} (0, (q_0)_2(b_2,d_2,u_2), 1 - (q_0)_2(b_2,d_2,u_2)) = q_0(b_2,d_2,u_2)$, if $(q_0)_2(b_1,d_1,u_1) =$

$(q_0)_2(b_2, d_2, u_2)$ and $(q_0)_1(b_1, d_1, u_1) < (q_0)_1(b_2, d_2, u_2)$ then $q_0(b_1, d_1, u_1) = ((q_0)_1(b_1, d_1, u_1), 0, 1 - (q_0)_1(b_1, d_1, u_1)) <_{qr} ((q_0)_1(b_2, d_2, u_2), 0, 1 - (q_0)_1(b_2, d_2, u_2)) = q_0(b_2, d_2, u_2)$, if $(q_0)_2(b_1, d_1, u_1) = (q_0)_2(b_2, d_2, u_2)$ and $(q_0)_1(b_1, d_1, u_1) = (q_0)_1(b_2, d_2, u_2)$ then $q_0(b_1, d_1, u_1) = q_0(b_2, d_2, u_2)$ i.e. also $q_0(b_1, d_1, u_1) \leq_{qr} q_0(b_2, d_2, u_2)$, hence q_0 keeps the ordering $\leq_{qr}$. We can easily show a counter example for the ordering $\leq_{pr}$: let $x_1 = (0.5, 0.1, 0.4)$, and $x_2 = (0.6, 0.3, 0.1)$, $p(x_1) = p(0.5, 0.1, 0.4) = (0.7, 0.3, 0) > (0.65, 0.35, 0) = p(0.6, 0.3, 0.1) = p(x_2)$, hence $x_1 >_{pr} x_2$, further $q_0(x_1) = q_0(0.5, 0.1, 0.4) = (\frac{0.4}{0.8}, 0, \frac{0.4}{0.8}) = (0.5, 0, 0.5) <_{pr} (0.75, 0, 0.25) = (\frac{0.3}{0.4}, 0, \frac{0.1}{0.4}) = q_0(0.6, 0.3, 0.1) = q_0(x_2)$, hence q_0 does not keep the ordering $\leq_{pr}$.
Note that $(S_1 \cup S_2) \subset (D_0 - G)$ is not closed with respect to $\circledcirc$, thus q_0 is not a homomorphism $D_0 - G$ onto $S_1 \cup S_2$.

(iii) S is closed, see Lemma 2 (i), and $x \circledcirc 0' = 0'$ for any x from $D_0 - G$, hence $S \cup 0'$ is also closed with respect to consensus operator $\circledcirc$. $r(x) = r(b, d, u) = (\frac{1-u}{2-u}, \frac{1-u}{2-u}, \frac{u}{2-u}) \in S$ if $u > 0$, $r(b, 1-b, 0) = (\frac{1-0}{2-0}, \frac{1}{2}, 0) = 0'$, $r(0, 0, 0) = (\frac{1-1}{2-1}, \frac{0}{1}, \frac{1}{1}) = 0$, $r(0') = 0'$.
$r(b_1, d_1, u_1) \circledcirc r(b_2, d_2, u_2) = (\frac{1-u_1}{2-u_1}, \frac{1-u_1}{2-u_1}, \frac{u_1}{2-u_1}) \circledcirc (\frac{1-u_2}{2-u_2}, \frac{1-u_2}{2-u_2}, \frac{u_2}{2-u_2}) =$

$$(\frac{\frac{1-u_1}{2-u_1}\frac{u_2}{2-u_2} + \frac{1-u_2}{2-u_2}\frac{u_1}{2-u_1}}{\frac{u_1}{2-u_1} + \frac{u_2}{2-u_2} - \frac{u_1}{2-u_1}\frac{u_2}{2-u_2}}, \frac{\frac{1-u_1}{2-u_1}\frac{u_2}{2-u_2} + \frac{1-u_2}{2-u_2}\frac{u_1}{2-u_1}}{\frac{u_1}{2-u_1} + \frac{u_2}{2-u_2} - \frac{u_1}{2-u_1}\frac{u_2}{2-u_2}}, \frac{\frac{u_1}{2-u_1}\frac{u_2}{2-u_1}}{\frac{u_1}{2-u_1} + \frac{u_2}{2-u_2} - \frac{u_1}{2-u_1}\frac{u_2}{2-u_2}}) =$$

$$(\frac{(1-u_1)u_2 + (1-u_2)u_1}{u_1(2-u_2) + u_2(2-u_1) - u_1u_2}, \frac{\frac{1-u_1}{2-u_1}\frac{u_2}{2-u_1} + \frac{1-u_2}{2-u_2}\frac{u_1}{2-u_2}}{\frac{1-u_1}{2-u_1} + \frac{1-u_2}{2-u_2} - \frac{1-u_1}{2-u_1}\frac{1-u_2}{2-u_2}}, \frac{u_1u_2}{u_1(2-u_2) + u_2(2-u_1) - u_1u_2}) =$$

$$(\frac{u_2 - u_1u_2 + u_1 - u_1u2}{2u_1 - u_1u_2 + 2u_2 - u_1u_2 - u_1u_2}, \frac{u_1 + u_2 - 2u_1u_2}{2(u_1+u_2) - 3u_1u_2}, \frac{u_1u_2}{2(u_1+u_2) - 3u_1u_2}),$$

$$r((b_1, d_1, u_1) \circledcirc (b_2, d_2, u_2)) = r(\frac{b_1u_2 + b_2u_1}{u_1 + u_2 - u_1u_2}, \frac{d_1u_2 + d_2u_1}{u_1 + u_2 - u_1u_2}, \frac{u_1u_2}{u_1 + u_2 - u_1u_2}) =$$

