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Abstract:

Based on a new demand — the commutativity of belief functions combination with refinement / coarsening of the frame of discernment — the role of the disjunctive rule of combination has increased. To compare the nature of this rule with a more frequent but also more controversional one, i.e. with Dempster's rule, an algebraic analysis was used.

The basic necessary definitions both from the Dempster-Shafer theory and from algebra are recalled. An algebraic investigation of the Dempster's semigroup — the algebraic structure of binary belief functions with the Dempster's rule of combination is briefly recalled as well.

After this, a new algebraic structure of binary belief functions with the disjunctive rule of combination is defined. The structure is studied, and the results are discussed in a comparison with those ones of the classical Dempster's rule.

In the end, an impact of new algebraic results to the field of decision making and some ideas for future research are presented.

Keywords:
Belief functions, Dempster-Shafer theory, Combination of belief functions, Dempster's semigroup, Expert systems, Decision making under uncertainty. .

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1 Introduction

When combining two or more belief functions, there are generally accepted requirements of associativity and commutativity of an operation of their combination. A new requirement of commutativity of a combination with refinement/coarsening of the frame of discernment was introduced in [8]. There are three sources of this requirement: it arises from some applications of belief functions (namely in cases of subjective beliefs which are not constructed from probabilities), furthermore it arises from logical studies on belief functions, see [11], and it is motivated by the utilization of a method of representing a n-dimensional belief function by a set of two-dimensional ones, see [7].

The classical Dempster-Shafer theory uses the Dempster's (conjunctive) rule of combination $\oplus$, while the Transferable Belief Model [16, 17] uses its non-normalized version $\textcircled{\cap}$. To meet the new requirement it is necessary to use the disjunctive rule of combination $\textcircled{\cup}$, which is the only known associative and commutative combination of belief functions which commutes with coarsening of the frame of discernment (while $\oplus$ and $\textcircled{\cap}$ commute with refinement only).

An algebraic structure of binary belief functions with Demspter's rule $\oplus$, called the Dempster's semigroup, was in detail studied in a series of publications, e.g. [1, 4, 12, 13, 18]. The new importance of the disjunctive rule of combination $\textcircled{\cup}$ is the motivation for a study of algebraic structures of belief functions with $\textcircled{\cup}$ to obtain a better theoretical comparison of both approaches.

The next section briefly recalls the basic definitions. An algebraic analysis of the Dempster's semigroup which is used as a methodology for the presented investigation is overviewed in the third section.

In Section 4, a new algebraic structure — the algebraic structure of belief functions with operation of combination $\textcircled{\cup}$ (disjunctive rule of combination) — is defined. The structure is analyzed there. The results are discussed and compared with those of the Dempster's semigroup in Section 5.

In Section 6, the disjunctive rule of combination $\textcircled{\cup}$ is considered from a decision making approach and its impact to this area is presented. In the end, some ideas for future research are outlined as well.

2 Preliminaries

Let us recall some basic algebraic notions and some basic notions from the Dempster-Shafer theory before we begin a description of its algebra.

A *commutative semigroup* (called also an *Abelian semigroup*) is a structure $\mathbf{X} = (X, \oplus)$ formed by the set X and a binary operation $\oplus$ on X which is commutative and associative ($x \oplus y = y \oplus x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ holds for all $x, y, z \in X$). A *commutative group* is a structure $\mathbf{X} = (X, \oplus, -, o)$ such that $(Y, \oplus)$ is a commutative semigroup, o is a neutral element ($x \oplus o = x$) and $-$ is a unary operation of the inverse ($x \oplus -x = o$). An *ordered Abelian (semi)group* consists of a commutative (semi)group $\mathbf{X}$ as above and a linear ordering $\leq$ of its elements satisfying monotonicity ($x \leq y$ implies $x \oplus z \leq y \oplus z$ for all $x, y, z \in X$). A subset of X which is a (semi)group itself is called a *sub(semi)group*. A subsemigroup $(\{x | x \geq o, x \in X\}, \oplus, o)$ is called *a positive cone* of the ordered Abelian group (OAG) X, similarly *a negative cone* of OAG Y for $x \leq o$.

For uncertainty processing, we extend OAG with *extremal elements* $\top$ and $\bot$ representing *True* and *False*, $\top \oplus x = \top$, $\bot \oplus x = \bot$, $\top \oplus \bot$ not defined. [3]

A *homomorphism* $p : (X, \oplus_1) \longrightarrow (Y, \oplus_2)$ is a mapping which preserves structure, i.e. $p(x \oplus_1 y) = p(x) \oplus_2 p(y)$ for each $x, y \in X$. The special cases are *automorphisms*, which are bijective morphisms from a structure onto itself, while *endomorphisms* are morphisms to a substructure of the original one.

Morphisms which also preserve ordering of elements are called *ordered morphisms*, see [10]. *Ordered automorphisms (o-automorphisms)* are ordered morphisms back to the original structures, analogically *o-endomorphisms* are ordered morphisms to a substructure of the original one.

Ordered structures and ordered morphisms are very important for a comparative approach to uncertainty management and decision making.

[3] Some examples are OAG$^+$ $\mathbf{PP} = ([0,1], \oplus_{PP}, 1-x, \frac{1}{2}, \leq)$ and $\mathbf{MC} = ([-1,1], \oplus_{MC}, -, 0, \leq)$ corresponding to the combining structures of the classical expert systems PROSPECTOR and EMYCIN, see [12], where $x \oplus_{PP} y = \frac{xy}{xy+(1-x)(1-y)}$ and $x \oplus_{MC} y = x + y - xy$ for $x, y \geq 0$, $x + y + xy$ for $x, y \leq 0$ and $\frac{x+y}{1-min(|x|,|y|)}$ for $xy \leq 0$.

Let us consider a two-element frame of discernment $\Theta = \{0,1\}$. A *basic belief assignment* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$. A *belief function* is a mapping $bel : \mathcal{P}(\Theta) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. In our special case $bel(1) = m(1)$, $bel(0) = m(0)$, $bel(\{0,1\}) = m(1) + m(0) + m(\{0,1\}) = 1$.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are singletons (i.e. one-element subsets of Ω), then we speak about *Bayesian belief function.* A *Bayesian transformation* is a mapping $t : Bel_\Omega \longrightarrow Prob_\Omega$, such that $bel(x) \leq t(bel)(x) \leq 1-bel(\overline{x})$. Thus the Bayesian transformation assigns a Bayesian belief function (i.e. probability fuction) to every general one. The fundamental example of Bayesian transformation is the pignistic transformation introduced by Smets.

The Dempster's conjunctive rule of combination is given as $(bel_1 \circledcirc bel_2)(A) = \sum_{X\cap Y=A} \frac{1}{K} m_1(X)m_2(Y)$, where $K = \sum_{X\cap Y\neq\emptyset} m_1(X)m_2(Y)$, see [15], while *the disjunctive rule of combination* is given by the formula $(bel_1 \circledcirc bel_2)(A) = \sum_{X\cup Y=A} m_1(X)m_2(Y)$, see [9]. Specially for $(m_1(1), m_1(0)) = (a,b)$, $(m_2(1), m_2(0)) = (c,d)$ we have $(a,b) \oplus (c,d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ and $(a,b) \circledcirc (c,d) = (ac, bd)$.

