Steps Towards a Conflicting Part of a Belief Function

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Abstract:

Belief functions usually contain some internal conflict. Based on Hájek-Valdés algebraic analysis of belief functions, a unique decomposition of a belief function into its conflicting and non-conflicting part was introduced at ISIPTA'11 symposium for belief functions defined on two-element frame of discernment. This study looks for the conditions under which such a decomposition exists for belief functions defined on three-element frame. A generalisation of important Hájek-Valdés homomorphism f of semigroup of belief functions onto its subsemigroup of indecisive belief functions is found and presented. A class of quasi-Bayesian belief functions, for which the decomposition into conflicting and non-conflicting parts exists is specified. A series of other partial results is presented. Several open problems from algebra of belief functions which are related to the investigated topic and are necessary for general solution of the issue of decomposition are formulated.

Keywords:
Belief function, Dempster-Shafer theory, Dempster's semigroup, conflict between belief functions, uncertainty, non-conflicting part of belief function, conflicting part of belief function.

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1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing that enable representation of incomplete and uncertain knowledge, belief updating, and combination of evidence. They were originally introduced as a principal notion of the Dempster-Shafer Theory or the Mathematical Theory of Evidence [26].

When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear, which are assigned to $\emptyset$ by non-normalized conjunctive rule $\circledcirc$ or normalized by Dempster's rule of combination $\oplus$. Combination of conflicting BFs and interpretation of conflicts is often questionable in real applications, thus a series of alternative combination rules was suggested and a series of papers on conflicting BFs was published, e.g. [2, 6, 17, 23, 24, 25, 28].

In [10, 15], new ideas concerning interpretation, definition, and measurement of conflicts of BFs were introduced. We presented three new approaches to interpretation and computation of conflicts: the combinational conflict, the plausibility conflict, and the comparative conflict. Later, the pignistic conflict — a pignistic analogy of plausibility conflict — was introduced in [16]. Differences were made between mutual conflicts between BFs and internal conflicts of single BFs; a conflict between BFs was distinguished from the difference between BFs.

When analysing mathematical properties of the three approaches to conflicts of BFs, there appears a possibility of expression of a BF Bel as Dempster's sum of non-conflicting BF Bel_0 with the same plausibility decisional support as the original BF Bel has and of indecisive BF Bel_S which does not prefer any of the elements of frame of discernment.

A unique decomposition to such BFs Bel_0 and Bel_S was demonstrated for BFs on 2-element frame of discernment in [11]. The present study analyses its generalisation and conditions under which such a decomposition of belief function on a 3-element frame of discernment exists. Three classes of BFs on a 3-element frame for which such decomposition exists are described; it remains an open problem for other BFs. Several other steps to a solution of this problem are also presented here.

As the idea of the decomposition is based on Hájek-Valdés analysis of BFs on 2-element frame of discernment [21, 22] and its generalisation [13, 14], the study begins with belief functions and algebraic preliminaries in Section 2. The present state of the art is briefly recalled in Section 3: the idea of decomposition on 2-element frame and hypothesis on general frame. This is followed by discussion and suggestion of generalisation of important Hájek-Valdés homomorphism f of semigroup of belief functions onto its subsemigroup of indecisive ones in Section 4; the main issue, i.e., the decomposition on 3-element frame is studied then. Several open problems from algebra of belief functions which are related to the investigated topic and necessary for general solution of the issue of decomposition are formulated in Section 5.

2 Preliminaries

2.1 General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* (BFs) [26] on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$, see also [4–9]; for illustration or simplicity, we often use 2- or 3-element frames Ω_2 and Ω_3. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed, then we speak about *normalized bba*. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$. There is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as about belief function.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF), it is a probability distribution on Ω in fact. If all the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a *quasi-Bayesian belief function* (qBBF), it is something like 'non-normalized probability distribution'. If all focal elements are nested, we speak about *consonant belief function.*

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [26]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$, see e. g. [27]. *The disjunctive rule of combination* is given by the formula $(m_1 \textcircled{\cup} m_2)(A) = \sum_{X \cup Y = A} m_1(X) m_2(Y)$, see [19].

Yager's rule of combination $\textcircled{Y}$, see [30], is given as $(m_1 \textcircled{Y} m_2)(A) = \sum_{X,Y \subseteq \Theta,\, X \cap Y = A} m_1(X) m_2(Y)$ for $\emptyset \neq A \subset \Theta$, $(m_1 \textcircled{Y} m_2)(\emptyset) = 0$, and $(m_1 \textcircled{Y} m_2)(\Theta) = m_1(\Theta) m_2(\Theta) + \sum_{X,Y \subseteq \Theta,\, X \cap Y = \emptyset} m_1(X) m_2(Y)$.

Dubois-Prade's rule of combination $\textcircled{D}$ is given as $(m_1 \textcircled{D} m_2)(A) = \sum_{X,Y \subseteq \Theta,\, X \cap Y = A} m_1(X) m_2(Y) + \sum_{X,Y \subseteq \Theta,\, X \cap Y = \emptyset, X \cup Y = A} m_1(X) m_2(Y)$ for $\emptyset \neq A \subseteq \Theta$, and $(m_1 \textcircled{D} m_2)(\emptyset) = 0$, see [18].

We say that BF Bel is *non-conflicting* (or conflict free, i.e., it has no internal conflict), when it is *consistent*, i.e., whenever $Pl(\omega_i) = 1$ for some $\omega_i \in \Omega_n$. Otherwise, BF is *conflicting*, i.e., it contains some internal conflict [10]. We can observe that Bel is non-conflicting if and only if the conjunctive combination of Bel with itself does not produce any conflicting belief masses[3] (when $(Bel \textcircled{\cap} Bel)(\emptyset) = 0$, i.e., $Bel \textcircled{\cap} Bel = Bel \oplus Bel$).

Let us recall U_n the *uniform Bayesian belief function*[4] [10], i.e., the uniform probability distribution on Ω_n, and *normalized plausibility of singletons*[5] of Bel: the BBF (probability distribution) $Pl\_P(Bel)$ such, that $(Pl\_P(Bel))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [3, 8].

Let us define an *indecisive (or nondiscriminative) BF* as a BF, which does not prefer any $\omega_i \in \Omega_n$, i.e., BF which gives no decisional support for any ω_i, i.e., BF such that $h(Bel) = Bel \oplus U_n = U_n$, i.e., $Pl(\{\omega_i\}) = const.$, i.e., $(Pl\_P(Bel))(\{\omega_i\}) = \frac{1}{n}$. Let us further define an *exclusive BF* as a BF Bel such[6] that $Pl(X) = 0$ for some $\emptyset \neq X \subset \Omega$; BF is *non-exclusive* otherwise.

2.2 Belief Functions on a 2-Element Frame of Discernment; Dempster's Semigroup

Let us suppose, that the reader is slightly familiar with basic algebraic notions like *a semigroup* (an algebraic structure with an associative binary operation), *a group* (a structure with an associative binary operation, with a unary operation of inverse, and with a neutral element), *a neutral element* n ($n * x = x$), *an absorbing element* a ($a * x = a$), *a homomorphism* f ($f(x * y) = f(x) * f(y)$), etc. (Otherwise, see e.g., [4, 7, 21, 22]; or any algebraic textbook of course. Nevertheless these algebraic notions are necessary only for deeper understanding the used algebraic methods; they are unnecessary for understanding of the issue conflicting part of BFs.)

We assume $\Omega_2 = \{\omega_1, \omega_2\}$, in this subsection. There are only three possible focal elements $\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}$ and any normalized *basic belief assignment (bba)* m is defined[7] by a pair $(a, b) = (m(\{\omega_1\}), m(\{\omega_2\}))$ as $m(\{\omega_1, \omega_2\}) = 1 - a - b$; this is called *Dempster's pair* or simply *d-pair* [21, 22] (it is a pair of reals such that $0 \leq a, b \leq 1, a + b \leq 1$).

Extremal d-pairs are the pairs corresponding to BFs for which either $m(\{\omega_1\}) = 1$ or $m(\{\omega_2\}) = 1$, i.e., exclusive d-pairs $(1, 0)$ and $(0, 1)$. The set of all non-extremal d-pairs is denoted as D_0; the set of all non-extremal *Bayesian d-pairs* (i.e. d-pairs corresponding to Bayesian BFs, where $a + b = 1$) is denoted as G; the set of d-pairs such that $a = b$ is denoted as S (set of indecisive[8] d-pairs), the set where $b = 0$ as S_1, and analogically, the set where $a = 0$ as S_2 (simple support BFs). Vacuous BF is denoted as $0 = (0, 0)$ and there is a special BF (d-pair) $0' = (\frac{1}{2}, \frac{1}{2}) = U_2$, see Figure 2.1.

The *(conjunctive) Dempster's semigroup* $\mathbf{D}_0 = (D_0, \oplus, 0, 0')$ is the set D_0 endowed with the binary operation $\oplus$ (i.e., with the Dempster's rule) and two distinguished elements 0 and $0'$. Dempster's rule can be expressed by the formula $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ for d-pairs [21]. In D_0 it is defined further: $-(a, b) = (b, a)$, $h(a, b) = (a, b) \oplus 0' = (\frac{1-b}{2-a-b},\ \frac{1-a}{2-a-b})$, $h_1(a, b) = \frac{1-b}{2-a-b}$, $f(a, b) = (a, b) \oplus (b, a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2},\ \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$; $(a, b) \leq (c, d)$ iff $[h_1(a, b) < h_1(c, d)$ or $h_1(a, b) =$

[3] Martin calls $(m \textcircled{\cap} m)(\emptyset)$ autoconflict of the BF [25].

[4] U_n which is idempotent w.r.t. Dempster's rule $\oplus$, and moreover neutral on the set of all BBFs, is denoted as ${}_{nD}0'$ in [8], $0'$ comes from studies by Hájek & Valdés.

[5] Plausibility of singletons is called *contour function* by Shafer [26], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

[6] BF Bel excludes all ω_i such, that $Pl(\{\omega_i\}) = 0$.

[7] It is an enumeration of m-values.

[8] BFs (a, a) from S are called *indifferent* BFs by Haenni [20].