$$(\frac{1 - \frac{u_1u_2}{u_1+u_2-u_1u_2}}{2 - \frac{u_1u_2}{u_1+u_2-u_1u_2}}, \frac{1 - \frac{u_1u_2}{u_1+u_2-u_1u_2}}{2 - \frac{u_1u_2}{u_1+u_2-u_1u_2}}, \frac{\frac{u_1u_2}{u_1+u_2-u_1u_2}}{2 - \frac{u_1u_2}{u_1+u_2-u_1u_2}}) = (\frac{\frac{u_1+u_2-2u_1u_2}{u_1+u_2-u_1u_2}}{\frac{2(u_1+u_2)-3u_1u_2}{u_1+u_2-u_1u_2}}, \frac{\frac{u_1+u_2-2u_1u_2}{u_1+u_2-u_1u_2}}{\frac{2(u_1+u_2)-3u_1u_2}{u_1+u_2-u_1u_2}}, \frac{\frac{u_1u_2}{u_1+u_2-u_1u_2}}{\frac{2(u_1+u_2)-3u_1u_2}{u_1+u_2-u_1u_2}}) =$$

$$(\frac{u_1 + u_2 - 2u_1u_2}{2(u_1+u_2) - 3u_1u_2}, \frac{u_1 + u_2 - 2u_1u_2}{2(u_1+u_2) - 3u_1u_2}, \frac{2u_1u_2}{2(u_1+u_2) - 3u_1u_2}),$$

thus $r(x_1) \circledcirc r(x_2) = r(x_1 \circledcirc x_2)$ and r is a homomorphismus of the opinion space to $S \cup 0'$.
Let it be $a < b$. $(0, a, 1-a) >_{qr} (0, b, 1-b)$ while $r(0, a, 1-a) = (\frac{1-(1-a)}{2-(1-a)}, \frac{a}{1+a}, \frac{1-a}{1+a}) <_{qr} (\frac{b}{1+b}, \frac{b}{1+b}, \frac{1-b}{1+b}) = r(0, b, 1-b)$ $(q_0(\frac{1-(1-a)}{2-(1-a)}, \frac{a}{1+a}, \frac{1-a}{1+a}) = q_0(\frac{b}{1+b}, \frac{b}{1+b}, \frac{1-b}{1+b})$ and $\frac{a}{1+a}) < (\frac{b}{1+b})$, hence r is not an ordered homomorphism with respect to $\leq_{qr}$. Similarly $(0, a, 1-a) >_{pr} (0, b, 1-b)$ because $p(0, a, 1-a) = (\frac{1-a}{2}, \frac{1+a}{2}, 0) > (\frac{1-b}{2}, \frac{1+b}{2}, 0) = p(0, b, 1-b)$ and $r(0, a, 1-a) <_{pr} r(0, b, 1-b)$ because of $p(r(0, a, 1-a)) = p(\frac{a}{1+a}, \frac{a}{1+a}, \frac{1-a}{1+a}) = 0' = p(\frac{b}{1+b}, \frac{b}{1+b}, \frac{1-b}{1+b}) = p(r(0, b, 1-b)$ and $\frac{a}{1+a} < \frac{b}{1+b}$ again. Hence is nor an ordered homomorphism with respect to $\leq_{pr}$.

(iv) All the sets S, S_1, S_2 and $S_{(k)}$ are closed, see Lemma 2 (i) and (iii). Commutativity and associatiovity follow general commutativity and associativity. The neutral elements 0 is an element of all the sets S, S_1, S_2 and $S_{(k)}$.
Let us define $S^-, S_1^-, S_2^-, S_{(k)}^-$ as sets of all $(-b, -d, u)$ where $(b, d, u) \in S, S_1, S_2, S_{(k)}$, thus we obtain OAGs with neutral element 0 and with inverse $inv(b, d, u) = (-b, -d, u)$. All the sets $S \cup S^-$, $S_1 \cup S_1^-$, $S_2 \cup S_2^-, S_{(k)} \cup S_{(k)}^-$ are dense as they are segments of straights lines and all of them are fully ordered both with $\leq_{pr}$ and $\leq_{qr}$, where $(-b_1, -d_1, u_1) \leq_{pr} (-b_2, -d_2, u_2)$ *iff* $(b_1, d_1, u_1) \geq_{pr} (b_2, d_2, u_2)$ and analogically $(-b_1, -d_1, u_1) \leq_{qr} (-b_2, -d_2, u_2)$ *iff* $(b_1, d_1, u_1) \geq_{qr} (b_2, d_2, u_2)$. Thus all of them are isomorphic to additive group of reals **Re**, see e.g. Chapter 7 in [11]. Hence all of them are isomoprphic to MC.
Because the negative cones of **Re**, **MC** are isomorphic to its negative cones (isomorphism -), all the Abelian semigroups S, S_1, S_2 and $S_{(k)}$ are isomorphic to the positive cones of OAGs **Re** and **MC**. Nevertheless we have to note that S, S_1, and $S_{(k)}$ for $k \leq 1$ are positive cones of OAGs $S \cup S^-$, $S_1 \cup S_1^-, S_{(k)} \cup S_{(k)}^-$ while S_2 and $S_{(k)}$ for $k > 1$ are negative cones of $S_2 \cup S_2^-, S_{(k)} \cup S_{(k)}^-$.