3 On the Dempster's semigroup

Now we introduce some principal notions according to [12]. For a two-valued frame of discernment $\Theta = \{0,1\}$ each basic belief assignment determines a d-pair $(m(1), m(0))$ and conversely, each d-pair determines a basic belief assignment:

Definition 1 *A Dempster's pair (or d-pair) is a pair of reals such that $a, b \geq 0$ and $a + b \leq 1$. A d-pair (a,b) is Bayesian if $a+b = 1$, (a,b) is simple if $a = 0$ or $b = 0$, in particular, extremal d-pairs are pairs (1,0) and (0,1). (Definitions of Bayesian and simple d-pairs correspond evidently to the usual definitions of Bayesian and simple belief assignments [12], [15]).*

Definition 2 *The* (standard/conjunctive) Dempster's semigroup $\mathbf{D}_0 = (D_0, \oplus)$ *is the set of all non-extremal Dempster's pairs, endowed with the operation $\oplus$ and two distinguished elements $0 = (0,0)$ and $0' = (\frac{1}{2}, \frac{1}{2})$, where the operation $\oplus$ is defined by*

$$(a,b) \oplus (c,d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)}). \tag{3.1}$$

Remark 1 *It is well known that this operation corresponds to the Dempster's rule (non-normalized conjunctive rule) of combination of basic belief assignments on a binary frame of discernment, see [12].*

Remark 2 *$\oplus$-sum of two d-pairs $(a,b) \oplus (c,d)$ is not defined if and only if (a,b) and (c,d) are two different extremal d-pairs (the denominators are zeros). We can simply derive expressions of Dempster's rule for Bayesian d-pairs (i), simple d-pairs with the same (ii) and different (iii) focal elements, and for d-pairs such that $a = b$ (iv), from the basic form of the rule.*
(i) $(a, 1-a) \oplus (c, 1-c) = (\frac{ac}{ac+(1-a)(1-c)}, \frac{(1-a)(1-c)}{ac+(1-a)(1-c)})$,
(ii) $(a,0) \oplus (c,0) = (a+c-ac, 0)$,
$(0,b) \oplus (0,d) = (0, b+d-bd)$,
(iii) $(a,0) \oplus (0,d) = (\frac{a-ad}{1-ad}, \frac{d-ad}{1-ad})$,
(iv) $(a,a) \oplus (c,c) = (\frac{a+c-3ac}{1-2ad}, \frac{a+c-3ac}{1-2ad})$.

Definition 3 *For $(a,b) \in \mathbf{D}_0$ we define*

$$\begin{aligned}
-(a,b) &= (b,a),\\
h(a,b) &= (a,b) \oplus 0' = (\tfrac{1-b}{2-a-b},\ \tfrac{1-a}{2-a-b}),\\
h_1(a,b) &= \tfrac{1-b}{2-a-b},\\
f(a,b) &= (a,b) \oplus (b,a) = (\tfrac{a+b-a^2-b^2-ab}{1-a^2-b^2},\ \tfrac{a+b-a^2-b^2-ab}{1-a^2-b^2}).
\end{aligned}$$

For $(a, b), (c, d) \in \mathbf{D}_0$ we further define
$(a, b) \leq (c, d)$ *iff* $h_1(a, b) < h_1(c, d)$ *or if* $h_1(a, b) = h_1(c, d)$ *and* $a \leq c$.

Let G denote the set of all Bayesian non-extremal d-pairs. Let us denote the set of all simple d-pairs such that $b = 0$ ($a = 0$) as S_1 (S_2). Furthermore, put
$S = \{(a, a) : 0 \leq a \leq 0.5\}$.
(Note: $h(a, b)$ is an abbreviation for $h((a, b))$, etc.)

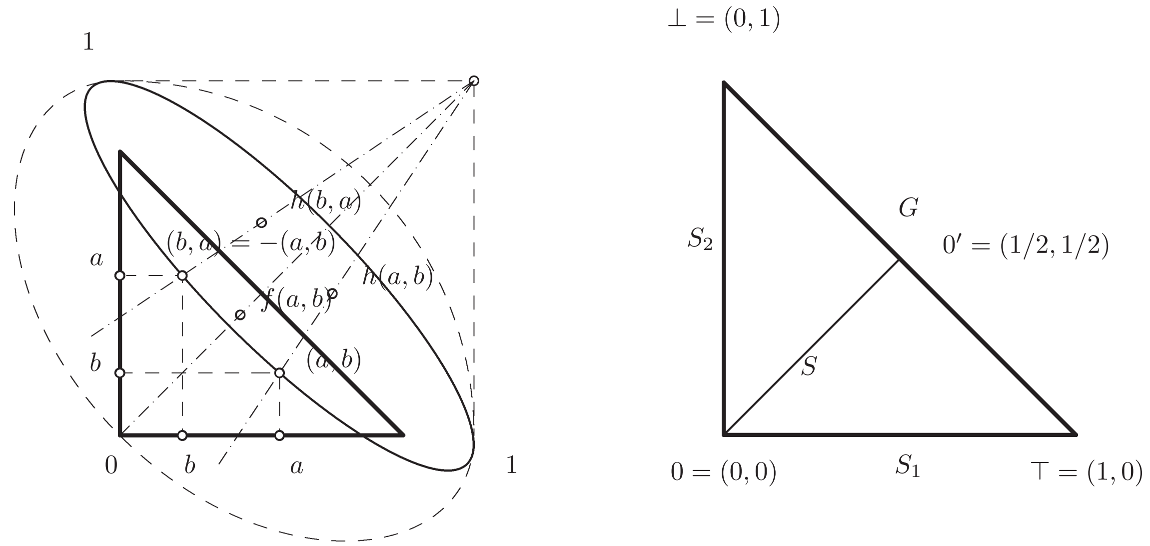

Figure 3.1: **Dempster's semigroup.** Homomorphism h is, in this representation, a projection to group G along the straight lines running through the point $(1, 1)$. All Dempster's pairs lying on the same ellipse are, by homomorphism f, mapped to the same d-pair in semigroup S.

Lemma 1 *Let x, y (or $(a, b), (c, d)$) be elements of the Dempster's semigroup. The following holds:*
(i) $-(x \oplus y) = -x \oplus -y$ *(i.e.* $-((a, b) \oplus (c, d)) = (b, a) \oplus (d, c)$*),*
(ii) $-(-x) = x$ *(i.e.* $-(-(a, b)) = (a, b)$*),*
(iii) $-x$ is not an inverse to x, i.e. the equation $(a, b) \oplus (c, d) = (0, 0)$ has no solution in $\mathbf{D}_0$ for $(a, b) \neq (0, 0)$,
(iv) $h(x) = 0'$ if and only if $x = -x$ if and only if $0 \leq x \leq 0'$ if and only if $x \in S$.

For proofs see [13, 18].

Theorem 1

(i) The Dempster's semigroup with the relation $\leq$ is an ordered commutative semigroup with the neutral element 0; $0'$ is the only nonzero idempotent of it.

(ii) The set G with the ordering $\leq$ is an ordered Abelian group $(G, \oplus, -, 0', \leq)$ which is isomorphic to the PROSPECTOR group **PP** *(cf. [12]) and consequently isomorphic to the additive group of reals with usual ordering.*

(iii) The sets S, S_1 and S_2 with the operation $\oplus$ and the ordering $\leq$ form ordered commutative semigroups with neutral element 0, and are all isomorphic to the semigroup of nonnegative elements (positive cone) of the MYCIN group **MC**.