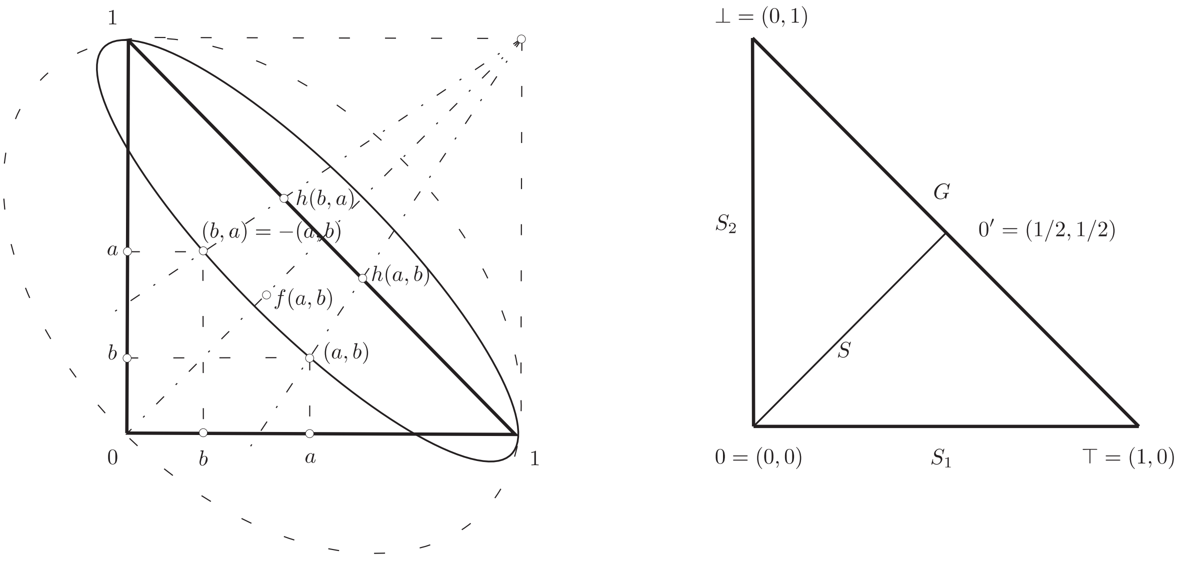

Figure 2.1: Dempster's semigroup D_0. Homomorphism h is in this representation a projection of the triangle representing semigroup D_0 to group G along the straight lines running through the point $(1, 1)$. All the Dempster's pairs lying on the same ellipse are mapped by homomorphism f to the same d-pair in semigroup S.

$h_1(c, d)$ and $a \le c]$ [9].

The principal properties of $\mathbf{D}_0$ are summarized by the following theorem:

Theorem 1 *(i) The Dempster's semigroup* $\mathbf{D}_0$ *with the relation* $\le \mathbf{D}_0 = (D_0, \oplus, 0, 0', \le)$ *is an ordered commutative (Abelian) semigroup with the neutral element* 0*;* $0'$ *is the only non-zero idempotent of* $\mathbf{D}_0$.

(ii) $\mathbf{G} = (G, \oplus, -, 0', \le)$ *is an ordered Abelian group, isomorphic to the additive group of reals with the usual ordering* $\mathbf{Re} = (Re, +, -, 0, \le)$.

(iii) The sets S, S_1, S_2 *with the operation* $\oplus$ *and the ordering* $\le$ *form ordered commutative semigroups with neutral element* 0*; they are all isomorphic to the positive cone of the group of reals* $\mathbf{Re}^{\ge 0} = (Re^{\ge 0}, +, -, 0, \le)$ *(or to* $\mathbf{Re}^{\ge 0+}$ *extended with* ∞ *in the case of* S *which includes absorbing element* $0'$*).*

(iv) h *is an ordered homomorphism:* $(D_0, \oplus, -, 0, 0', \le) \longrightarrow (G, \oplus, -, 0', \le)$*;* $h(Bel) = Bel \oplus 0' = Pl\_P(Bel)$*, i.e., the normalized plausibility probabilistic transformation.*

(v) f *is a homomorphism:* $(D_0, \oplus, -, 0, 0') \longrightarrow (S, \oplus, -, 0)$*; (but, not an ordered one).*

For proofs see [21, 22, 29].

Notice, that '$-$' is an inverse on G (on BBFs) only, not in general. There is $-Bel \oplus Bel = 0'$ for any BBF Bel. This does not hold for general BFs. The operation '$-$' is some kind of symmetry only; in the case of representation on Fig. 2.1, it is the symmetry along the axis S.

Let us denote $h^{-1}(a) = \{x \mid h(x) = a\}$ and similarly $f^{-1}(a) = \{x \mid f(x) = a\}$. Using the theorem, see (iv) and (v), we can express[10] Dempster's sum $\oplus$ of two general BFs (d-pairs) from $\mathbf{D}_0$ using homomorphisms f and h and Dempster's sum on subalgebras of Bayesian and indecisive BFs G and S:

$$(a \oplus b) = h^{-1}(h(a) \oplus h(b)) \cap f^{-1}(f(a) \oplus f(b)). \tag{2.1}$$

[9] Note, that $h(a, b)$ is an abbreviation for $h((a, b))$, similarly for $h_1(a, b)$ and $f(a, b)$.

[10] This is neither a new nor alternative definition of Dempster's rule $\oplus$. It is an important relationship of Dempster's combination of general d-pairs (combination on Dempster's semigroup $\mathbf{D}_0$) with Dempster's combination of special cases: Bayesian and symmetric d-pairs (combination on subalgebras G and S of $\mathbf{D}_0$) based on homomorphisms h and f and their preimages. For detail see [4].

Let us denote $D_0^{\geq 0} = \{(a,b) \in D_0 \,|\, (a,b) \geq 0\}$ and analogically $D_0^{\leq 0'} = \{(a,b) \leq 0'\}$. Let us further denote negative and positive cones of group G as $G^{\leq 0'}$ and $G^{\geq 0'}$.

Besides the classic results by Hájek & Valdés [21, 22, 29] we will use also our new result from [12] motivated by [11] (in fact also a special case of automorphisms of Dempster's semigroup investigated by the author of this study in 90's [4, 5]):

Theorem 2 *Mapping* $- : \mathbf{D}_0 \longrightarrow \mathbf{D}_0$, $-(a,b) = (b,a)$ *for* $(a,b) \in \mathbf{D}_0$ *is an automorphism of* $\mathbf{D}_0$, *i.e., a bijective homomorphism:* $(D_0, \oplus, -, 0, 0', \leq) \longrightarrow (D_0, \oplus, -, 0', \leq)$.

For proof see [12].

2.3 BFs on an n-Element Frame of Discernment

Analogically to the case of Ω_2, we can represent a BF on any n-element frame of discernment Ω_n by an enumeration of its m-values (bbms), i.e., by a (2^n-2)-tuple $(a_1, a_2, ..., a_{2^n-2})$, or as a (2^n-1)-tuple $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1})$ when we want to explicitly mention also the redundant value $m(\Omega) = a_{2^n-1} = 1 - \sum_{i=1}^{2^n-2} a_i$. For BFs on Ω_3 we use $(a_1, a_2, ..., a_6; a_7) = (m(\{\omega_1\}), m(\{\omega_2\}), m(\{\omega_3\}), m(\{\omega_1, \omega_2\}), m(\{\omega_1, \omega_3\}), m(\{\omega_2, \omega_3\}); m(\{\Omega_3\}))$.

2.4 On Dempster's Semigroup on Ω_3 (on a 3-Element Frame of Discernment)

There is significant increase of complexity considering 3-element frame of discernment Ω_3. While we can represent BFs on Ω_2 by a 2-dimensional triangle, we need a 6-dimensional simplex in the case of Ω_3. Further, all the dimensions are not equal: there are 3 independent dimensions corresponding to singletons from Ω_3, but there are other 3 dimensions corresponding to 2-element subsets of Ω_3, each of them is somehow related to 2 dimensions corresponding to singletons (dimension corresponding to $\{\omega_1, \omega_2\}$ is related to those corresponding to singletons $\{\omega_1\}$ and $\{\omega_2\}$, etc.).

Dempster's semigroup $\mathbf{D}_3$ on Ω_3 is defined analogously to $\mathbf{D}_0$ on Ω_2. First results on algebraic structures related to BFs on Ω_3 were recently presented at IPMU'12 (a quasi-Bayesian case, the dimensions related to singletons only, $\mathbf{D}_{3-0}$, see Figure 2.2) [13] and at WUPES'12 (a general case, all six dimensions, $\mathbf{D}_3$, see Figure 2.3) [14].

Let us briefly recall the following results on $\mathbf{D}_3$ which are related to our topic.

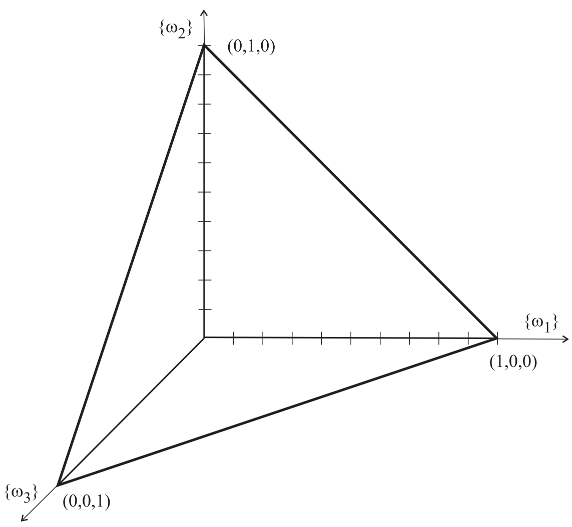

Figure 2.2: Quasi-Bayesian BFs on 3-element frame Ω_3.

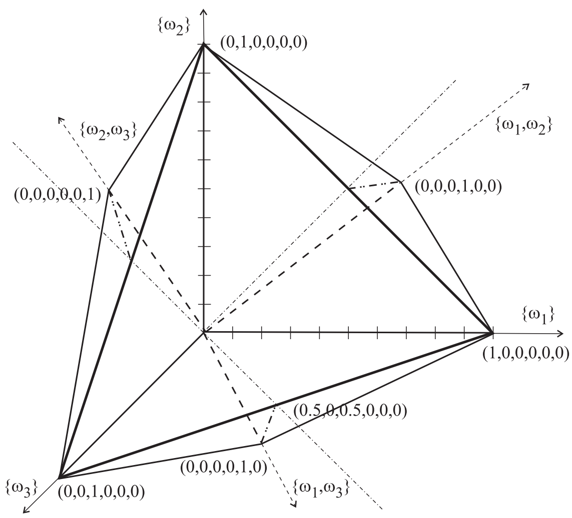

Figure 2.3: General BFs on 3-element frame Ω_3.