(v) Let N $= (\mathrm{b}_N, d_N, u_N) = (b_N, 1 - b_N, 0)$ be a fixed neutral element of $\circledcirc$ in G. Thus for any

$(b,1-b,0) \in G$ holds that $(b,1-b,0)⊚(b_N,1-b_N,0) = (b,1-b,0)$. We can compute $(b,1-b,0)⊚(b_N,1-b_N,0) = (\frac{b+b_N}{2}, \frac{1-b+1-b_N}{2}, 0) = (\frac{b+b_N}{2}, \frac{2-b-b_N}{2}, 0)$, hence $\frac{b+b_N}{2} = b$ and $b_N = b$ for any b what is a contradiction. Thus there is neither inverse nor neutral element of ⊚ in G.

(vi) D_0 is closed with respect to ⊚ and q_0 maps D_0 to $S_1 \cup S_2$ thus $S_0 = S_1 \cup S_2$ is closed with respect to $⊚_{S_0}$. Commutativity and associativity of $⊚_{S_0}$ follow commutativity and associativity of ⊚. $(b,d,u)⊚_{S_0}(0,0,0) = q_0(0,0,0) = (0,0,0)$, $(b,0,1-b)⊚_{S_0}(0,b,1-b) = q_0(\frac{b(1-b)}{(1+b)(1-b)}, \frac{b(1-b)}{(1+b)(1-b)}, \frac{(1-b)^2}{(1+b)(1-b)}) = q_0(\frac{b}{1+b}, \frac{b}{1+b}, \frac{1-b}{1+b}) = (0,0,0)$, and similarly $(0,b,1-b)⊚_{S_0}(b,0,1-b) = (0,0,0)$.
Let it be $(b,0,1-b) \leq_{qr} (B,0,1-B)$, i.e. $b \leq B$. $(a,0,1-a)⊚_{S_0}(b,0,1-b) = (\frac{a+b-2ab}{1-ab}, 0, \frac{1-a-b+ab}{1-ab}) = (\frac{a+(1-2a)b}{1-ab}, 0, \frac{1-a-(1-a)b}{1-ab})$ and $(a,0,1-a)⊚_{S_0}(B,0,1-B) = (\frac{a+(1-2a)B}{1-aB}, 0, \frac{1-a-(1-a)B}{1-aB})$. $b \leq B$ implies $\frac{a+(1-2a)b}{1-ab} \leq \frac{a+(1-2a)B}{1-aB}$ and $(a,0,1-a)⊚_{S_0}(b,0,1-b) \leq_{qr}$(a,0,1-a) $⊚_{S_0}(B,0,1-B)$. Similarly it holds $(0,a,1-a)⊚_{S_0}(0,b,1-b) \leq_{qr}$(0,a,1-a) $⊚_{S_0}(0,B,1-B)$ for $(0,b,1-b) \leq_{qr} (0,B,1-a)$. Thus $\leq_{qr}$ keeps monotonicity on both S_1 and S_2.
$(a,0,1-a)⊚_{S_0}(B,0,1-B) = q_0(\frac{a+B-2aB}{1-aB}, 0, \frac{1-a-B+aB}{1-aB}) = (\frac{a+B-2aB}{1-aB}, 0, \frac{1-a-B+aB}{1-aB})$. $(a,0,1-a)⊚_{S_0}(0,b,1-b) = q_0(\frac{a-ab}{1-ab}, \frac{b-ab}{1-ab}, \frac{1-a-b+ab}{1-ab})$, $\frac{a-ab}{1-ab} \leq \frac{b-ab}{1-ab}$ *iff* $a \leq b$, thus for $a \leq b$ we obtain $(a,0,1-a)⊚_{S_0}(0,b,1-b) = (0, \frac{a-b}{2a-ab-1}, \frac{a+b-ab-1}{2a-ab-1}) < (a,0,1-a)⊚_{S_0}(B,0,1-B)$, and for $a \geq b$ we obtain the following: $(a,0,1-a)⊚_{S_0}(0,b,1-b) = (\frac{\frac{b-ab}{1-ab} - \frac{a-ab}{1-ab}}{2\frac{b-ab}{1-ab}-1}, 0, \frac{\frac{a-ab}{1-ab} + \frac{b-ab}{1-ab} - 1}{2\frac{b-ab}{1-ab}-1}) = (\frac{b-ab-a+ab}{2b-2ab-1+ab}, 0, \frac{a-ab+b-ab-1+ab}{2b-2ab-1+ab}) = (\frac{b-a}{2b-ab-1}, 0, \frac{a-ab+b-1}{2b-ab-1}) \leq_{qr} (a,0,1-a) \leq_{qr} (a,0,1-a)⊚_{S_0}(B,0,1-B)$. Analogically for $(0,a,1-a)$, $(0,b,1-b)$ and $(B,0,1-B)$, $((0,a,1-a)⊚_{S_0}(0,b,1-b) \leq_{qr} (0,a,1-a) \leq_{qr} (0,a,1-a)⊚_{S_0}(B,0,1-B))$. Hence $\leq_{qr}$ keeps monotonicity on whole $S_1 \cup S_2$. Both the orderings $\leq_{pr}$ and $\leq_{qr}$ are the same on S_0, thus $\leq_{pr}$ also keeps monotonicity on whole $S_0 = S_1 \cup S_2$.

□

r preimage of $r(b,d,u)$ is just the intersection of horizontal straight line $y = u$ with the opinion triangle (it is just the intersection of triangle representing the Dempster's semigroup with a straigt line parallel to G, in both the cases it is an abscissa parallel to abscissa representing Bayesian d-pairs / opionions). The consensus of two opinions laying on the straigth lines (r-lines) $y = u_1$ and $y = u_2$ is an opinion laying on the r-line $y = \frac{u_1 u_2}{u_1+u_2-u_1u_2}$. Thus the consensus operator is effected 'per r-lines'.