(iv) The mapping h is an ordered homomorphism of the ordered Dempster's semigroup onto its subgroup G (i.e. onto **PP***).*

(v) The mapping f is a homomorphism of the Dempster's semigroup onto its subsemigroup S (but it is not an ordered homomorphism).

For proofs see [12], [13], [18].

Note: ad (ii): A mapping $p_1(a, a-1) = a$ is the ordered isomorphism of G onto the PROSPECTOR group **PP** on interval $(0, 1)$,
ad (iii): 0 is the least element of the semigroups; it is extended semigroup (with the greatest and absorbing element $0'$) in the case of S. A mapping $p_1(a, 0) = a$ is the ordered isomorphism of S_1 onto a semigroup of non-negative, non-extremal elements of the MYCIN group **MC**, while $p_2(0, b) = b$ is the ordered isomorphism of S_2 to the same semigroup (the positive cone of **MC**).
ad (iv): h-preimages of a d-pair $(a, 1-a)$ lie on the straight line running through the pairs $(a, 1-a)$ and $(1, 1)$, i.e. $h^{-1}(a, 1-a)$ is the intersection of this line with D_0.
ad (v): Pairs with the same f-image $f(a, b)$ lie on an ellipse running through pairs $(0, 1)$, $(1, 0)$, and (a, b), where the main axis of the ellipse is parallel with the abscissa (segment of straight line) $(0, 1), (1, 0)$, hence $f^{-1}(f(a, b))$ is the intersection of this ellipse with D_0.

Using the theorem, see (iv) and (v), we can see that the Dempster's rule is effected 'per h-lines and f-ellipses', and we can express $(a \oplus b) \in h^{-1}(h(a) \oplus h(b))$, and $(a \oplus b) \in f^{-1}(f(a) \oplus f(b))$, hence we obtain the following equation

$$(a \oplus b) = h^{-1}(h(a) \oplus h(b)) \cap f^{-1}(f(a) \oplus f(b)). \tag{3.2}$$

Lemma 2 *Let x, y, z (or $(a, b), (c, d), (e, f)$) be elements of the Dempster's semigroup. The following holds:*
(i) $x \oplus z < y \oplus z$ for $x < y$ & $z \notin G$,
(ii) if $x < y$ and $h(x) = h(y)$ then $f(x) < f(y)$.
(iii) $\lambda(a, b) = (h(a, b), f(a, b))$ is an one-to-one mapping of $\mathbf{D}_0$ into $G \times S$ but not onto $G \times S$.
Let $(g, 1-g) \in G$ and $(s, s) \in S$. There exist $(a, b) \in \mathbf{D}_0$ such that $\lambda(a, b) = ((g, 1-g), (s, s))$ if and only if it holds $\frac{1-2s}{2-3s} \le g \le \frac{1-s}{2-3s}$.

For proofs see [13].

Corollary 1 *All the three semigroups $S - \{0'\}$, S_1, S_2 are ordered Abelian semigroups with subtraction, i.e. for $(a, b), (c, d) \in S_i$, $(a, b) < (c, d)$ there exist d-pair (e, f) in S_i such that $(a, b) \oplus (e, f) = (c, d)$.*

The statement follows part (iii) from the previous theorem.

A generalization of a notion of the Dempster's semigroup is described in [18], see also [12]. The resulting algebraic structure is called a dempsteroid. It has a similar relation to the Dempster's semigroup as OAG has to **PP** or **MC**.

Definition 4 *A dempsteroid is an algebra $\mathbf{D} = (D, \oplus, \ominus, o, o', \le)$ satisfying the following:*

1. *$(D, \oplus, o, \le)$ is an ordered commutative semigroup with o as a neutral element,*
2. *$\ominus(\ominus x) = x$,*
   $\ominus(x \oplus y) = (\ominus x) \ \oplus \ (\ominus y)$ for each $x, y \in D$,
3. *$o \le o'$,*
4. *for each $x \in D$: $o \le x \le o'$ iff $x \oplus o' = o'$ iff $x = \ominus x$, the set of all x satisfying any of these conditions is denoted by S.*
5. *for each $x, y \in S$ such that $x \le y$ there exists $z \in S$ such that $x \oplus y = z$ (subtraction in S).*

Let us define mappings h and f on dempsteroid D: $h(x) = x \oplus o'$, $f(x) = x \oplus (\ominus x)$ and let us denote G the set $G = \{x \oplus o' : x \in D\}$.

Definition 5 *The standard dempsteroid $\mathbf{D}_0 = (D_0, \oplus, -, 0, 0', \le)$ is the dempsteroid defined by the Dempster's semigroup, by the operation $-$, and by the ordering $\le$, see definition 3.*

In order to use dempsteroids in uncertain information processing, it is necessary to enrich the defined algebraic structures with extremal elements:

Definition 6 *An extended dempsteroid* $\mathbf{D}^+ = (D \cup \{\perp, \top\}, \oplus, \ominus, o, o', \leq)$ *is an algebraic structure resulting from taking a dempsteroid and adding extremal elements* $\perp, \top$ *in the following way:*

$$\begin{array}{cccc} x \oplus \perp = \perp & \text{and} & x \oplus \top = \top & \text{for all } x \in D, \\ \ominus\perp = \top & \text{and} & \ominus\top = \perp, & \\ \perp \leq x \leq \top & & & \text{for all } x \in D. \end{array}$$

For the standard dempsteroid let us define $\perp = (0, 1)$*,* $\top = (1, 0)$*.*

4 The Disjunctive Dempster's semigroup

Let us turn our attention to an algebra of belief functions on a binary frame of discernment with the disjunctive rule of combination $\circledcirc$. As $\circledcirc$ is a commutative and associative operation, see lemma bellow, we can speak about an Abelian semigroup again.

Because of the different nature of the operation, $0' = (\frac{1}{2}, \frac{1}{2})$ does not play the analogical role as in the case of the (standard) Dempster's semigroup ($0'$ is not as idempotent). The other idempotent $0 = (0, 0)$ of the Dempster's semigroup is idempotent again, but it is not neutral element in this case. To obtain a neutral element we add to D_0 a technical pair $1 = (1, 1)$ which is not a d-pair (it does not correspond to any basic belief assignment). Analogically, it is useful to consider all pairs (a, a) for $a \geq 1$ (or for all $a \geq 0$, where (a, a) for $\frac{1}{2} < a < 1$ do not play any important role in the presented theory).

Lemma 3 *The disjunctive rule of combination (the operation* $\circledcirc$*) is commutative and associative.*

Proof: In a general case we have: $(m_1 \circledcirc m_2)(A) = \sum_{X \cup Y = A} m_1(X)m_2(Y) = \sum_{X \cup Y = A} m_2(Y)m_1(X) = (m_2 \circledcirc m_1)(A)$, and
$((m_1 \circledcirc m_2) \circledcirc m_3)(A) = \sum_{W \cup Z = A} (m_1 \circledcirc m_2)(W)m_3(Z) = \sum_{W \cup Z = A} (\sum_{X \cup Y = W} m_1(X)m_2(Y))m_3(Z) =$
$\sum_{W \cup Z = A} \sum_{X \cup Y = W} m_1(X)m_2(Y)m_3(Z) = \sum_{W \cup Z = A \& X \cup Y = W} m_1(X)m_2(Y)m_3(Z) =$
$\sum_{X \cup V = A} \sum_{Y \cup Z = V} m_1(X)m_2(Y)m_3(Z) = \sum_{X \cup V = A} m_1(X)(\sum_{Y \cup Z = V} m_2(Y)m_3(Z)) =$
$\sum_{X \cup V = A} m_1(X)(m_2 \circledcirc m_3)(V) = (m_1 \circledcirc (m_2 \circledcirc m_3))(A)$.
Specially for $\Omega = \{0, 1\}$ the following holds: $(a, b) \circledcirc (c, d) = (ac, bd) = (ca, db) = (c, d) \circledcirc (a, b)$, and $((a, b) \circledcirc (c, d)) \circledcirc (e, f) = (ac, bd) \circledcirc (e, f) = (ace, bdf) = (a, b) \circledcirc (ce, df) = (a, b) \circledcirc ((c, d) \circledcirc (e, f))$. □