Theorem 3 *(i) Dempster's semigroup* $\mathbf{D}_3 = (D_3, \oplus, 0, U_3)$ *of non-exclusive BFs on* Ω_3 *is a commutative semigroup with neutral element* $0 = (0,0,0,0,0,0)$ *(i.e. it is monoid), and with just four other idempotents* $0' = U_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$, $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$, $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$, *and* $(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0)$.
$\mathbf{D}_{3-0} = (D_{3-0}, \oplus, 0, U_3)$ *is its subalgebra, where* D_{3-0} *is set of non-exclusive quasi-Bayesian belief functions* $\mathbf{D}_{3-0} = \{(a,b,c,0,0,0) \,|\, 0 \le a+b+c \le 1;\ 0 \le a,b,c;\ a+b<1;\ a+c<1;\ b+c<1\}$.

(ii) Subalgebra of non-exclusive Bayesian BFs $G_3 = (\{(a,b,c,0,0,0) \,|\, a+b+c=1;\ 0<a,b,c\}, \oplus, \text{'}-\text{'}, U_3)$ *is a subgroup of* $\mathbf{D}_3$*, where '−' is given*[11] *by* $-(d_1, d_2, 1-(d_1+d_2), 0,0,0) = (x_1, \frac{d_1}{d_2}x_1, \frac{d_1}{1-(d_1+d_2)}x_1, 0,0,0;0)$*, and* $x_1 = 1/(1 + \frac{d_1}{d_2} + \frac{d_1}{1-(d_1+d_2)})$.

(iii a) The sets of non-exclusive BFs $S_0 = \{(a,a,a,0,0,0) \,|\, 0 \le a \le \frac{1}{3}\}$, $S_1 = \{(a,0,0,0,0,0) \,|\, 0 \le a < 1\}$, $S_2, S_3, S_{1-2} = \{(0,0,0,a,0,0) \,|\, 0 \le a < 1\}$, S_{1-3}, S_{2-3} *with the operation* $\oplus$ *and VBF* 0 *form commutative semigroups with neutral element* 0 *(monoids); they are all isomorphic*[12] *to the positive cone of the additive group of reals* $\mathbf{Re}^{\ge 0}$ *(to* $\mathbf{Re}^{\ge 0+}$ *extended with* ∞ *in the case of* S_0 *which includes absorbing element* U_3*).*

(iii b) There are another subsemigroups $S = (\{(a,a,a,b,b,b) \in D_3\}, \oplus)$ *and* $S_{Pl} = (\{(d_1, d_2, ..., d_{23}) \in D_3 \,|\, Pl(d_1, d_2, ..., d_{23}) = U_3\}, \oplus)$ *which are alternative generalisations of Hájek-Valdés S, both with neutral idempotent* 0 *and absorbing one* U_3*. (note that set of BFs* $\{(a,a,a,a,a,a) \in D_3\}$ *is not closed under* $\oplus$*, thus it does not form a semigroup).*

(iv) Mapping h is a homomorphism: $(D_3, \oplus, 0, U_3) \longrightarrow (G_3, \oplus, \text{'}-\text{'}, U_3)$*;* $h(Bel) = Bel \oplus U_3 = Pl\_P(Bel)$*, i.e., the normalized plausibility of singletons.*

For detail and proofs of the assertions from the theorem see [13, 14], proof of (iv) already in [11].

Unfortunately, a full generalisation of $-$ or f was not yet found [13, 14].

3 State of the Art

Let us introduce a unique decomposition of a BF on a 2-element frame of discernment and a unique non-conflicting part of a general BF on a general finite frame in this section.

3.1 Non-conflicting and Conflicting Parts of Belief Functions on a 2-Element Frame of Discernment

For BFs on a 2-element frame discernment Ω_2 the following holds true:

Proposition 1 *BF Bel on* Ω_2 *is non-conflicting iff* $Bel \in S_1 \cup S_2$.

For proof of this and other assertions in this Section see [11].

Using the important property of Dempster's sum (2.1), which is respecting the homomorphisms h and f (i.e., respecting the h-lines and f-ellipses, when two BFs are combined on two-element frame of discernment [21, 22]), we obtain the following statement.

Proposition 2 *(i) Any belief function* $(a,b) \in \Omega_2$ *is the result of Dempster's combination of BF* $(a_0, b_0) \in S_1 \cup S_2$ *and a BF* $(s,s) \in S$*, such that* (a_0, b_0) *has the same plausibility decision support (same normalized plausibility) for the elements of* Ω_2 *as* (a,b) *does.*

(ii) $(a_0, b_0) \in S_1 \cup S_2$ *has no internal conflict, and* (s,s) *does not prefer any of the elements of* Ω_2*. Let us call* (a_0, b_0) *a* non-conflicting part *of* (a,b)*. There is* $(a_0, b_0) = (\frac{a-b}{1-b}, 0)$ *for* $a \ge b$ *and* $(a_0, b_0) = (0, \frac{b-a}{1-a})$ *for* $a \le b$.

[11] An alternative expression for group operation '−' on G_3 is $-(a,b,c,0,0,0) = (\frac{bc}{ab+ac+bc}, \frac{ac}{ab+ac+bc}, \frac{ab}{ab+ac+bc}, 0,0,0)$ [11].

[12] o-isomorphic as in the case of $\mathbf{D}_0$ in fact, see Theorem 1. Nevertheless, there is no ordering of elements of Ω_3, thus we are either not interesting in ordering of algebras S_i in this text.

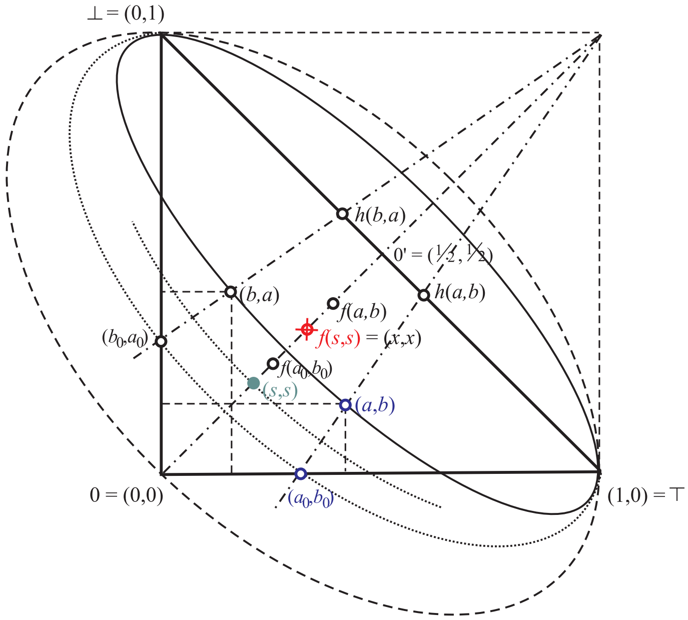

Figure 3.1: Conflicting and non-conflicting parts of BF on a 2-element frame of discernment.

Let us look for (s,s) from the proposition now. It holds true that $(a,b) = (a_0,b_0) \oplus (s,s)$, thus it also holds true $f(a,b) = f(a_0,b_0) \oplus f(s,s)$. Let us denote $f(a_0,b_0) = (u,u)$, $f(a,b) = (v,v)$, $f(s,s) = (x,x)$ for a moment, thus we have $(u,u) \oplus (x,x) = (v,v)$, where $v = 1 - \frac{(1-u)(1-x)}{1-2ux} = \frac{u+x-3ux}{1-2ux}$, hence $u + x - 3ux = v - 2vux$ and $x = \frac{v-u}{1-3u+2uv}$. We can express this as Lemma 1 (i), further we have Lemma 1(ii), (iii). Finally, we obtain a summarization in Theorem 4.

Lemma 1 *(i) For any BFs (u,u), (v,v) on S, such that $u \leq v$, we can compute their Dempster's 'difference' (x,x) such that $(u,u) \oplus (x,x) = (v,v)$, as $(x,x) = (\frac{v-u}{1-3u+2uv}, \frac{v-u}{1-3u+2uv})$.*
(ii) For any BF (w,w) on S, we can compute its Dempster's 'half' (s,s) such that $(s,s) \oplus (s,s) = (w,w)$, as $(s,s) = (\frac{1-\sqrt{1-3w+2w^2}}{3-2w}, \frac{1-\sqrt{1-3w+2w^2}}{3-2w}) = (\frac{1-\sqrt{(1-w)(1-2w)}}{3-2w}, \frac{1-\sqrt{(1-w)(1-2w)}}{3-2w})$.
(iii) There is no Dempster's 'difference' on D_0 in general.

Theorem 4 *Any BF (a,b) on a 2-element frame of discernment Ω_2 is Dempster's sum of its unique* non-conflicting part *$(a_0,b_0) \in S_1 \cup S_2$ and of its unique* conflicting part *$(s,s) \in S$, which does not prefer any element of Ω_2, i.e.,*

$$(a,b) = (a_0,b_0) \oplus (s,s).$$

It holds true that

$$(a,b) = (\frac{a-b}{1-b}, 0) \oplus (s,s) \ \text{for}\ a \geq b, \ \text{where}\ s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2};$$

and similarly that

$$(a,b) = (0, \frac{b-a}{1-a}) \oplus (s,s) \ \text{for}\ a \leq b, \ \text{where}\ s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}.$$

For proofs see [11] again.