Analogically the consensus is effected 'per q-lines' (straight lines containing $0' = (\frac{1}{2}, \frac{1}{2}, 0)$). In this case it is not possible to speak about q preimages of $q(b,d,u) = 0'$ but the idea is analogous. Because of q maps all non-Bayesian opinions to the same opinion $0'$, we are interesting in the other ends of intersections of q-lines with the opinion triangle. They are formalized by the mapping q_0.

Examples: both theoretical and numerical.

Remark 1 *Because of computation of consensus of Bayesian opinions as an average, there is no neutral element in G, there is no inverse on G, i.e. there is no relation to any group. Moreover consensus of Bayesian opinions is not associative in general, see the next subsection.*

6.1 Associativity of computing of the consensus of Bayesian opinions

The consensus operator of non-Bayesian opinions is computed as a increased weighted mean. Both belief and disbelief components are weighted by u of the other opinion and resulting mean is increased by a factor $\frac{u_1+u_2}{u_1+u_2-u_1u_2} > 1$, i.e. we can express consensus of non-Bayesian opinions as $(b_1,d_1,u_1) ⊚ (b_2,d_2,u_2) = (\frac{b_1u_2+b_2u_1}{u_1+u_2}\frac{u_1+u_2}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2}\frac{u_1+u_2}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2})$ for $u_1u_2 \neq 0$.
While the consensus of Bayesian opinions without any additional information correspond just to non-associative arithmetical mean. To overcome it additional tools required additional information are used to obtain $\gamma \neq 1$, see the definition 5. If there is no additional information we have to distinguish whether the opinions to be combined are 'single', i.e. not a results of consensus, or howmany times the concensus was already used. We use $\gamma = 1$ for two 'single' opinions, $\gamma = n$ in the case where $(b_1,d_1,0)$ is a result of just n applications of the consensus operator and $(b_2,d_2,0)$ is a 'single' one. In the case where the first argument $(b_1,d_1,0)$ is a 'single' and the second one is already consensed we

use $\gamma = \frac{1}{n}$. Hence, the computation of the consensus corresponds to stepwise computation of n-ary arithmetic mean. For an example of associative combination of three Bayesian opinions see [8].

Using the above procedure, we are able to compute the consensus of several Bayesian opinions in associative way. But this method is not general. We have to always remember and handle the history of the opinions (how many times the consensus operator was used). And it is not always easy. In the case of subjective opinions it is often even for opinion agent himself quite difficult to deside whether his opinion is 'single' or it is already implicitly consensed from two or several ones.

6.2 A formal definition of Jøsang's semigroup

From the algebraical point of view we have obtained instead of the operator on opinions a new one defined on the Cartesian product of the set of opinions with the set of positive integers or reals if we admit non-integer γ based on a different additional information. Because nor this method is not completely general, we do not include it into the following formal definition of Jøsang's semigroup, and we stay limited to non-Bayesian opinions.

Definition 8 *Jøsang's semigroup* $\mathbf{J}_0 = (J_0, \circledcirc)$ *is the set of all non Bayesian Dempster's pairs (opinions), endowed with the operation* $\circledcirc$ *and with a distinguished element* $0 = (0,0,1)$*, where the operation* $\circledcirc$ *is defined by*

$$(b_A, d_A, u_A) \circledcirc (b_B, d_B, u_B) = \left(\frac{b_A u_B + b_B u_A}{u_A + u_B - u_A u_B}, \frac{d_A u_B + d_B u_A}{u_A + u_B - u_A u_B}, \frac{u_A u_B}{u_A + u_B - u_A u_B}\right).$$

Theorem 2 *(i) Jøsang's semigroup with the relation* $\leq_{qr}$ *is an ordered commutative semigroup with the neutral element* $0 = (0,0,1)$*;* 0 *is the only idempotent of it. (It does not hold for* $\leq_{pr}$*).*

(ii) The sets S, S_1, S_2 *and* $S_{(k)}$ *with the operation* $\circledcirc$ *and the ordering* $\leq_{qr}$ *(or* $\leq_{pr}$ *respectively) form ordered commutative semigroups with neutral element* 0*, and they are all isomorphic to the semigroup of nonnegative elements (positive cone) of the MYCIN group* **MC***.*

(iii) The set $S_0 = S_1 \cup S_2$ *with the operations* $\circledcirc_{S_0} = \circledcirc \circ q_0$ *and* $-$*, and with the ordering* $\leq_{qr}$ *form ordered Abelian group with neutral element* 0*:* $S_0 = (S_0, \circledcirc_{S_0}, -, 0, \leq_{qr})$*.* S_0 *is isomorphic to the MYCIN group* **MC***.*

(iv) The mapping q_0 *is an ordered homomorphism of Jøsang's semigroup onto group* S_0*, it preserves the ordering* $\leq_{qr}$ *(*S_0 *is subset of* J_0 *but it is not a subalgebra of* J_0*).*

(v) The mapping r *is a homomorphism of Jøsang's semigroup onto its subsemigroup* S *(but it is not an ordered homomorphism).*

(vi) The mapping q *is a trivial homomorphism from* J_0 *to the set* G *of Bayesian opinions. There does not exist any homomorhic Bayesian transformation of* J_0*. There does not exist any homomorhic Bayesian transformation of the whole opinion space which is homomorphic with respect to the consensus operator.*

Using the theorem, see (iv) and (v), we can express

$$(x \circledcirc y) = q_0^{-1}(q_0(x) \circledcirc q_0(y)) \cap r^{-1}(r(x) \circledcirc r(y)) \tag{6.1}$$

for every couple of non-Bayesian opinions x, y.