Definition 7 *The disjunctive Dempster's semigroup* $\mathbf{D}_{\circledcirc} = (D_0 \cup \{(1, 0), (0, 1), (1, 1)\}, \circledcirc)$ *is the set of all Dempster's pairs extended by* $1 = (1, 1)$*, endowed with the operation* $\circledcirc$ *and two distinguished elements* $0 = (0, 0)$ *and* $1 = (1, 1)$*, where the operation* $\circledcirc$ *is defined by*

$$(a, b) \circledcirc (c, d) = (ac,\ bd). \tag{4.1}$$

Remark 3 *(i)* $\circledcirc$*-sum of two d-pairs* $(a, b) \circledcirc (c, d)$ *is defined for all d-pairs from* $\mathbf{D}_{\circledcirc}$*.*
(ii) $0 = (0, 0)$ *is not a neutral element in* $\mathbf{D}_{\circledcirc}$ *(*$(0, 0) \circledcirc (a, b) = (0a, 0b) = (0, 0)$*),*
(iii) $0' = (\frac{1}{2}, \frac{1}{2})$ *is not idempotent in* $\mathbf{D}_{\circledcirc}$ *(*$(\frac{1}{2}, \frac{1}{2}) \circledcirc (a, b) = (\frac{a}{2}, \frac{b}{2})$*).*

Remark 4 *Analogically to the Dempster's semigroup, we can simply derive expressions of disjunctive rule for Bayesian d-pairs (i), simple d-pairs with the same (ii) and different (iii) focal elements, and for d-pairs such that* $a = b$ *(iv), from the basic form of the rule.*

(i) $(a, 1 - a) \circledcirc (c, 1 - c) = (ac, (1 - a)(1 - c))$,
(ii) $(a, 0) \circledcirc (c, 0) = (ac, 0)$,
$(0, b) \circledcirc (0, d) = (0, bd)$,
(iii) $(a, 0) \circledcirc (0, d) = (0a, 0d) = (0, 0)$,
(iv) $(a, a) \circledcirc (c, c) = (ac, ac)$.

Definition 8 *For* $(a, b) \in \mathbf{D}_{\circledcirc}$ *we define*

$$
\begin{aligned}
-(a,b) &= (b,a),\\
u(a,b) &= (a,b)\circledcirc(\tfrac{1}{a+b},\ \tfrac{1}{a+b}) = (\tfrac{a}{a+b},\ \tfrac{b}{a+b}),\\
u_1(a,b) &= \tfrac{a}{a+b},\\
v(a,b) &= (a,b)\circledcirc(b,a) = (ab,ab).
\end{aligned}
$$

For $(a,b),(c,d) \in \mathbf{D}_{\circledcirc}$ *we further define*
$(a,b) \leq_{\circledcirc} (c,d)$ *iff* $u_1(a,b) < u_1(c,d)$ *or if* $u_1(a,b) = u_1(c,d)$ *and* $a \leq c$.

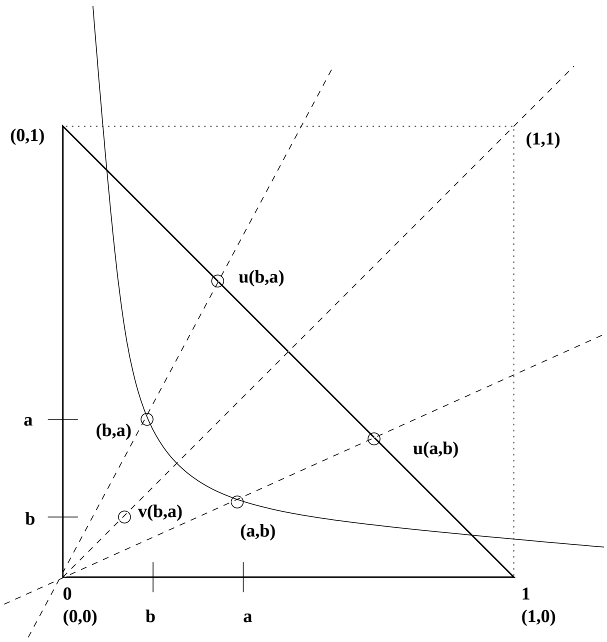

Figure 4.1: ***Disjunctive Dempster's semigroup*** *The homomorphism u is, in this representation, a projection to group G along the straight lines running through the point* $(0,0)$*. All Dempster's pairs lying on the same hyperbole are, by the homomorphism v, mapped to the same d-pair in semigroup S.*

Remark 5 $u(a,b)$ *is defined on* $(D_0 - -\{(0,0)\}) \cup \{\perp, \top\}$*, i.e. it is not defined* $u(0,0)$ *which should be* $(\frac{0}{0}, \frac{0}{0})$*.*
u is equal to identity on G, $u(a,1-a) = (\frac{a}{a+1-a}, \frac{1-a}{a+1-a}) = (a, 1-a)$*.*

Lemma 4 *Let* x, y *(or* $(a,b),(c,d)$*) be elements of the disjunctive Dempster's semigroup. The following holds:*

(o) $1 = (1,1)$ *is neutral element in* $\mathbf{D}_{\circledcirc}$*, while* $0 = (0,0)$ *is an absorbing idempotent there,* $\perp = (0,1)$ *and* $\top = (1,0)$ *are idempotents which are neither neutral nor absorbing in*

(i) $-(x \circledcirc y) = -x \circledcirc - y$ *(i.e.* $-((a,b)\circledcirc(c,d)) = (b,a)\circledcirc(d,c)$*),*

(ii) $-(-x) = x$ *(i.e.* $-(-(a,b)) = (a,b)$*),*

(iii) $-x$ *is not an inverse to* x*, i.e. the equation* $(a,b)\circledcirc(c,d) = (1,1)$ *has no solution in* $\mathbf{D}_{\circledcirc}$ *for* $(a,b) \neq (1,1)$*,*

(iv) $u(x) = 0'$ *if and only if* $0 \neq x = -x$ *if and only if* $0 < x \leq 0'$ *if and only if* $x \in S - \{0\}$*,*

(v) $x \circledcirc \top = (p_1(x), 0)$*, i.e.* $(a,b)\circledcirc(1,0) = (p_1(a,b), 0) = (a, 0)$*,*
$x \circledcirc \perp = (0, p_2(x))$*, i.e.* $(a,b)\circledcirc(0,1) = (0, p_2(a,b)) = (0, b)$*,*
projection.