We can summarize formulas from the theorem as it follows

$$(a,b) = (a_0,b_0) \oplus (s,s) =$$

$$(max(\tfrac{a-b}{1-b},0), max(\tfrac{b-a}{1-a},0)) \oplus (\tfrac{min(a,b)(1-max(a,b))}{1-ab+min(a,b)-2max(a,b)-max^2(a,b)}, \tfrac{min(a,b)(1-max(a,b))}{1-ab+min(a,b)-2max(a,b)-max^2(a,b)}) =$$

$$(max(\frac{a-b}{1-b},0), max(\frac{b-a}{1-a},0)) \oplus (\frac{min(a,b)(1-min(a,b))}{1+ab-max(a,b)-min^2(a,b)}, \frac{min(a,b)(1-min(a,b))}{1+ab-max(a,b)-min^2(a,b)}).$$

3.2 Non-conflicting Part of BFs on General Finite Frames of Discernment

We would like to verify that Theorem 4 holds true also for BFs defined on general finite frames, i.e., to verify the following hypothesis:

Hypothesis 1 *We can represent any BF Bel on an n-element frame of discernment* $\Omega_n = \{\omega_1, ..., \omega_n\}$ *as Dempster's sum* $Bel = Bel_0 \oplus Bel_S$ *of non-conflicting BF* Bel_0 *and of indecisive conflicting BF* Bel_S *which has no decisional support, i.e. which does not prefer any element of* Ω_n *to the others, see Figure 3.2.*

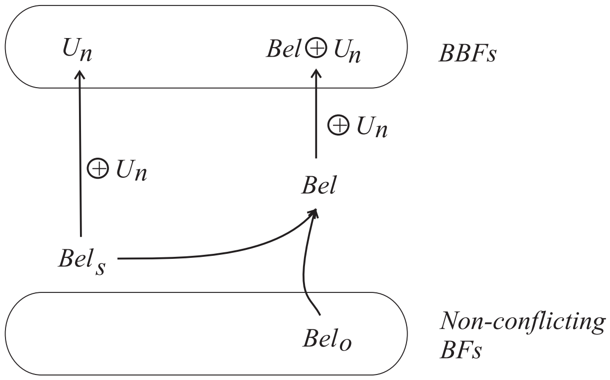

Figure 3.2: Schema of Hypothesis 1.

Analogously to the 2-element case we have:

Proposition 3 *The set of non-conflicting BFs is just the set of all BFs such, that all focal elements of a BF have non-empty intersection, i.e., the set of consistent BFs.*
Consonant BFs are a special case of non-conflicting BFs.

We would like to follow the idea from the case of two-element frames, see Figure 3.3. Unfortunately, we have only a simple description of the most basic algebraic substructures and homomorphism h on Dempster's semigroup on Ω_3. We have not yet any generalisation or analogy of $-Bel$ and of homomorphism f, we have not group properties of the set of indecisive BFs.

Using group properties of G_3, structure of Bayesian BFs (including inverse $Bel \oplus -Bel = U_3$) and homomorphic properties of h we have a partial generalisation of mapping '$-$' to sets of Bayesian and consonant BFs, thus we have $-h(Bel)$ and $-Bel_0$.

Theorem 5 *(i) For any BF Bel defined on* Ω_n *there exists unique consonant BF* Bel_0 *such that,*

$$h(Bel_0 \oplus Bel_S) = h(Bel)$$

for any BF Bel_S *such that* $Bel_S \oplus U_n = U_n$.
(ii) If for $h(Bel) = (h_1, h_2, ..., h_n, 0, 0, ..., 0)$ *holds true that,* $0 < h_i < 1$, *then further exist unique BFs* $-Bel_0$ *and* $-h(Bel_0)$ *such that,*

$$h(-Bel_0 \oplus Bel_S) = -h(Bel) = h(-Bel_0), \quad \textit{and} \quad h(Bel_0) \oplus -h(Bel_0) = U_n.$$

Corollary 1 *(i) For any consonant BF Bel such that* $Pl(\{\omega_i\}) > 0$ *there exists a unique BF* $-Bel$*;* $-Bel$ *is consonant in this case.*
(ii) There is one-to-one correspondence between Bayesian BFs and consonant BFs.

The construction of Bel_0 is a projection of the set of all BFs to consonant BFs, i.e., Bel_0 is a consonant approximation of Bel such that $h(Bel_0) = h(Bel)$. For any BBF we have its '$-$' inverse,

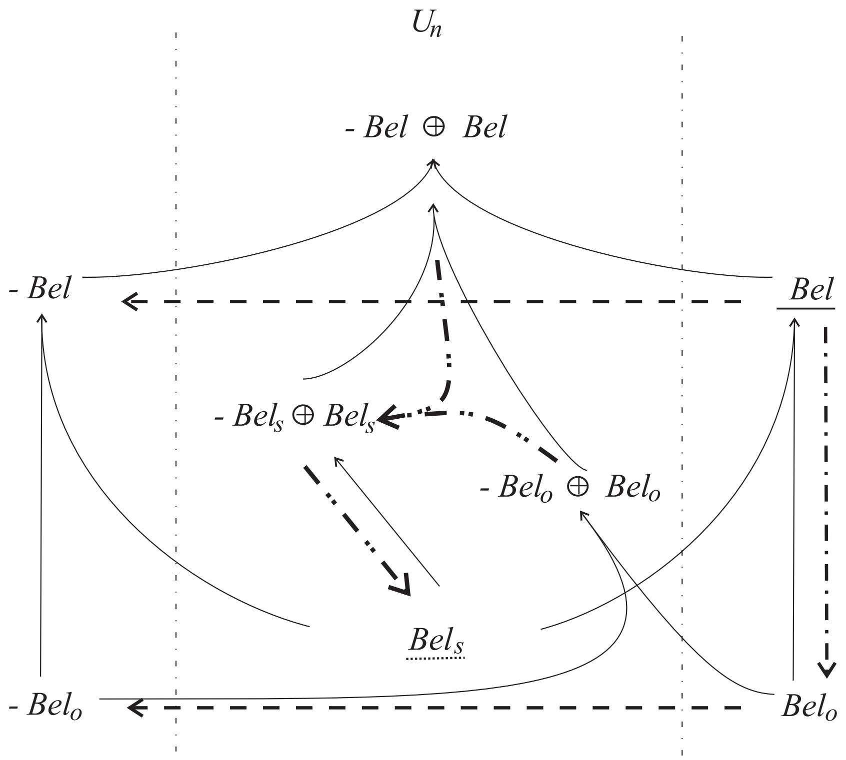

Figure 3.3: Schema of a decomposition of BF Bel.

thus also for BBF $h(Bel)$: $h(Bel) \oplus -h(Bel) = U_n$. $-Bel_0$ is then constructed as a non-conflicting part of $-h(Bel)$, i.e. $-Bel_0 = (-h(Bel))_0$. For detail of proofs see [11]. There was also verified that the above partial definition of $-Bel$ using $-h(Bel)$ satisfies: $-m(X) = m(\Omega \setminus X)$ for $X \subset \Omega$ and SSF m.

Let us notice the importance of the consonance property here: that a stronger statement for general consistent (non-conflicting) BFs does not hold true on Ω_3. There could be several different non-conflicting BFs Bel_i (and usially there are many Bel_i) such that $h(Bel_i \oplus Bel_S) = h(Bel)$ for any indecisive BF B_S, but there is just one consonant BF Bel_0 among them. For an example see [11]; see also the following example.

Example 1. To BF $Bel = (0.25, 0.175, 0.075, 0.35, 0.15, 0)$ with $h(Bel) = (0.25, 0.175, 0.075, 0.35, 0.15, 0) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0) = (0.50, 0.35, 0.15, 0, 0, 0)$ there are following non-conflicting BFs: $Bel_0 = (0.3, 0, 0, 0.4, 0, 0; 0.3)$, $Bel_1 = (0.2, 0, 0, 0.5, 0.1, 0; 0.2)$; $Bel_2 = (0.1, 0, 0, 0.6, 0.2, 0; 0.1)$, $Bel_3 = (0, 0, 0, 0.7, 0.3, 0; 0)$, $Pl_i(\{\omega_1\}) = 1$, thus Bel_is are all non-conflicting, we can simply verify that $h(Bel_i) = h(Bel)$, thus $(Bel_i \oplus Bel_S) \oplus U_3 = Bel_i \oplus (Bel_S \oplus U_3) = Bel_i \oplus U_3 = h(Bel)$.

There are numerous other such Bel_i's: e.g., any BF $Bel_i = (0.3 - j, 0, 0, 0.4 + j, j, 0; 0.3 - j)$ for $0 \leq j \leq 0.3$ and any $Bel_i = (0.2 - k, 0, 0, 0.5 + k, 0.1 + k, 0; 0.2 - k)$ for $0 \leq k \leq 0.2$ have this property. $Bel_0 = (0.3, 0, 0, 0.4, 0, 0; 0.3)$ is the unique consonant BFs among all such BFs.

Including Theorem 5 into the diagram of decomposition we obtain Figure 3.4. We still have only partial results; to complete the diagram, we need a definition of $-Bel$ for general BFs on Ω_3 and Ω_n to compute $Bel \oplus -Bel$, we further need an analysis of indecisive BFs (i.e. BFs Bel such that, $h(Bel) = U_n$) to compute $Bel_S \oplus -Bel_S$ and resulting Bel_S and to specify conditions under which a unique Bel_S exists. Hence an algebraic analysis of BFs on a general finite frame of discernment was required in [11].

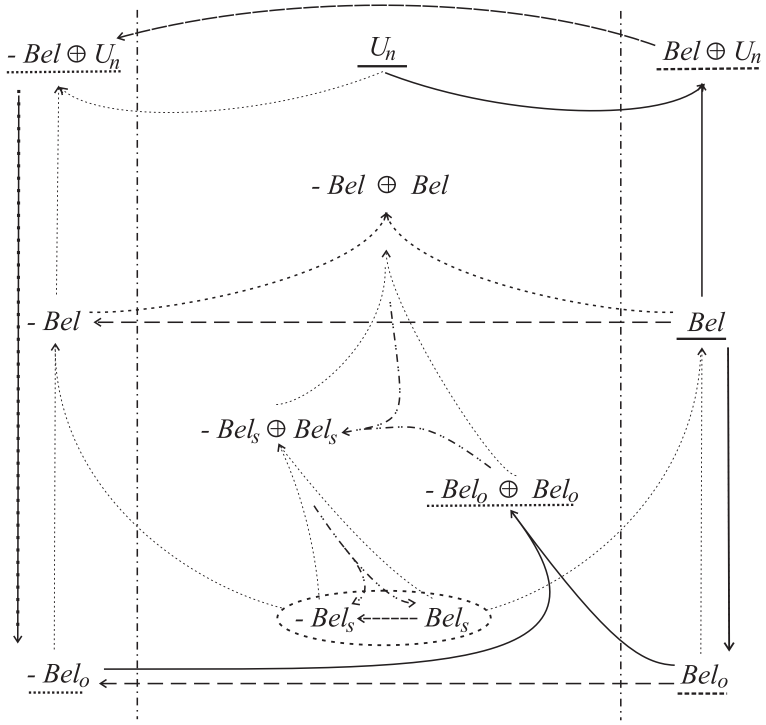

Figure 3.4: Detailed schema of a decomposition of BF Bel.