Before a proof of the theorem we present the following lemma:

Lemma 5 *(i) Let us define* J_1 *as* $\{(b,d,u) \in J_0 \mid b \geq d\} = D_1 - G$ *and* J_2 *as* $\{(b,d,u) \in J_0 \mid b \leq d\} = D_2 - G$*. Let further* $x, y \in J_0$*. It holds that* $x \circledcirc y \in J_1$ *iff* $q_0(x) \circledcirc q_0(y) \in J_1$ *and* $x \circledcirc y \in J_2$ *iff* $q_0(x) \circledcirc q_0(y) \in J_2$*.*
(ii) The operation $\circledcirc_{S_0} = \circledcirc \circ q_0$ *commutes with* q_0*, i.e. the following holds:* $q_0(x \circledcirc_{S_0} y) = q_0(x) \circledcirc_{S_0} q_0(y)$*.*

Proof: (i) Let $(b,d,u) = x = x_1 \circledcirc x_2 = (b_1,d_1,u_1) \circledcirc (b_2,d_2,u_2) = (\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2})$. If $b_i \geq d_i$ then $(b_1-d_1)u_2+(b_2-d_2)u_1 \geq 0$, $b_1u_2+b_2u_1 \geq d_1u_2+d_2u_1$, and $b \geq d$. Analogically $b \leq d$ for $b_i \leq d_i$. Hence both J_1 and J_2 are closed with respect to the consensus operator $\circledcirc$. From Lemma 4(ii), we know that q_0 maps J_i to S_i and S_i are also closed with respect to $\circledcirc$. Thus it satisfies to prove the statement for $x_1 \in J_1$ and $x_2 \in J_2$.
Let us compute $x \circledcirc y$ and $q_0(x_1) \circledcirc q_0(x_2)$ and compare their belief components with disbelief ones: $x_1 \circledcirc x_2 = (b_1,d_1,u_1) \circledcirc (b_2,d_2,u_2) = (\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2})$, all b_i, d_i, u_i, and all the fractions are positive less or equal to 1 thus all enumerators and denominators are positive and $\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2} \geq \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}$ *iff* $b_1u_2+b_2u_1 \geq d_1u_2+d_2u_1$ *iff* $(b_1-d_1)u_2+ \geq (d_2-b_2)u_1$;
$q_0(b_1,d_1,u_1) = (\frac{d_1-b_1}{2d_1-1}, 0, \frac{b_1+d_1-1}{2d_1-1})$, $q_0(b_2,d_2,u_2) = (0, \frac{b_2-d_2}{2b_2-1}, \frac{b_2+d_2-1}{2b_2-1})$, and $q_0(b_1,d_1,u_1) \circledcirc q_0(b_2,d_2,u_2) =$

$$(\frac{\frac{d_1-b_1}{2d_1-1}\frac{b_2+d_2-1}{2b_2-1}+0}{\frac{b_1+d_1-1}{2d_1-1}+\frac{b_2+d_2-1}{2b_2-1}-\frac{b_1+d_1-1}{2d_1-1}\frac{b_2+d_2-1}{2b_2-1}}, \frac{0+\frac{b_2-d_2}{2b_2-1}\frac{b_1+d_1-1}{2d_1-1}}{\frac{b_1+d_1-1}{2d_1-1}+\frac{b_2+d_2-1}{2b_2-1}-\frac{b_1+d_1-1}{2d_1-1}\frac{b_2+d_2-1}{2b_2-1}}, \frac{\frac{b_1+d_1-1}{2d_1-1}\frac{b_2+d_2-1}{2b_2-1}}{\frac{b_1+d_1-1}{2d_1-1}+\frac{b_2+d_2-1}{2b_2-1}-\frac{b_1+d_1-1}{2d_1-1}\frac{b_2+d_2-1}{2b_2-1}}) =$$

$$(\frac{(d_1-b_1)(b_2+d_2-1)}{(b_1+d_1-1)(2b_2-1)+(b_2+d_2-1)(2d_1-1)-(b_1+d_1-1)(b_2+d_2-1)}, \frac{(b_2-d_2)(b_1+d_1-1)}{(b_1+d_1-1)(2b_2-1)+(b_2+d_2-1)(2d_1-1)-(b_1+d_1-1)(b_2+d_2-1)}, \frac{(b_1+d_1-1)(b_2+d_2-1)}{(b_1+d_1-1)(2b_2-1)+(b_2+d_2-1)(2d_1-1)-(b_1+d_1-1)(b_2+d_2-1)}) =$$

$$(\frac{(d_1b_2-b_1b_2+d_1d_2-b_1d_2-d_1+b_1)}{(2b_1b_2+2b_2d_1-2b_2-b_1-d_1+1)+(2b_2d_1+2d_1d_2-2d_1-b_2-d_2+1)-(b_1b_2+b_1d_2-b_1+b_2d_1+d_1d_2-d_1-b_2-d_2+1)}, \frac{(b_1b_2-b_1d_2+b_2d_1-d_1d_2-b_2+d_2)}{(b_1b_2-b_1d_2+3b_2d_1+d_1d_2-2b_2-2d_1+1)}, \frac{(b_1b_2+b_1d_2-b_1+b_2d_1+d_1d_2-d_1-b_2-d_2+1)}{(b_1b_2-b_1d_2+3b_2d_1+d_1d_2-2b_2-2d_1+1)}) =$$

$$(\frac{(b_1-d_1)(1-b_2-d_2)}{(b_1b_2-b_1d_2+3b_2d_1+d_1d_2-2b_2-2d_1+1)}, \frac{(d_2-b_2)(1-b_1-d_1)}{(b_1b_2-b_1d_2+3b_2d_1+d_1d_2-2b_2-2d_1+1)}, \frac{(1-b_1-d_1)(1-b_2-d_2)}{(b_1b_2-b_1d_2+3b_2d_1+d_1d_2-2b_2-2d_1+1)}) =$$