Proof: (o): $(a,b)\circledcirc(1,1) = (1a,1b) = (a,b)$,
$(0,0)\circledcirc(0,0) = (0\cdot 0, 0\cdot 0) = (0,0)$, $(a,b)\circledcirc(0,0) = (0a,0b) = (0,0)$;
$(1,0)\circledcirc(1,0) = (1\cdot 1,\ 0\cdot 0) = (1,0)$, $(a,b)\circledcirc(1,0) = (1a,0b) = (a,0)$,
$(0,1)\circledcirc(0,1) = (0\cdot 0,\ 1\cdot 1) = (0,1)$, $(a,b)\circledcirc(0,1) = (0a,1b) = (0,a)$;
(i): $-((a,b)\circledcirc(c,d)) = -(ac,bd) = (bd,ac) = (b,a)\circledcirc(d,c)$;
(ii): $-(-(a,b)) = -(b,a) = (a,b)$, (the same proof as in the case of $\mathbf{D}_0$);
(iii): $(a,b)\circledcirc(c,d) = (ac,bd) = (1,1)$ iff $ac = 1$ & $bd = 1$ iff $a = c = b = d = 1$ because of $a,b,c,d \in < 0,1 >$.
(iv): $(a,b)\circledcirc(\frac{1}{a+b},\frac{1}{a+b}) = (\frac{a}{a+b},\frac{b}{a+b}) = (\frac{1}{2},\frac{1}{2})$ iff $\frac{a}{a+b} = \frac{1}{2},\ \frac{b}{a+b} = \frac{1}{2}$ iff $2a = a+b,\ 2b = a+b$ and $a+b \neq 0$ *iff* $a = b \neq 0$;
(v): $(a,b)\circledcirc(1,0) = (1a,0b) = (a,0) = (p_1(a,b),0)$,
$(a,b)\circledcirc(0,1) = (0a,1b) = (0,b) = (0,p_2(a,b))$. □

Theorem 2

(i) *The disjunctive (Dempster's) semigroup with the relation $\leq_{\circledcirc}$ is an ordered commutative semigroup with the neutral element 1; where 0, $\perp$, and $\top$ are all the other idempotents of it.*

(ii a) *The set G of Bayesian d-pairs is not closed under the operation $\circledcirc$. Hence it is not subalgebra of $\mathbf{D}_{\circledcirc}$ with respect the operation $\circledcirc$. ($\circledcirc$ maps back to G only the following d-pairs: $\perp\circledcirc\perp = \perp$ and $\top\circledcirc\top = \top$, i.e. only idempotents are mapped back to).*

(ii-b) *The set G with the ordering $\leq_{\circledcirc}$ and with the operation $\circledcirc_G = \circledcirc \circ u$, where $(a,b)\circledcirc_G(c,d) = u(ac,bd) = (\frac{ac}{ac+bd},\frac{bd}{ac+bd})$, is an ordered Abelian group $\mathbf{G}_{\circledcirc_G} = (G,\circledcirc_G,-,0',\leq_{\circledcirc})$ which is isomorphic to the PROSPECTOR group* **PP** *(cf. [12]) and consequently, it is isomorphic to the additive group of reals with usual ordering.*

(iii) *The sets $S \cup 1$, $S_1 \cup \top$ and $S_2 \cup \perp$ with the operation $\circledcirc$ and the ordering $\leq_{\circledcirc}$ form ordered commutative semigroups with neutral elements $(1,1)$, $(1,0)$, or $(0,1)$ respectively. $S \cup 1$ is isomorphically embeddable*[4] *and $S_1 \cup \top$ and $S_2 \cup \perp$ are isomorphic to the negative cone of the extended (with 0) multiplicative group of positive reals.*
Consequently they are isomorphic to the negative cone of the additive group of reals with usual ordering and to the negative cone of the PROSPECTOR group **PP** *as well.*

(iv a) *The mapping u is not a homomorphism of the disjunctive Dempster's semigroup onto its subalgebra G.*

(iv-b) *The mapping u is an ordered homomorphism of the disjunctive Dempster's semigroup onto group $\mathbf{G}_{\circledcirc_G} = (G,\circledcirc_G,-,0',\leq_{\circledcirc}$ (i.e. onto* **PP***), which isa subset of $\mathbf{D}_{\circledcirc}$.*

(v) *The mapping v is a homomorphism of the disjunctive Dempster's semigroup onto its subsemigroup S (but it is not an ordered homomorphism).*

Proof: (i): $a,b,c,d \in [0,1]$, thus $ac, bd \in [0,1]$, $ac \leq a,c$ and $bd \leq b,d$ for all $a,b,c,d \in [0,1]$, thus $ac+bd \leq a+b, c+d \leq \frac{1}{2}$, hence $D_{\circledcirc}$ is closed with respect to $\circledcirc$. Associativity and commutativity follow properties of $\circledcirc$. Neutral element 0 and idempotency of $0'$ follow (o) from the previous lemma. $(a,b)\circledcirc(a,b) = (aa,bb) = (a,b)$ iff $aa = a,\ bb = b$ iff $a,b \in \{0,1\}$, thus there are just four idempotents $0,\ 1,\ \perp$, and $\top$, $(a,b)\circledcirc(1,0) = (a,0)$, hence $\top$ is neither neutral nor absorbing, similarly for $\perp$.
(ii-a): $(a,1-a)\circledcirc(b,1-b) = (ab,(1-a)(1-b)) = (X,Y),\ X+Y = ab+1-a-b+ab,\ X+Y = 1$ iff $1 = 1+2ab-a-b$ iff $2ab-a = b$ iff $a = \frac{b}{2b-1}$ iff $a = b = 0$ or $a = b = 1$, ($a > 1$ for $\frac{1}{2} < b < 1$ and $a < 0$ for $0 < b < \frac{1}{2}$).
(ii-b): $\circledcirc_G = \circledcirc \circ u$, where $(a,b)\circledcirc_G(c,d) = u((a,b)\circledcirc(c,d)) = u(ac,bd) =$
$(\frac{ac}{ac+bd},\frac{bd}{ac+bd})$;

[4]Note: $(\{(a,a)|0 \leq a \leq 1\},\circledcirc,1,\leq)$ is isomorphic to the negative cone of the extended (with 0) multiplicative group of positive reals.