4 Towards Conflicting Parts of BFs on a 3-Element Frame Ω_3

4.1 A General Idea

An introduction to the algebra of BFs on a 3-element frame of discernment was performed, but a full generalisation of basic homomorphisms of Dempster's semigroup '$-$' and f is still missing [11, 12, 13, 14]. We need $f(Bel) = -Bel \oplus Bel$ to complete the decomposition diagram (Figure 3.4) according to the original idea from [11] trying to follow the 2-element case as close as possible.

Let us forget for a moment a meaning of '$-$' and its relation to group 'minus' in subgroups G and G_3; and look at its construction $-(a, b) = (b, a)$. It is a simple transposition of m-values of ω_1 and ω_2 in fact. Generally on Ω_3 we have:

Lemma 2 *Any transposition τ of a 3-element frame of discernment Ω_3 is an automorphism of D_3.* $\tau_{12}(\omega_1, \omega_2, \omega_3) = (\omega_2, \omega_1, \omega_3)$, $\tau_{23}(\omega_1, \omega_2, \omega_3) = (\omega_1, \omega_3, \omega_2)$, $\tau_{13}(\omega_1, \omega_2, \omega_3) = (\omega_3, \omega_2, \omega_1)$.

Proof. Bijection of D_3 onto D_3 is obvious. Thus proof is a verification of homomorphic properties of τ, i.e. $\tau(Bel_1 \oplus Bel_2)(X) = (\tau(Bel_1) \oplus \tau(Bel_2))(X)$ for individual subsets X of Ω_3.

Let us start with τ_{12} and $\{\omega_1\}$. $(m_1 \oplus m_2)(\{\omega_1\}) = K(m_1(\{\omega_1\})m_2(\{\omega_1\}) + m_1(\{\omega_1\})m_2(\{\omega_1, \omega_2\}) + m_1(\{\omega_1\})m_2(\{\omega_1, \omega_3\}) + m_1(\{\omega_1\})m_2(\{\omega_1, \omega_2, \omega_3\}) + m_2(\{\omega_1\})m_1(\{\omega_1, \omega_2\}) + m_2(\{\omega_1\})m_1(\{\omega_1, \omega_3\}) + m_2(\{\omega_1\})m_1(\{\omega_1, \omega_2, \omega_3\}) + m_1(\{\omega_1, \omega_2\})m_2(\{\omega_1, \omega_3\}) + m_1(\{\omega_1, \omega_3\})m_2(\{\omega_1, \omega_2\}))$, where K is the corresponding normalisation constant. Thus there is:
$\tau_{12}(m_1 \oplus m_2)(\{\omega_1\}) = K(m_1(\{\omega_2\})m_2(\{\omega_2\}) + m_1(\{\omega_2\})m_2(\{\omega_1, \omega_2\}) + m_1(\{\omega_2\})m_2(\{\omega_2, \omega_3\}) + m_1(\{\omega_2\})m_2(\{\omega_1, \omega_2, \omega_3\}) + m_2(\{\omega_2\})m_1(\{\omega_1, \omega_2\}) + m_2(\{\omega_2\})m_1(\{\omega_2, \omega_3\}) + m_2(\{\omega_2\})m_1(\{\omega_1, \omega_2, \omega_3\}) + m_1(\{\omega_1, \omega_2\})m_2(\{\omega_2, \omega_3\}) + m_1(\{\omega_2, \omega_3\})m_2(\{\omega_1, \omega_2, \}))$.
$(\tau_{12}(m_1) \oplus \tau_{12}(m_2))(\{\omega_1\}) = K'(\tau_{12}(m_1)(\{\omega_1\})\tau_{12}(m_2)(\{\omega_1\}) + \tau_{12}(m_1)(\{\omega_1\})\tau_{12}(m_2)(\{\omega_1, \omega_2\}) + \tau_{12}(m_1)(\{\omega_1\})\tau_{12}(m_2)(\{\omega_1, \omega_3\}) + \tau_{12}(m_1)(\{\omega_1\})\tau_{12}(m_2)(\{\omega_1, \omega_2, \omega_3\}) + \tau_{12}(m_2)(\{\omega_1\})\tau_{12}(m_1)(\{\omega_1, \omega_2\}) + \tau_{12}(m_2)(\{\omega_1\})\tau_{12}(m_1)(\{\omega_1, \omega_3\}) + \tau_{12}(m_2)(\{\omega_1\})\tau_{12}(m_1)(\{\omega_1, \omega_2, \omega_3\}) + \tau_{12}(m_1)(\{\omega_1, \omega_2\}) \tau_{12}(m_2)(\{\omega_1, \omega_3\}) + \tau_{12}(m_1)(\{\omega_1, \omega_3\})\tau_{12}(m_2)(\{\omega_1, \omega_2, \})) = K'(m_1(\{\omega_2\})m_2(\{\omega_2\}) + m_1(\{\omega_2\})$

$m_2(\{\omega_1,\omega_2\})+m_1(\{\omega_2\})m_2(\{\omega_2,\omega_3\})+m_1(\{\omega_2\})m_2(\{\omega_1,\omega_2,\omega_3\})+m_2(\{\omega_2\})m_1(\{\omega_1,\omega_2\})+m_2(\{\omega_2\})$ $m_1(\{\omega_2,\omega_3\})+m_2(\{\omega_2\})m_1(\{\omega_1,\omega_2,\omega_3\})+m_1(\{\omega_1,\omega_2\})m_2(\{\omega_2,\omega_3\})+m_1(\{\omega_2,\omega_3\})m_2(\{\omega_1,\omega_2,\}))=$ $\frac{K'}{K}\tau_{12}(m_1\oplus m_2)(\{\omega_1\})$. In the same way we can show equality upto normalisation constants also for $\{\omega_2\}$ and $\{\omega_3\}$.

For $\{\omega_1,\omega_2\}$ we simply obtain $\tau_{12}(m_1\oplus m_2)(\{\omega_1,\omega_2\}) = (m_1\oplus m_2)(\{\omega_1,\omega_2\}) = (\tau_{12}(m_1)\oplus \tau_{12}(m_2))(\{\omega_1,\omega_2\})$. Analogously to the singleton case, for τ_{12} and $\{\omega_1,\omega_3\}$ we obtain: $(m_1\oplus m_2)(\{\omega_1,\omega_3\}) = K(m_1(\{\omega_1,\omega_3\})m_2(\{\omega_1,\omega_3\}) + m_1(\{\omega_1,\omega_3\})m_2(\{\omega_1,\omega_2,\omega_3\}) + m_2(\{\omega_1,\omega_3\})$ $m_1(\{\omega_1,\omega_2,\omega_3\}))$. Thus there is: $\tau_{12}(m_1\oplus m_2)(\{\omega_1,\omega_3\}) = K(m_1(\{\omega_2,\omega_3\})m_2(\{\omega_2,\omega_3\})+$ $m_1(\{\omega_2,\omega_3\})m_2(\{\omega_1,\omega_2,\omega_3\}) + m_2(\{\omega_2,\omega_3\})m_1(\{\omega_1,\omega_2,\omega_3\}))$.
$(\tau_{12}(m_1)\oplus\tau_{12}(m_2))(\{\omega_1,\omega_3\}) = K'(\tau_{12}(m_1)(\{\omega_1,\omega_3\})\tau_{12}(m_2)(\{\omega_1,\omega_3\}) + \tau_{12}(m_1)(\{\omega_1,\omega_3\})$ $\tau_{12}(m_2)(\{\omega_1,\omega_2,\omega_3\}) + \tau_{12}(m_2)(\{\omega_1,\omega_3\})\tau_{12}(m_1)(\{\omega_1,\omega_2,\omega_3\})) = K'(m_1(\{\omega_2,\omega_3\})m_2(\{\omega_2,\omega_3\}) +$ $m_1(\{\omega_2,\omega_3\})(m_2)(\{\omega_1,\omega_2,\omega_3\}) + m_2(\{\omega_2,\omega_3\})(m_1)(\{\omega_1,\omega_2,\omega_3\})) = \frac{K'}{K}\tau_{12}(m_1\oplus m_2)(\{\omega_1,\omega_3\})$. In the same way we can show equality up to normalisation constants also for $\{\omega_2,\omega_3\}$.

Finally, we simply obtain $\tau_{12}(m_1\oplus m_2)(\{\omega_1,\omega_2,\omega_3\}) = \tau_{12}K(m_1(\{\omega_1,\omega_2,\omega_3\})m_2(\{\omega_1,\omega_2,\omega_3\}) =$ $K(m_1(\{\omega_1,\omega_2,\omega_3\})m_2(\{\omega_1,\omega_2,\omega_3\})$. And $(\tau_{12}(m_1)\oplus\tau_{12}(m_2))(\{\omega_1,\omega_2,\omega_3\}) = K'(\tau_{12}(m_1)(\{\omega_1,\omega_2,\omega_3\})$ $\tau_{12}(m_2)(\{\omega_1,\omega_2,\omega_3\})) = K'(m_1(\{\omega_1,\omega_2,\omega_3\})m_2(\{\omega_1,\omega_2,\omega_3\})) = \frac{K'}{K}\tau_{12}(m_1\oplus m_2)(\{\omega_1,\omega_2,\omega_3\})$.

We have equality up to normalisation constants for all subsets of Ω_3. Both results are normalised, thus the normalisation constant must be same, i.e., $K = K'$ and $\frac{K'}{K} = 1$. Thus equality holds for all the subsets of the frame of discernment and $\tau_{12}(m_1\oplus m_2) = \tau_{12}(m_1)\oplus\tau_{12}(m_2)$ holds true.