$(\frac{(b_1-d_1)u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}, \frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}, \frac{u_1u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2})$. Again, all b_i, d_i, u_i, and all the fractions are positive less or equal to 1 thus all enumerators and denominators are positive and $\frac{(b_1-d_1)u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2} \geq \frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}$ *iff* $(b_1-d_1)u_2 \geq u_1(1-2b_2)$ *iff* $\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2} \geq \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}$. Hence $x_1 \circledcirc x_2 \in J_1$ *iff* $q_0(x_1) \circledcirc q_0(x_2) \in J_1$, and similarly $x_1 \circledcirc x_2 \in J_2$ *iff* $q_0(x_1) \circledcirc q_0(x_2) \in J_2$.
(ii) From Lemma 4(ii), we know that q_0 is an ordered homomorphism from J_1 to S_1 and from J_2 to S_2, thus it again satisfies to prove the statement for $x_1 \in J_1$ and $x_2 \in J_2$. Let $x_1 \circledcirc x_2 \in J_1$, i.e. $b \geq d$, $(b_1-d_1)u_2 \geq (d_2-b_2)u_1$. $q_0(x_1 \circledcirc_{S_0} x_2) = q_0(q_0(x_1 \circledcirc x_2)) = q_0(x_1 \circledcirc x_2) =$

$$q_0(\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}, \frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}, \frac{u_1u_2}{u_1+u_2-u_1u_2}) = (\frac{\frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}-\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}}{2\frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}-1}, 0, \frac{\frac{b_1u_2+b_2u_1}{u_1+u_2-u_1u_2}+\frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}-1}{2\frac{d_1u_2+d_2u_1}{u_1+u_2-u_1u_2}-1}) =$$

$$(\frac{d_1u_2+d_2u_1-b_1u_2-b_2u_1}{2d_1u_2+2d_2u_1-u_1-u_2+u_1u_2}, 0, \frac{b_1u_2+b_2u_1+d_1u_2+d_2u_1-u_1-u_2+u_1u_2}{2d_1u_2+2d_2u_1-u_1-u_2+u_1u_2}) =$$

$$(\frac{(d_1-b_1)u_2+(d_2-b_2)u_1}{(2d_1-1)u_2+(2d_2-1)u_1+u_1u_2}, 0, \frac{(b_1+d_1-1)u_2+(b_2+d_2-1)u_1+u_1u_2}{(2d_1-1)u_2+(2d_2-1)u_1+u_1u_2}) =$$

$$(\frac{(b_1-d_1)u_2+(b_2-d_2)u_1}{(1-2d_1)u_2+(1-2d_2)u_1-u_1u_2}, 0, \frac{(1-b_1-d_1)u_2+(1-b_2-d_2)u_1-u_1u_2}{(1-2d_1)u_2+(1-2d_2)u_1-u_1u_2}) =$$

$$(\frac{(b_1-d_1)u_2+(b_2-d_2)u_1}{(1-b_1-d_1+(b_1-d_1))u_2+(1-b_2-d_2+(b_2-d_2)u_1-u_1u_2}, 0, \frac{(u_1)u_2+(u_2)u_1-u_1u_2}{(1-b_1-d_1+(b_1-d_1))u_2+(1-b_2-d_2+(b_2-d_2)u_1-u_1u_2}) =$$

$$(\frac{(b_1-d_1)u_2+(b_2-d_2)u_1}{u_1u_2+(b_1-d_1)u_2+u_1u_2+(b_2-d_2)u_1-u_1u_2}, 0, \frac{u_1u_2}{u_1u_2+(b_1-d_1)u_2+u_1u_2+(b_2-d_2)u_1-u_1u_2}) =$$

$$(\frac{(b_1-d_1)u_2+(b_2-d_2)u_1}{(b_1-d_1)u_2+(b_2-d_2)u_1+u_1u_2}, 0, \frac{u_1u_2}{(b_1-d_1)u_2+(b_2-d_2)u_1+u_1u_2});$$

$q_0(x_1) \circledcirc_{S_0} q_0(x_2) = q_0(q_0(x_1) \circledcirc q_0(x_2)) =$

$$q_0(\frac{(b_1-d_1)u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}, \frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}, \frac{u_1u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}) =$$

$$(\frac{\frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}-\frac{(b_1-d_1)u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}}{2\frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}-1}, 0, \frac{\frac{(b_1-d_1)u_2}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}+\frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}-1}{2\frac{(d_2-b_2)u_1}{u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}-1}) =$$

$$(\frac{(d_2-b_2)u_1-(b_1-d_1)u_2}{2(d_2-b_2)u_1-u_1(1-2b_2)-u_2(1-2d_1)+u_1u_2}, 0, \frac{(b_1-d_1)u_2+(d_2-b_2)u_1-u_1(1-2b_2)-u_2(1-2d_1)+u_1u_2}{2(d_2-b_2)u_1-u_1(1-2b_2)-u_2(1-2d_1)+u_1u_2}) =$$

$$(\frac{(b_2-d_2)u_1+(b_1-d_1)u_2}{2(b_2-d_2)u_1+u_1(1-2b_2)+u_2(1-2b_1)-u_1u_2}, 0, \frac{-(b_1-d_1)u_2+(b_2-d_2)u_1+u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}{2(b_2-d_2)u_1+u_1(1-2b_2)+u_2(1-2d_1)-u_1u_2}) =$$