closeness: $(a,b),(c,d) \in D_{\circledcup}$, hence $(a,b)\circledcup(c,d) \in D_{\circledcup}$, thus $(a,b)\circledcup_G(c,d) = u((a,b)\circledcup(c,d)) \in G$, i.e. $\circledcup_G$ maps $\mathbf{D}_{\circledcup}$ to G, hence $G \subset \mathbf{D}_{\circledcup}$ is closed with respect to $\circledcup_G$,
commutativity: $(c,d)\circledcup_G(a,b) = u(ca,db) = (\frac{ca}{ca+db}, \frac{db}{ca+db}) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd}) = (a,b)\circledcup_G(c,d)$,
associativity: $((a,b)\circledcup_G(c,d))\circledcup_G(e,f) = (\frac{\frac{ace}{ac+bd}}{\frac{ace}{ac+bd}+\frac{bdf}{ac+bd}}, \ \frac{\frac{bdf}{ac+bd}}{\frac{ace}{ac+bd}+\frac{bdf}{ac+bd}}) =$
$(\frac{ace}{ace+bdf}, \ \frac{bdf}{ace+bdf}) = (\frac{a\frac{ce}{ce+df}}{\frac{ace}{ce+df}+\frac{bdf}{ce+df}}, \ \frac{b\frac{df}{ce+df}}{\frac{ace}{ce+df}+\frac{bdf}{ce+df}}) = (a,b)\circledcup_G((c,d)\circledcup_G(e,f))$,
neutral element: $(a,1-a)\circledcup_G(\frac{1}{2},\frac{1}{2}) = (\frac{\frac{a}{2}}{\frac{a}{2}+\frac{1-a}{2}}, \ \frac{\frac{1-a}{2}}{\frac{a}{2}+\frac{1-a}{2}}) =$
$(\frac{a}{a+1-a}, \ \frac{1-a}{a+1-a}) = (a,1-a)$,
inverse: $(a,1-a)\circledcup_G(1-a,a) = (\frac{a(1-a)}{a(1-a)+(1-a)a}, \frac{(1-a)a}{(1-a)a+a(1-a)} = (\frac{1}{2}, \frac{1}{2})$;
An isomorphism from G to $\mathbf{PP}$ is the projection $p_1(a,1-a) = a$:
$p_1((a,1-a)\circledcup_G(b,1-b)) = p_1(\frac{ab}{ab+(1-a)(1-b)}, \frac{(1-a)(1-b)}{ab+(1-a)(1-b)}) = \frac{ab}{ab+(1-a)(1-b)} = a \oplus_{PP} b = p_1(a,1-a) \oplus_{PP} p_1(b,1-b)$;
$p_1(-(a,1-a)) = p_1(1-a,a) = 1-a = -_{PP}(a) = -_{PP}(p_1(a,1-a))$;
$p_1(0') = p_1(\frac{1}{2},\frac{1}{2}) = \frac{1}{2} = 0_{PP}$;
$(a,1-a) \le (b,1-b)$ iff $u_1(a,1-a) \le u_1(b,1-b)$ iff $p_1(a,1-a) \le p_1(b,1-b)$, hence p_1 is an ordered isomorphism.

(iii): Commutativity and associativity follow properties of $\circledcup$ in all the three cases. Ordered isomorhpisms onto the positive cone of $(\mathbf{Re}_m^{>0})^+ = (Re^{>0} \cup \{0,\infty\}, \cdot, \frac{1}{x}, 1, \le)$, (i.e. $([0,1], \cdot, 1/x, 1, \le)$), are the following projections $p_1(a,a) = a$ ($p_1(a,0) = a$ and $p_2(0,a) = a$) for S (S_1 and S_2) respectively. Proofs are analogous to the case (ii-b).
$S \cup \{(1,1)\}$:
$(a,a)\circledcup(b,b) = (ab,ab) \in S \cup \{(1,1)\}$, ($0 \le a,b \le 1$ hence $0 \le ab \le 1$), i.e. $S \cup \{(1,1)\}$ is closed with respect to $\circledcup$,
$(a,a)\circledcup(1,1) = (a,a)$ — neutral element $(1,1)$ — isomorphic to $1 \in \mathbf{Re}_m^{>0}$,
$(a,a)\circledcup(\frac{1}{a},\frac{1}{a}) = (1,1)$ — inverse which is isomorphic to an element $\frac{1}{a}$ of the positive cone of the group $\mathbf{Re}_m^{>0}$, i.e. outside of $S \cup \{(1,1)\}$,
$(a,a)\circledcup(0,0) = (0,0)$ — absorbing element — $(0,0)$ is isomorphic to an extremal element 0 from $(\mathbf{Re}_m^{>0})^+$.
$S_1 \cup \{(1,0)\}$:
$(a,0)\circledcup(b,0) = (ab,0) \in S_1 \cup \{(1,0)\}$ — closeness with respect to $\circledcup$,
$(a,0)\circledcup(1,0) = (a,0)$ — neutral element $(1,0)$,
$(a,0)\circledcup(\frac{1}{a},0) = (1,0)$ — inverse $(\frac{1}{a},0)$ outside of $S \cup \{(1,0)\}$,
$(a,0)\circledcup(0,0) = (0,0)$ — absorbing element $(0,0)$.
$S_2 \cup \{(0,1)\}$:
$(0,a)\circledcup(0,b) = (0,ab) \in S_2 \cup \{(0,1)\}$ — closeness with respect to $\circledcup$,
$(0,a)\circledcup(0,1) = (0,a)$ — neutral element $(0,1)$,
$(0,a)\circledcup(0,\frac{1}{a}) = (0,1)$ — inverse $(0,\frac{1}{a})$ outside of $S \cup \{(0,1)\}$,
$(0,a)\circledcup(0,0) = (0,0)$ — absorbing element $(0,0)$.

(iv a): We know from (ii-a) that G is not closed with respect to the operation $\circledcup$, hence it is not subalgebra of $\mathbf{D}_{\circledcup}$,
moreover: $u((a,b)\circledcup(c,d)) = u(ac,bd) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd})$;
while $u(a,b)\circledcup u(c,d) = (\frac{a}{a+b}, \frac{b}{a+b})\circledcup(\frac{c}{c+d}, \frac{d}{c+d}) = (\frac{ac}{(a+b)(c+d)}, \frac{bd}{(a+b)(c+d)}) =$
$(\frac{ac}{(ac+bd+ad+bc)}, \frac{bd}{(ac+bd+ad+bc)})$;
(iv b): $u((a,b)\circledcup(c,d)) = u(ac,bd) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd})$;
$u(a,b)\circledcup_G u(c,d) = (\frac{a}{a+b}, \frac{b}{a+b})\circledcup_G(\frac{c}{c+d}, \frac{d}{c+d}) =$
$(\frac{\frac{ac}{(a+b)(c+d)}}{\frac{ac}{(a+b)(c+d)}+\frac{bd}{(a+b)(c+d)}}, \ \frac{\frac{bd}{(a+b)(c+d)})}{\frac{ac}{(a+b)(c+d)}, \frac{bd}{(a+b)(c+d)}}) = (\frac{ac}{ac+bd}, \ \frac{bd}{ac+bd}) = u((a,b)\circledcup(c,d))$;
(v): $v((a,b)\circledcup(c,d)) = v(ac,bd) = (ac,bd)\circledcup(bd,ac) = (acbd,acbd)$;
$v(a,b)\circledcup v(c,d) = ((a,b)\circledcup(b,a))\circledcup((c,d),\circledcup(d,c)) = ((ab,ab))\circledcup((cd),(cd)) =$
$(acbd,acbd) = v((a,b)\circledcup(c,d))$. □

Note: ad (ii): A mapping $p_1(a, a-1) = a$ is the ordered isomorphism of G onto the PROSPECTOR group $\mathbf{PP}$ on interval $(0, 1)$,
ad (iii): 1 is the greatest element of the semigroups; they are extended semigroup (with the least and absorbing element 0). A mapping $p_1(a, a) = a$ is the ordered isomorphism of $S \cup \{(1, 1)\}$ onto the negative cone of the extended multiplicative group of positive reals, a mapping $p_1(a, 0) = a$ is the ordered isomorphism of $S_1 \cup \{(1, 0)\}$ onto the same semigroup (the negative cone of $(\mathbf{Re}_m^{>0})^+$, i.e. $([0, 1], \cdot, 1, \leq)$), while $p_2(0, b) = b$ is the ordered isomorphism of $S_2 \cup \{(0, 1)\}$ to the negative cone of $(\mathbf{Re}_m^{>0})^+$.
ad (iv): u-preimages of a d-pair $(a, 1 - a)$ lie on the straight line running through the pairs $(a, 1 - a)$ and $(0, 0)$, i.e. $u^{-1}(a, 1 - a)$ is the intersection of this line with D_0.
ad (v): Pairs with the same v-image $v(a, b)$ lie on an hyperbole running through pairs (a, b), (b, a), and $(\sqrt{ab}, \sqrt{ab})$, hence $v^{-1}(f(a, b))$ is the intersection of this hyperbole with D_0.