The proofs for τ_{23} and τ_{13} are completely analogous.

Alternative presentation of the proof for τ_{12} is the following. Let $Bel_a = (a_1, a_2, a_3, a_{12}, a_{13}, a_{23}; a_{123})$ and $Bel_b = (b_1, b_2, b_3, b_{12}, b_{13}, b_{23}; b_{123})$, thus $\tau_{12}(Bel_a) = (a_2, a_1, a_3, a_{12}, a_{23}, a_{13}; a_{123})$ and $\tau_{12}(Bel_b) = (b_2, b_1, b_3, b_{12}, b_{23}, b_{13}; b_{123})$. Thus $(Bel_a\oplus Bel_b)(\{\omega_1\}) = K[a_1(b_1 + b_{12} + b_{13} + b_{123}) + (a_{12} + a_{13} + a_{123})b_1 + a_{12}b_{13} + a_{13}b_{12}]$. There is $\tau_{12}(Bel_a\oplus Bel_b)(\{\omega_1\}) = (Bel_a\oplus Bel_b)(\{\omega_2\}) = K[a_2(b_2 + b_{12} + b_{23} + b_{123}) + (a_{12} + a_{23} + a_{123})b_2 + a_{12}b_{23} + a_{23}b_{12}]$. And $\tau_{12}(Bel_a)\oplus\tau_{12}(Bel_b)(\{\omega_1\}) = K'[a_2(b_2 + b_{12} + b_{23} + b_{123}) + (a_{12} + a_{23} + a_{123})b_2 + a_{12}b_{23} + a_{23}b_{12}] = \frac{K'}{K}\tau_{12}(Bel_a\oplus Bel_b)(\{\omega_1\})$. In the same way we obtain equality up to normalisation constant for $\{\omega_2\}$, $\{\omega_3\}$ and analogously for $\{\omega_1,\omega_2\}$, $\{\omega_1,\omega_3\}$, $\{\omega_2,\omega_3\}$ and $\{\omega_1,\omega_2,\omega_3\}$. From these equalities and assumed normality of BFs we obtain $K' = K$ and full equality for all subsets of the frame of discernment Ω_3 Hence the assertion holds true. □

Theorem 6 *Any permutation π of a 3-element frame of discernment Ω_3 is an automorphism of D_3.*

Proof. We can verify homomorphic properties of individual permutations analogously to the proof for transpositions. Or we can use that $\pi_{213}(\omega_1,\omega_2,\omega_3) = (\omega_2,\omega_1,\omega_3) = \tau_{12}$, $\pi_{231}(\omega_1,\omega_2,\omega_3) = (\omega_2,\omega_3,\omega_1) = \tau_{12}\tau_{23}$, $\pi_{132}(\omega_1,\omega_2,\omega_3) = (\omega_1,\omega_3,\omega_2) = \tau_{12}\tau_{23}\tau_{13}$, $\pi_{312}(\omega_1,\omega_2,\omega_3) = (\omega_3,\omega_1,\omega_2) = \tau_{12}\tau_{23}\tau_{13}\tau_{12}$, $\pi_{321}(\omega_1,\omega_2,\omega_3) = (\omega_3,\omega_2,\omega_1) = \tau_{12}\tau_{23}\tau_{13}\tau_{12}\tau_{23}$, $\pi_{123}(\omega_1,\omega_2,\omega_3) = (\omega_1,\omega_2,\omega_3) = \tau_{12}\tau_{23}$ $\tau_{13}\tau_{12}\tau_{23}\tau_{13}$ and keeping of homomorphic properties by composition. □

Considering function '$-$' as transposition (permutation), we have $f(a,b) = (a,b)\oplus(b,a)$ a Dempster's sum of all (both in the case of BFs on Ω_2) permutations of Bel given by (a,b) on Ω_2. Analogously we can define

$$f(Bel) = \bigoplus_{\pi\in\Pi_3}\pi(Bel) \tag{4.1}$$

where $\Pi_3 = \{\pi_{123}, \pi_{213}, \pi_{231}, \pi_{132}, \pi_{312}, \pi_{321}\}$, i.e.,

$$\begin{gathered} f(a,b,c,d,e,f;g) = \bigoplus_{\pi\in\Pi_3}\pi(a,b,c,d,e,f;g) = \\ (a,b,c,d,e,f;g)\oplus(b,a,c,d,f,e;g)\oplus(b,c,a,f,d,e;g)\oplus \\ (a,c,b,e,d,f;g)\oplus(c,a,b,e,f,d;g)\oplus(c,b,a,f,e,d;g). \end{gathered} \tag{4.2}$$

Theorem 7 *Function $f : D_3 \longrightarrow S$, $f(Bel) = \bigoplus_{\pi\in\Pi_3}\pi(Bel)$ is homomorphism of Dempster's semigroup $\mathbf{D}_3$ to its subsemigroup $S = (\{(a,a,a,b,b,b;1-3a-3b)\},\oplus)$.*

Proof. From homomorpic properties of permutations and commutativity of homomorhism with $\oplus$ we have also the homomorphic property of f. The rest is verification that $\bigoplus_{\pi\in\Pi_3}\pi(Bel)$ is in S. We can either compute $\bigoplus_{\pi\in\Pi_3}\pi(a,b,c,d,e,f;g)$ according to Equation 4.2, further, compute $h(f(Bel)$ and verify that it is equal to U_3, i.e. compute (using (4.2)) a verify that the following holds true: $h(f(Bel)) = \bigoplus_{\pi\in\Pi_3}\pi(a,b,c,d,e,f;g)\oplus U_3 = U_3$. Or alternatively, it follows the symmetry property of BFs from S: $m(\{\omega_1\}) = m(\{\omega_2\}) = m(\{\omega_3\})$ and $m(\{\omega_1,\omega_2\}) = m(\{\omega_1,\omega_3\}) = m(\{\omega_2,\omega_3\})$, which holds for any BF $Bel\in S$ and its corresponding bba m; and further the fact that Dempster's sum of all 6 permutations of any Bel on Ω_3 is symmetric. □

We have to note here, that Dempter's sum of 3 transpositions is not enough for a homomorphism to S. As a counterexample, we can use $Bel = (0.1,0,0,0,0.2,0.6)$: we obtain $(0.1,0,0,0,0.2,0.6)\oplus(0,0.1,0,0,0,0.6,0.2)\oplus(0,0,0.1,0.6,0.2,0) = (\frac{7}{95},\frac{11}{95},\frac{36}{95},0,\frac{20}{95},\frac{20}{95};\frac{1}{95})\oplus(0,0,0.1,0.6,0.2,0;0.1) = (\frac{303}{674},\frac{77}{674},\frac{205}{674},\frac{6}{674},\frac{62}{674},\frac{20}{674};\frac{1}{674})$. This is obvious, as we have used the original BF Bel, $\tau_{12}(Bel)$, and $\tau_{13}(Bel)$, where $\tau_{13}(Bel)\neq\tau_{23}(\tau_{12}(Bel))$. Thus the transpositions do not make a cycle and mutual interchange of m-values of ω_2 and ω_3 is not used in fact.

Having homomorphism f, we can leave a question of existence $-Bel$ such that $h(-Bel) = -h(Bel)$, where '$-$' from group of BBFs G_3 is used on the right hand side. Unfortunately, we have not an isomorphism of S to the additive group of reals as in the case of semigroup S of $\mathbf{D}_0$, thus we have an open question of subtraction there. Let us focus, at first, on the subsemigroup of quasi-Bayesian BFs for simplification.

4.2 Towards Conflicting Parts of Quasi-Bayesian Belief Functions on Ω_3

Let us consider qBBFs $(a,b,c,0,0,0;1-a-b-c)\in\mathbf{D}_{3-0}$ in this section. Following Theorem 7 we obtain the following formulation for qBBFs:

Theorem 8 *Function $f: D_{3-0}\longrightarrow S_0$, $f(Bel) = \bigoplus_{\pi\in\Pi_3}\pi(Bel)$ is homomorphism of Dempster's semigroup $\mathbf{D}_{3-0}$ to its subsemigroup $S_0 = (\{(a,a,a,0,0,0;1-3a)\},\oplus)$.*

Proof. $\mathbf{D}_{3-0}$ is subalgebra of $\mathbf{D}_3$, thus homomorphic properties are preserved. Further, $f(Bel)\in S$ as a Dempster's sum of elements from $\mathbf{D}_{3-0}$ must be in $\mathbf{D}_{3-0}$ again, i.e. in S_0 which is both restriction of S to $\mathbf{D}_{3-0}$ and also a subalgebra of S and $\mathbf{D}_{3-0}$. □

S_0 is isomorphic to the positive cone of the additive group of reals, see Theorem 3, thus there is subtraction which is necessary for completion of diagram from Figure 3.4. Utilizing isomorphism with reals, we have also existence of 'Dempster's sixth'[13] which is needed to obtain preimage of $f(Bel)$ in S_0:

Lemma 3 *'Dempster's sixth'. Having $f(Bel_S)$ in S_0, there is unique $f^{-1}(f(Bel_S))\in S_0$, such that $\bigoplus_{(6\text{-}times)} f^{-1}(f(Bel_S)) = f(Bel_S)$. If $Bel_S\in S_0$ then $f^{-1}(f(Bel_S)) = Bel_S$.*

Proof. Utilizing isomorphism of S_0 with the positive cone of the additive group of reals we obtain unique $'\frac{1}{6}Bel'\in S_0$ such that $\bigoplus_{(6\text{-}times)}{'\frac{1}{6}Bel'} = Bel$ for any $Bel\in S_0$. Specially also for $f(Bel_S)$ if it is in S_0. The second part of the statement follows uniqueness of $'\frac{1}{6}Bel'$. □

On the other hand there is a complication considering qBBFs on Ω_3 that their non-conflicting part is a consonant BF frequently out of $\mathbf{D}_{3-0}$. Hence we can simply use the advantage of properties of S_0 only for qBBFs with singleton simple support non-conflicting parts.