$(\frac{(b_2-d_2)u_1+(b_1-d_1)u_2}{2(b_2-d_2)u_1+u_1(1-b_2-d_2-(b_2-d_2))+u_2(1-b_1-d_1-(b_1-d_1))-u_1u_2}, 0,$
$\frac{-(b_1-d_1)u_2+(b_2-d_2)u_1+u_1(1-b_2-d_2-(b_2-d_2))+u_2(1-b_1-d_1+(b_1-d_1))-u_1u_2}{2(b_2-d_2)u_1+u_1(1-b_2-d_2-(b_2-d_2))+u_2(1-b_1-d_1-(b_1-d_1))-u_1u_2}) =$

$(\frac{(b_2-d_2)u_1+(b_1-d_1)u_2}{2(b_2-d_2)u_1+u_1u_2-u_1(b_2-d_2))+u_1u_2+u_2(b_1-d_1)-u_1u_2}, 0, \frac{-(b_1-d_1)u_2+(b_2-d_2)u_1+u_1u_2-u_1(b_2-d_2)+u_1u_2+u_2(b_1-d_1)-u_1u_2}{2(b_2-d_2)u_1+u_1u_2-u_1(b_2-d_2))+u_1u_2+u_2(b_1-d_1)-u_1u_2}) =$

$(\frac{(b_2-d_2)u_1+(b_1-d_1)u_2}{(b_2-d_2)u_1+u_2(b_1-d_1)+u_1u_2}, 0, \frac{u_1u_2}{(b_2-d_2)u_1+u_2(b_1-d_1)+u_1u_2}) = q_0(x_1 \circledcirc_{S_0} x_2)$.
Analogically we can compute $q_0(x_1) \circledcirc_{S_0} q_0(x_2)$ for $x_1 \circledcirc x_2 \in J_2$. The cases of $x_1 \in J_2$ & $x_2 \in J_1$ follow commutativity of $\circledcirc$. Hence $q_0(x_1 \circledcirc_{S_0} x_2) = q_0(x_1) \circledcirc_{S_0} q_0(x_2)$ for any $x_1, x_2 \in J_0$. □

Remark 2 *We can define $q_0(a, 1-a, 0) = (1, 0, 0)$ for $a > \frac{1}{2}$ and $q_0(a, 1-a, 0) = (0, 1, 0)$ for $a < \frac{1}{2}$ and extend the statements of Lemma 5 from J_0, J_1, J_2 to $D_0 - 0', D_1 - 0', D_2 - 0'$ and Theorem 2 (ii) and (iii) from S_i to S_i^+.*

Proof (of the theorem): (i) D_0 is closed with respect to the consensus operator $\circledcirc$, there are no Bayesian opinions in J_0, thus J_0 is also closed. Commutativity and associativity we have already proved on D_0. From Lemma 1(iv) we have that $0 = (0, 0, 1)$ is the only idempotent. Monotonicity of the ordering $\leq_{qr}$ based on homomorphism q_0 follows Lemata 4(ii) and 5(ii).
$\leq_{pr}$ does not keep the monotonicity condition: let $x_1 = (0.5, 0.1, 0.4)$, and $x_2 = (0.6, 0.3, 0.1)$, (as in the proof of Lemma 4(ii)), and $x = (0.4, 0.4, 0.2)$, thus $p(x_1) = p(0.5, 0.1, 0.4) = (0.7, 0.3, 0) >_{pr} (0.65, 0.35, 0) = p(0.6, 0.3, 0.1) = p(x_2)$, hence $x_1 > x_2$, $x_1 \circledcirc x + (\frac{0.5 \cdot 0.2 + 0.4 \cdot 0.4}{0.4+0.2-0.08}, \frac{0.02+0.16}{0.52}, \frac{0.08}{0.52}) = (\frac{26}{52}, \frac{18}{52}, \frac{8}{52})$, $x_2 \circledcirc x + (\frac{0.6 \cdot 0.2 + 0.4 \cdot 0.1}{0.1+0.2-0.02}, \frac{0.06+0.04}{0.28}, \frac{0.02}{0.28}) = (\frac{16}{28}, \frac{10}{28}, \frac{2}{28})$, and $p(x_1 \circledcirc x) = p(\frac{26}{52}, \frac{18}{52}, \frac{8}{52}) = p(\frac{30}{52}, \frac{22}{52}, 0) = (0.5769, 0.4231, 0)$, $p(x_2 \circledcirc x) = p(\frac{16}{28}, \frac{10}{28}, \frac{2}{28}) = p(\frac{17}{28}, \frac{11}{28}, 0) = (0.6071, 0.3929, 0)$, hence $p(x_1 \circledcirc x) < p(x_1 \circledcirc x)$ and $(\mathrm{x}_1 \circledcirc x) <_{pr} (x_1 \circledcirc x)$.
(ii) See Lemma 4(iv).
(iii) $S_0 = S_1 \cup S_2$ is subset of J_0, J_0 is closed and q_0 maps J_0 back to S_0, thus $S_0 = S_1 \cup S_2$ is closed with respect to $\circledcirc$. $\circledcirc$ is commutative and associative with neutral element 0. $(b, 0, 1-b) \circledcirc_{S_0} (0, b, 1-b) = q_0((b, 0, 1-b) \circledcirc (0, 1, 1-b)) = q_0(\frac{b(1-b)+0}{1-b+1-b-(1-b)(1-b)}, \frac{0+b(1-b)}{1-b^2}, \frac{(1-b)^2}{1-b^2} = (0, 0, 1)$. Monotonicity follows statement (i).
(iv) It again follows Lemata 4(ii) and 5(ii).
(v) It follows Lemma 4(iii) and closeness of J_0.
(vi) It holds that $q(x \circledcirc y) = 0' = 0' \circledcirc 0' = q(x) \circledcirc q(x)$ and $0' \in G$. Thus $q : J_0 \to G$ is a trivial homomorphism.
Let t be a homomorphic transformation. For any opinion x from J_0 we have: $t(x) = t(0 \circledcirc x) = t(0) \circledcirc t(x)$, there is no neutral element on G, hence it should be $t(x) = t(0)$. t is a trivial transformation from J_0 to G, thus it does not satisfy the definition of Bayesian transformation. For any opinion x from J_0 and Bayesian opinion y we have: $t(y) = t(x \circledcirc y) = t(x) \circledcirc t(y)$, there is no neutral element on G, hence it should be $t(x) = t(y)$. Thus we have $t(x) = t(y) = t(0)$ for $x \in J_0, y \in G$ (i.e. $t(z) = t(0)$ for $z \in D_0$, hence t is a trivial transformation again which is not a Bayesian transformation.
□