Using the theorem, see (iv) and (v), we can see that the disjunctive rule of combination is effected 'per u-lines and v-hyperboles, and analogically to the conjunctive case we can express $(a \circledcirc b) \in u^{-1}(u(a) \circledcirc u(b))$, and $(a \circledcirc b) \in v^{-1}(v(a) \circledcirc v(b))$, hence we obtain the following equation:

$$(a \circledcirc b) = u^{-1}(u(a) \circledcirc u(b)) \cap v^{-1}(v(a) \circledcirc v(b)). \qquad (4.2)$$

Remark 6 *(i) We can extend G with $\{\bot, \top\}$, $\bot$, $\top$ are extremal absorbing elements of G^+.*
(ii) $(a, b) \circledcirc_G \top = \top$,
(iii) $(a, b) \circledcirc_G \bot = \bot$,
(iv) $\bot \circledcirc_G \top =$ not defined. Despite, of $\bot \circledcirc \top$ is defined and equal to 0, $\circledcirc_G$ is not defined because $u(0)$ is not defined.

Proof: (ii): $(a, b) \circledcirc_G \top = u((a, b) \circledcirc (1, 0)) = u(a, 0) = (\frac{a}{a+0}, \frac{0}{a+0}) = (1, 0) = \top$,
(iii): $(a, b) \circledcirc_G \bot = u((a, b) \circledcirc (0, 1)) = u(0, a) = (\frac{0}{a+0}, \frac{a}{a+0}) = (0, 1) = \bot$,
(iv): $\bot \circledcirc_G \top = u((0, 1) \circledcirc (1, 0)) = u(0, 0) = (\frac{0}{0+0}, \frac{0}{0+0})$ ... it is not defined,
(i): for all $0 < a < 1$ we have: $u(0, 1) = (0, 1) < (a, 1 - a) = u(a, 1 - a) < (1, 0) = u(1, 0)$, the rest follows the previous proofs.

□

Lemma 5 *Let x, y, z (or $(a, b), (c, d), (e, f)$) be elements of disjunctive Dempster's semigroup. It holds the following:*
(i) $x \circledcirc z < y \circledcirc z$ for $x < y$,
(ii) if $x < y$ and $u(x) = u(y)$ then $v(x) < v(y)$,
(iii) $\kappa(a, b) = (u(a, b), v(a, b))$ is an one-to-one mapping of $\mathbf{D}_\circledcirc - (S_1 \cup S_2)$ into $G \times S$ but not onto $G \times S$.

Remark 7 *Unlike in the case of the conjunctive Demspter's semigroup, z can be an element of G in the case of the Dempster's disjunctive semigroup, see (i). (In the case of the conjunctive Dempster's semigroup $z \notin G$ or we must use $\leq$ instead of $<$.)*
Note that κ-image of $S_1 \cup S_2$ is vacuous belief function $0 = (0, 0)$. As we do not investigate automorphisms of the Dempster's disjunctive semigroup in this text, we do not formulate any analogy of the second part of Lemma 2 (iii) here.

Proof: (i): Let us denote $x = (x_1, x_2)$, $y = (y_1, y_2)$ $z = (z_1, z_2)$. $x < y$ iff $u(x) < u(y)$ or $u(x) = u(y)$ & $x_1 < y_2$. If $u(x) < u(y)$ then $u(x \circledcirc z) = u(x) \circledcirc u(z) < u(y) \circledcirc u(z) = u(y \circledcirc z)$. If $u(x) = u(y)$ & $x_1 < y_1$ then $u(x \circledcirc z) = u(x) \circledcirc u(z) = u(y) \circledcirc u(z) = u(y \circledcirc z)$ and $p_1(x \circledcirc z) = x_1 z_1 < y_1 z_1 = p_1(y \circledcirc z)$.
(ii): If $x < y$ and $u(x) = u(y)$ then $x_1 < y_1$ and also $x_2 < y_2$ because x and y lie on the same straight line going through $(0, 0)$ and $u(x) = u(y)$, hence $v(x) = (x_1, x_2) \circledcirc (x_2, x_1) = (x_1 x_2, x_1, x_2) < (y_1 y_2, y_1 y_2) = y \circledcirc (-y) = v(y)$.
(iii): A mapping to $G \times S$ is trivial. Let us suppose a hyperbole hyp such that its intersection v_{hyp} with straight line $x = y$ lies inside S. Let us denote $u^+_{hyp} > u^-_{hyp}$ its intersection with G. If we take any $u^+_{hyp} < u_i \leq 1$ and any $v_{hyp} < v_j < \frac{1}{2}$ then (u_i, v_j) has no κ-preimage in $\mathbf{D}_\circledcirc - (S_1 \cup S_2)$. □

Corollary 2 *Semigroups $S_1 \cup \top$, $S_2 \cup \bot$ are extended ordered Abelian semigroups with subtraction, i.e. for $(a,b),(c,d) \in S_i, (a,b)$, $(a,b) > (c,d)$ there exist d-pair (e,f) in S_i such that $(a,b) \circledcirc (e,f) = (c,d)$.*

Remark 8 *(i) Note that there is inequality $(a,b) > (c,d)$ here, while there is $(a,b) < (c,d)$ in Lemma 2. It is because $S_1 \cup \top$ and $S_2 \cup \bot$ are negative cones of the corresponding Abelian group in the disjunctive case.*
(ii) To say an analogy about S we need to take all (a,a) for $0 \leq a \leq 1$. ($S \cup 1$ is not enough here.)

5 A comparison of the disjunctive Dempster's semigroup with the standard (conjunctive) one

Both the algebraic structures have a lot of **similarities:**
Both of them are ordered Abelian semigroups with a neutral element.
There is the same operation $-$, which is not inverse in both cases.
Both the structures have subsemigroups S, S_1, S_2 respectively $S \cup 1, S_1 \cup \top, S_2 \cup \bot$ with neutral elements.
Both of them have a OAG defined on G.
Both of them have a surjective homomorphism $D_0 \longrightarrow G$.
Both of them have a surjective homomorphism $D_0 \longrightarrow S$.
Both the semigroup operations $\oplus$ and $\circledcirc$ are expressable using these homomorphisms, their preimages and operations restricted to S and G.

Differences:
$\oplus$ is not defined for $\top \oplus \bot$, while $\circledcirc$ is defined on the whole extended $D_0^+ \cup \{(1,1)\}$.
0 is a neutral element in $\mathbf{D}_0$, while it is an absorbing element in $\mathbf{D}_{\circledcirc}$.
Extremal elements are not absorbing in $\mathbf{D}_{\circledcirc}$.
The neutral element $1 = (1,1)$ of $\mathbf{D}_{\circledcirc}$ is out of D_0.
$0' = (\frac{1}{2}, \frac{1}{2})$ is not an idempotent of $\mathbf{D}_{\circledcirc}$.
The homomorphism u is not defined for $0 = (0,0)$.
If we add $1 = (1,1)$ into $\mathbf{D}_0$ we obtain a new absorbing element, where $(a, 1-a) \oplus 1$ is not defined for all $(a, 1-a) \in G$.
$\mathbf{G}_{\circledcirc_G}$ is not a subalgebra of $\mathbf{D}_{\circledcirc}$.