Lemma 4 *(i) Quasi-Bayesian belief functions which have quasi-Bayesian non-conflicting part are just BFs from the following sets $Q_1 = \{(a,b,b,0,0,0)\,|\,a\geq b\}$, $Q_2 = \{(b,a,b,0,0,0)\,|\,a\geq b\}$, $Q_3 = \{(b,b,a,0,0,0)\,|\,a\geq b\}$.*

(ii) Q_1,Q_2,Q_3 with $\oplus$ are subsemigroups of $\mathbf{D}_{3-0}$; their union $Q = Q_1\cup Q_2\cup Q_3$ is not closed w.r.t. $\oplus$ thus it is not a subalgebra of $\mathbf{D}_{3-0}$. $(Q_1,\oplus)$ is further subsemigroup of $D_{1-2=3} = (\{(d_1,d_2,d_2,0,0,0)\}$,

[13] Analogously we can show existence of general 'Demspter's k-th' for any natural k and any BF Bel from S_0, but we are interested in 'Dempster's sixth' in our case.

$\oplus, 0, U_3)$; *analogously* $(Q_2, \oplus)$ *is subsemigroup of* $D_{2-1=3}$, *and* $(Q_3, \oplus)$ *is subsemigroup of* $D_{3-1=2}$, *see [14]. Following this, we can denote* $(Q_i, \oplus)$ *as* $D^{i \geq j = k}_{i-j=k}$.

Proof. From construction of non-conflicting part Bel_0 of a BF Bel [11] we can see that for quasi-Bayesian Bel_0, i.e., singleton simple support belief function, there is $Pl = (x, y, y)$ where $x \geq y$ if $Bel_0 \in S_1$ (or $Pl = (y, x, y)$ or $Pl = (y, y, x)$ for Bel_0 from S_2 or S_3). From this we obtain $Bel = (a, b, b)$ or $Bel = (b, a, b)$ or $Bel = (b, b, a)$ where $a \geq b$, $a + b + b \leq 1$. The rest follows properties of $D_{i-j=k}$ see [13]. □

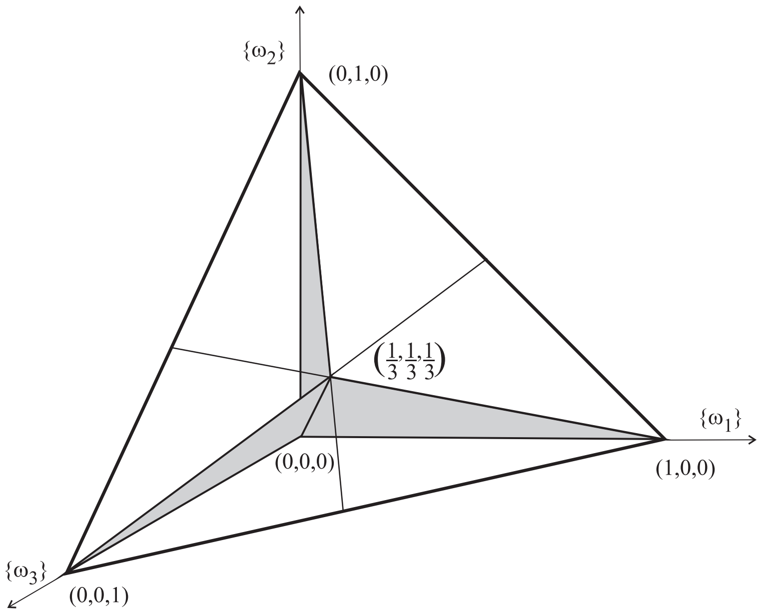

Figure 4.1: Quasi-Bayesian BFs with unique decomposition into $Bel_0 \oplus Bel_S$ on 3-element frame of discernment Ω_3.

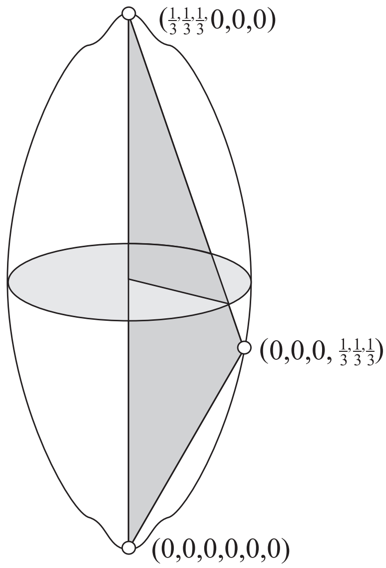

Figure 4.2: S_{Pl} — subsemigroup of general indecisive belief functions.

Theorem 9 *Belief functions Bel from* $Q = D^{1\geq 2=3}_{1-2=3} \cup D^{2\geq 1=3}_{2-1=3} \cup D^{3\geq 1=2}_{3-1=2}$ *have unique decomposition into their conflicting part* $Bel_S \in S_0$ *and non-conflicting part in* S_1 *(*S_2 *or* S_3 *respectively).*
For quasi-Bayesian BFs out of Q *(i.e. BFs from* $\mathbf{D}_{3-0} \setminus Q$*) we have not decomposition into conflicting and non-conflicting part according to Hypothesis 1, as we have not* $f(Bel_0) \in S_0$ *and have not subtraction in* S *in general.*

Proof. $Bel_0 \in S_i \subset D_{3-0}$ for $Bel \in D^{i\geq j=k}_{i-j=k}$, thus $f(Bel_0) \in S_0$ which is isomorphic to the positive cone of the additive group of reals. Hence there is subtraction and Dempster's 'sixth', which gives us unique $Bel_S \in S_0$. □

BFs from $\mathbf{D}_{3-0} \setminus Q$ either have their conflicting part in $S_{Pl} \setminus S_0$ or in $S_{Pl} \setminus S$ or have not conflicting part according to Hypothesis 1 (i.e. their conflicting part is a pseudo belief function out of D_3). Solution of the problem is related to a question of subtraction in subsemigroups S and S_{Pl}, as $f(Bel_0)$ is not in S_0 but in $S \setminus S_0$ for qBBFs out of Q. Thus we have to study these qBBFs together with general BFs from the point of view of their conflicting parts.

4.3 Towards Conflicting Parts of General Belief Functions on Ω_3

There is a special class of general BFs with singleton simple support non-conflicting part, i.e. BFs with $f(Bel_0) \in S_0$. Nevertheless due to the generality of Bel, we have $f(Bel) \in S$ in general, thus there is a different special type of belief 'subtraction' $((a, a, a, b, b, b) \ominus (c, c, c, 0, 0, 0, 0)$ for $f(Bel) \ominus f(Bel_0))$.

We are interested to follow the idea of the decomposition schema from Figure 3.4 as much as possible. What do we already have?

We have the entire right part: given Bel, $Bel \oplus U_3$, and non-conflicting part Bel_0 (Theorem 5 (i)); in the left part we have $-Bel \oplus U_3 = -(Bel \oplus U_3)$ using G_3 group '$-$' (Theorem 3 (ii)) and $-Bel_0 = (-Bel \oplus U_3)_0$ (a non-conflicting part of $-Bel \oplus U_3$). In the central part of the figure, we only have U_3 and $-Bel_0 \oplus Bel_0$ in fact. As we have not $-Bel$ we have not $-Bel \oplus Bel$, we use $f(Bel) = \bigoplus_{\pi\in\Pi_3} \pi(Bel)$ instead of it; $f(Bel) \in S$ in general, (in the special case of qBBF Bel: $f(Bel) \in S_0$).

We can also compute[14] $-Bel_0 \oplus Bel_0$; is it equal to $f(Bel_0)$? If not, what is their relation then?

One of the important questions is: When $f(Bel_S)$ is computable from $f(Bel)$ and $f(Bel_0)$ as $f(Bel) \ominus f(Bel_0)$?

The other important question is: What is a relation of $f(Bel_S)$ and Bel_S? It is not possible to compute Bel_S only from $f(Bel_S) \in S \setminus S_0$ as there should be multiple pre-images of $f(Bel_S)$ out of S_0. There is the simple one-dimensional abscissa (segment of line) S_0 in $\mathbf{D}_{3-0}$, similarly to S in classic Dempsters' semigroup $\mathbf{D}_0$ on Ω_2. Besides S_0, there is also two-dimensional triangle S (given by vertices $0, U_3, (0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3})$) and six-dimensional S_{Pl} of indecisive BFs (with $Pl(Bel) = U_3$) in $\mathbf{D}_3$ in general, see Figure 4.2.

What about 'Dempster's sixth'? We know that the unique Dempster's sixth exits for BFs from S_0. Is it unique for $f(Bel) \in S$? Does it always exist there (also out of S_0)? What it its relation to Bel and Bel_S?

Thinking about these structures, a new question arises: is the homomorphism f, as it is defined in Theorem 7, the only generalisation of f, i.e. does it hold that $f(Bel) = \bigoplus_{\pi\in\Pi_3} \pi(Bel) = -Bel \oplus Bel$? We can include these questions into the diagram of decomposition of a BF Bel into its conflicting and non-conflicting part as it is in Figure 4.3.

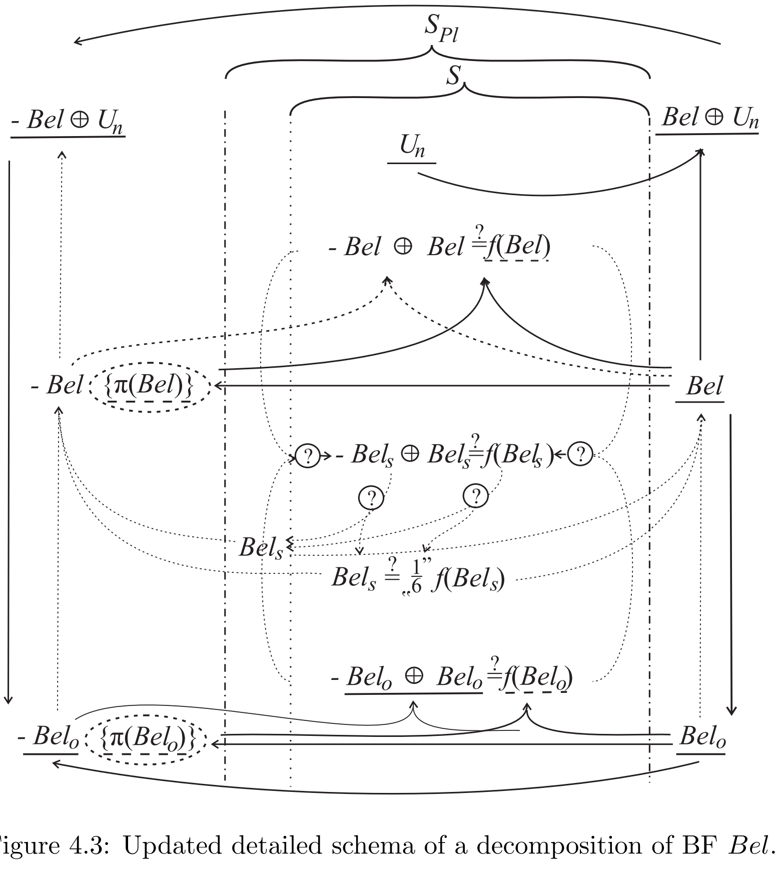

Figure 4.3: Updated detailed schema of a decomposition of BF Bel.