7 A Comparison of Jøsang's Semigroup with Dempster's one

Both the algebraic structures have the following **similarities:**
Both of them are ordered Abelian semigroups with neutral element $(0, 0, 1)$.
There is the same unary operation minus $-$ which is not inverse in both the cases.
Both the structures have subsemigroups S, S_1, S_2 with neutral elements.
Both of them have a surjective homomorphism $D_0 \longrightarrow S$.
We can define group S_0 on subsets $S_0 = S_1 \cup S_2$ of both the structures (with $\oplus \circ h_0$, -, and $\leq$, in the case of D_0, while $\circledcirc \circ q_0$, -, and $\leq_{qr}$ in the case of J_0).
In both the cases there exist surjective ordered homomorphism onto group S_0.
Both the operations $\oplus$ and $\circledcirc$ are expressable using the pair of homomorphisms.

Differences:
Demspter's semigroup is defined on all non-extremal d-pairs, $\oplus$ is not defined for $\top \oplus \bot$, while Jøsang's semigroup is defined on Bayesian d-pairs only, i.e. on $D_0 - G$.
On the other hand the consensus operator $\circledcirc$ is defined on the whole extended D_0^+ but it is necessary to use additional information to obtain its associativity.
$S_0^{\oplus} = (S_0, \oplus_{S_0}, -, 0, \leq)$ is isomorphic to $G = (G, \oplus, -, 0', \leq)$ while
$S_0^{\circledcirc} = (S_0, \circledcirc_{S_0}, -, 0, \leq_{qr})$ collapses to $\{0'\}$.

$\oplus$ forms a OAG on G, while behaviour of $\circledcirc$ is completely different on G:
$\circledcirc$ is not associative on the set G,
$0' = (\frac{1}{2}, \frac{1}{2})$ is not neutral element,
all Bayesian opinions are absorbing with respect to non-Bayesian,
extremal elements (absolute opinions) are not absorbing with respect to Bayesian.

We have to remember a different interpretations of uncertainty here. In the Dempster's semigroup certainty / uncertainty of d-pair x is defined as $h(x)$, especially for Bayesian d-pair $y = (b, d, 0)$ the value b is just the certainty / uncertainty of y. Value $f(x)$ corresponds to vagueness / impreciseness of x. Bayesian d-pairs are precise, while $0 = (0, 0, 1)$ is the most vague d-pair. This corresponds also to general consideration of probability as a tool for uncertainty processing.

In the opinion space interpretation Bayesian opinions have no uncertainty, they are considered to be certain. And uncertainty increases with a distance from Bayesians.

The principal is the following.
$\oplus$ combination of any two elements (d-pairs / opinions) is on an ellipse further from 0 (closer to G), and similarly, $\circledcirc$ combination of any two elements is on a straight line (r-line) further from 0. I.e. the measure $u = 1 - b - d$ is decreased by the combination, regardless on its interpretation (vagueness / uncertainty).
Both combinations $\oplus$ and $\circledcirc$ of two elements $\geq 0'$ (or two ones ≤ 0) are on homomorphic straight lines (h-lines, q-lines) further from S. In the case of the Dempster's semigroup, we can interpret it as that big values (close to $(1, 0, 0)$, d-pairs $\geq 0'$) are increased (closer to $(1, 0, 0)$), while small values are decreased (closer to $(0, 1, 0)$). It is caused by a cumulative nature of the Dempster's rule $\oplus$. There is no such an interpretation in the case of Jøsang's semigroup. It is caused by an averaging nature of the consensus operator $\circledcirc$.
Example: $(0.8, 0.1, 0.1) \circledcirc (0.6, 0.3, 0.1) = (\frac{14}{19}, \frac{4}{19}, \frac{1}{19},) = (0.7368, 0.2105, 0.0526)$, thus Bayesian transformation of the beliefs should be in the following intervals $[0.8, 0.9]$, $[0.6, 0.7]$ and $[0.7368, 0.7895]$ respectively, hence the value of the first opinion is decreased.

8 Conclusions and Perspectives

A new algebraic structure — Jøsang's semigroup — is defined on a binary frame of discernment. Jøsang's semigroup and related structures are analysed in this text. It is compared with the analogically constructed Dempster's semigroup.

The analysis of an algebraic nature of the consensus operator moves us on to better and deeper understanding of this operator and also understanding of combining of several beliefs in general.

The main theoretical disadvantage of the present state of the consensus operator is its non-associativity on dogmatic beliefs. This problem was already partially solved with using of an additive information, see an example of associative consensus of three dogmatic beliefs in [8]. On the other hand theoretically clean associative consensus of several dogmatic beliefs is still an interesting open problem.

The other interesting topic for a future research is a comparison of the focusing of a frame of discernment introduced by Jøsang, see [13], [14], with the approach of refinement / coarsening of a frame of discernment suggested in [5] and used in [6].

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