The pricipal is the following.
Both combinations $\oplus$ and $\circledcirc$ of two elements (d-pairs) $\geq 0'$ (or two ones ≤ 0) are on homomorphic straight lines further from S (than those, which contain the original elements (d-pairs)). We can reformulate this as that the certainty which is represented by belief functions is increased by both combining rules $\oplus$ and $\circledcirc$.

$\oplus$ combination of any two elements (d-pairs) is on an ellipse further from 0, i.e. vagueness is decreased by the Dempster's rule $\oplus$, while $\circledcirc$ combination of of any two elements is on a hyperbole closer to 0, i.e. vagueness is increased by the disjunctive rule $\circledcirc$.

$\mathbf{S}_{\oplus} = (S, \oplus, 0, \leq)$ is o-isomorphic to a positive cone of OAGs, while $\mathbf{S}_{\circledcirc} = (S \cup \{1\}, \circledcirc, 1, \leq)$ is o-isomorphic to a negative cone of OAGs (ordered such that $0 \leq 0'$). In another words, inverse elements of S in a group defined by $\mathbf{S}_{\oplus}$ are in $\{(a,a) | a \leq 0\}$. While inverse elements of S in a group defined by $\mathbf{S}_{\circledcirc}$ are in $\{(a,a) | a \geq 1\}$.

6 Impact to decision making

Summarizing the results of comparison of the disjunctive Dempster's semigroup with the standard (conjunctive) one, we obtain the following originally surprising theorem:

Theorem 3 *The groups $\mathbf{G}_{\oplus} = (G, \oplus, -, 0', \leq)$ and $\mathbf{G}_{\circledcirc_G} = (G, \circledcirc_G, -, 0', \leq_{\circledcirc})$ are identical; especially $x \oplus y = x \circledcirc_G y$ for all $x, y \in G$. (The same holds also for $\mathbf{G}_{\oplus}^+$ and $\mathbf{G}_{\circledcirc_G}^+$.)*

Proof: G is same in $\mathbf{G}_\oplus$ and $\mathbf{G}_{\circledcirc_G}$;
$(a,1-a)\oplus(b,1-b) = (1-\frac{(1-a)(1-b)}{1-(a(1-b)+(b(1-a))}, 1-\frac{ab}{1-(a(1-b)+(b(1-a))}) =$
$(\frac{1-(a-ab+b-ab)-(1-a-b+ab)}{1-(a-ab+b-ab)}, \frac{1-(a-ab+b-ab)-ab}{1-(a-ab+b-ab)}) = (\frac{ab}{1-a-b+2ab}, \frac{1-a-b+ab}{1-a-b+2ab}) =$
$(\frac{ab}{ab+(1-a)(1-b)}, \frac{(1-a)(1-b)}{ab+(1-a)(1-b)})$,
$\circledcirc_G = \circledcirc \circ u:$
$(a,1-a)\circledcirc_G(b,1-b) = u((a,1-a)\circledcirc(b,1-b)) = u((ab,(1-a)(1-b)) =$
$(\frac{ab}{ab+(1-a)(1-b)}, \frac{(1-a)(1-b)}{ab+(1-a)(1-b)})$;
hence $x\oplus y = x\circledcirc_G y$ for all $x,y\in G$, this holds also for extremal elements:
$(a,1-a)\oplus(1,0) = (1,0)$, $(a,1-a)\circledcirc_G(1,0) = u(a,0) = (\frac{a}{a},\frac{0}{a}) = (1,0)$, analogically for $(0,1)$,
$(1,0)\oplus(0,1)$ is not defined, $(1,0)\circledcirc_G(0,1) = u((1,0)\circledcirc(0,1)) = u(0,0)$, and it is also not defined;
the operation '$-$' and $0'$ are same for both $\mathbf{G}_\oplus$ and $\mathbf{G}_{\circledcirc_G}$;
$(a,1-a)\le(b,1-b)$ *iff* $h_1((a,1-a)) < h_1((b,1-b))$ or if $h_1((a,1-a)) = h_1((b,1-b))$ and $a\le b$
iff $\frac{1-(1-a)}{2-(a+1-a)} = \frac{a}{1} < \frac{b}{1} = \frac{1-(1-b)}{2-(b+1-b)}$ or $a=b$ and $a\le b$ *iff* $a\le b$;
$(a,1-a)\le_\circledcirc(b,1-b)$ *iff* $u_1((a,1-a)) < u_1((b,1-b))$ or if $u_1((a,1-a)) = u_1((b,1-b))$ and $a\le b$
iff $\frac{a}{a+1-a} = a < b = \frac{b}{b+1-b}$ or $a=b$ and $a\le b$ *iff* $a\le b$;
hence $(a,1-a)\le(b,1-b)$ *iff* $a\le b$ *iff* $(a,1-a)\le_\circledcirc(b,1-b)$ for all $a,b\in[0,1]$, i.e. $x\le y$ *iff* $p_1(x)\le p_1(y)$ *iff* $x\le_\circledcirc y$ for all $x,y\in G\cup\{(0,1),(1,0)\}$.
Thus we have $\mathbf{G}_\oplus$ is just the same as $\mathbf{G}_{\circledcirc_G}$, and $\mathbf{G}^+_\oplus$ is the same as $\mathbf{G}^+_{\circledcirc_G}$.

Corollary 3 *It holds that:*

$$\begin{array}{rclclcl} h(a\oplus b) & = & h(a)\oplus h(b) & = & h(a)\circledcirc_G h(b) & & \\ a\circledcirc_G b & = & u(a\circledcirc b) & = & u(a)\circledcirc_G u(b) & = & u(a)\oplus u(b) \end{array}$$

From the point of view of decision making, the difference between $\oplus$ and $\circledcirc$ is given by their homomorphic projections h and u from D_0 onto G. (There is no difference on G because it holds $h(x) = u(x) = x$ for all $x\in G$).

The theorem and its corollary express the importance of projection of D_0 onto G from the point of view of decision making. There are two homomorphic projections h and u (homomorphic with respect to operations $\oplus$ and $\circledcirc$). The another such a projection is the pignistic transformation defined in Transferable Belief Model (TBM), see e.g. [16]. Such projections are useful for decision making using belief functions. Hence, it would be an interesting and useful task to make a comparative study of these projections.

7 Conclusion

A new algebraic structure — the disjunctive Dempster's semigroup — is defined on a binary frame of discernment and analyzed in this text. It is compared with the standard Dempster's semigroup. The high principal importance of homomorphic projections of general belief (d-pairs) onto Bayesian ones was shown. And consequently great importance, from the point of view of decision making, of general Bayesian projections was mentioned.

8 Perspectives for future research

The present research can continue with defining and studying of disjunctive analogy of dempsteroids, i. e. with algebraic generalization of the notion of the disjunctive Dempster's semigroup, further with a disjunctive analogy of Lemma 2 (iii), and with a research of automorphisms and endomorphisms of the disjunctive Dempster's semigroup, analogically to the study of the standard conjunctive case, see [1, 2, 3, 4, 5, 6].

Moreover there are the following other fields for future research.

An investigation of algebraic structures related to combination of belief functions on general n-element frames of discernment.

A comparative study of Bayesian projections which could be motivated both by this algebraic analysis and by looking for a combination of belief functions which commutes with refinement coarsening, see [8].

An algebraic study of subjective logic by Jøsang [14] and the comparison of the algebraic structure given on a binary frame of discernment by Jøsang's Consensus operator (an algebraisation of his Opinion space) with both the standard and the disjunctive Dempster's semigroup. This topic is just under development.

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