[14]For examples of computation of -$Bel_0 \oplus Bel_0$ and consequential results and new open problems see the Appendix.

4.4 Example of Decomposition with Conflicting Part out of S

Example 2. A general BF $Bel = (\frac{50}{128}, \frac{28}{128}, \frac{4}{128}, \frac{22}{128}, \frac{4}{128}, \frac{2}{128}; \frac{8}{128})$ has its non-conflicting part $Bel_0 = (\frac{2}{12}, 0, 0, \frac{6}{12}, 0, 0; \frac{4}{12}) = (0.166, 0, 0, 0.5, 0, 0; 0.333)$ and conflicting part $Bel_S = (\frac{3}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{3}{12}; \frac{2}{12}) = (0.2500, 0.0833, 0.0833, 0.0833, 0.0833, 0.2500; 0.1666)$ which is in $S_{Pl} = \{Bel \mid Pl\_P = U_3\}$ out of S.

5 Open Problems for a Future Research

A series of open questions were suggested in the last subsection of the previous section. The particular open problems are really numerous, nevertheless each of them is a subproblem of one of 3 main general open problems.

- Improvement of the algebraic analysis, is really necessary, especially of sugbroup S_{Pl} of indecisive BFs.
- A new question is related to two approaches to generalisation of f:
  (i) homomorphism f is defined by $f(Bel) = -Bel \oplus Bel$, respecting '$-$' on G_3,
  (ii) the presented approach which is based on permutations of elements of frame of discernment.
  Produce these approaches same f and same Bel_S (if it exists)? Or there are two different generalisations of homomorphism f and of conflicting part Bel_S from the 2-element case of frame of discernment?
- And a principal question of the study: a specification of sets (or of subalgebras) of BFs which are decomposable into $Bel_0 \oplus Bel_S$ and which are not.

6 Summary and Conclusions

New approach to understanding operation '$-$' and homomorphism f from $\mathbf{D}_0$ (a transposition of elements instead of some operation related to group 'minus' of G, G_3). is introduced in this study.

First generalisation of Hájek-Valdés homomorphism f is presented. Specification of first classes of BFs (on Ω_3) which are decomposable into $Bel_0 \oplus Bel_S$. And several other partial results were obtained.

The presented results improve general understanding of conflicts of belief functions and the entire nature of belief functions. Correct understanding of conflicts may consequently improve a combination of conflicting belief functions. These results can be also used as one of the mile-stones to further study of conflicts between belief functions.

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Appendix

7 Two Different Generalisations of Hájek-Valdés homomorphism f

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From the previous text, we have two ways of generalisation of Hájek-Valdés homomorphism f:
(i) The original way using simple generalisation of Hájek-Valdés definition from Demspter's semigroup on a two-element frame; $f(Bel) = -Bel \oplus Bel$. As it was mentioned this definition is only partial, defined for consonant and Bayesian BFs only; as we have not yet full generalisation of the operation '$-$'.
(ii) The presented new approach which is based on permutations of elements of the frame of discernment.

There is a question, whether these approaches mutually coincide, i.e., whether they produce the same generalisation of homomorphism f or not. There are two different classes of BFs, where classic way is defined, i.e., two classes where we can make a comparison. We can show that the approaches coincide on Baeysian BFs, whereas we can simply find counterexamples for consonant BFs.

Lemma 5 *$-Bel \oplus Bel = \bigoplus_{\pi\in\Pi_3} \pi(Bel)$ for Bayesian belief functions on $\mathbf{D}_3$ (i.e., for BFs on G_3).*

Proof. $-Bel$ is defined such that, $-Bel \oplus Bel = U_3$ on G_3. From definition of $\bigoplus_{\pi\in\Pi_3}$we can easy see that it is symmetric BF (i.e., $\bigoplus_{\pi\in\Pi_3} \in S$). All the permutations of a Bayesian BF are Bayesian again, thus $\bigoplus_{\pi\in\Pi_3} = U_3$, as U_3 is the only symmetric Bayesian BF. (Alternatively, we can take any general consonant BF, e.g., $(a, 0, 0, b, 0, 0; 1-a-b)$, all its permutations: $(a, 0, 0, 0, b, 0; 1-a-b)$, $(0, 0, 0, b, 0, 0; 1-a-b)$, ..., and combine them; we obtain U_3 as a result again). □

Example 3. A counterexample: a general consonant belief function.
Let us take consonant BFs $Bel_{cons} = (\frac{1}{4}, 0, 0, \frac{2}{4}, 0, 0; \frac{1}{4})$. We obtain corresponding $Pl_{cons} = (\frac{4}{4}, \frac{3}{4}, \frac{1}{4}, \frac{4}{4}, \frac{4}{4}, \frac{3}{4}; \frac{4}{4})$ and $Pl\_P_{cons} = (\frac{4}{8}, \frac{3}{8}, \frac{1}{8})$, further $h(Bel_{cons}) = (\frac{4}{8}, \frac{3}{8}, \frac{1}{8}, 0, 0, 0; 0)$. Thus $-h(Bel_{cons}) = (\frac{3}{19}, \frac{4}{19}, \frac{12}{19}, 0, 0, 0; 0)$ (as $(-h(Bel_{cons}))(\{\omega_1\}) = \frac{3\cdot 1}{3\cdot 1+4\cdot 1+4\cdot 3} = \frac{3}{19}$, etc.) and $-Bel_{cons} = (-h(Bel_{cons}))_0 = (0, 0, \frac{8}{12}, 0, 0, \frac{1}{12}; \frac{3}{12})$.

Hence, we can easily compute $Bel_{cons} \oplus -Bel_{cons} = (\frac{1}{4}, 0, 0, \frac{2}{4}, 0, 0; \frac{1}{4}) \oplus (0, 0, \frac{8}{12}, 0, 0, \frac{1}{12}; \frac{3}{12}) = (\frac{3}{48}, \frac{2}{48}, \frac{8}{48}, \frac{6}{48}, 0, \frac{1}{48}; \frac{3}{48} | \frac{8+1+16}{48}) = (\frac{3}{23}, \frac{2}{23}, \frac{8}{23}, \frac{6}{23}, 0, \frac{1}{23}; \frac{3}{23})$. We can verify, that $h(Bel_{cons} \oplus -Bel_{cons}) = U_3$, because of related plausibility is equal to $(\frac{3+6+3}{23}, \frac{2+6+1+3}{23}, \frac{8+1+3}{23}, \frac{15}{23}, \frac{20}{23}, \frac{20}{23}; \frac{23}{23})$. Unfortunately, it is not a symmetric BF which $\bigoplus_{\pi\in\Pi_3} \pi(Bel_{cons})$ should be.

Let us compute $\bigoplus_{\pi\in\Pi_3} \pi(Bel_{cons})$ now. It is equal to $((\frac{1}{4}, 0, 0, \frac{2}{4}, 0, 0; \frac{1}{4}) \oplus (\frac{1}{4}, 0, 0, 0, \frac{2}{4}, 0; \frac{1}{4})) \oplus ((0, \frac{1}{4}, 0, \frac{2}{4}, 0, 0; \frac{1}{4}) \oplus (\frac{1}{4}, 0, 0, 0, 0, \frac{2}{4}; \frac{1}{4})) \oplus ((0, 0, \frac{1}{4}, 0, \frac{2}{4}, 0; \frac{1}{4}) \oplus (0, 0, \frac{1}{4}, 0, 0, \frac{2}{4}; \frac{1}{4}))$. Dempster's sum of the first couple is equal to $(\frac{4+6+1}{16}, 0, 0, \frac{2}{16}, \frac{2}{16}, 0; \frac{1}{16})$, analogously sums of the second and third couples are $(0, \frac{11}{16}, 0, \frac{2}{16}, 0, \frac{2}{16}; \frac{1}{16})$ and $(0, 0, \frac{11}{16}, 0, \frac{2}{16}, \frac{2}{16}; \frac{1}{16})$. The rest is Dempster's sum of these three partial results; thus we obtain[15] $(\frac{11\cdot 3+2\cdot 2}{256}, \frac{2\cdot 13+1\cdot 11}{256}, \frac{2\cdot 2}{256}, \frac{2\cdot 3+1\cdot 2}{256}, \frac{2\cdot 1}{256}, \frac{2\cdot 1}{256}; \frac{1\cdot}{256} | \frac{11\cdot 13+2\cdot 11}{256}) = (\frac{37}{91}, \frac{37}{91}, \frac{4}{91}, \frac{8}{91}, \frac{2}{91}, \frac{2}{91}; \frac{1}{91})$ and $(\frac{37}{91}, \frac{37}{91}, \frac{4}{91}, \frac{8}{91}, \frac{2}{91}, \frac{2}{91}; \frac{1}{91}) \oplus (0, 0, \frac{11}{16}, 0, \frac{2}{16}, \frac{2}{16}; \frac{1}{16}) = (\frac{2\cdot 45+37}{91\cdot 16}, \frac{2\cdot 45+37}{1456}, \frac{11\cdot 9+2\cdot 6+2\cdot 6+4}{1456}, \frac{8}{1456}, \frac{2\cdot 3+2}{1456}, \frac{2\cdot 3+2}{1456}; \frac{1\cdot}{1456} | \frac{11\cdot 37+11\cdot 37+11\cdot 8+2\cdot 37+2\cdot 37}{1456}) = (\frac{127}{406}, \frac{127}{406}, \frac{127}{406}, \frac{8}{406}, \frac{8}{406}, \frac{8}{406}; \frac{1}{406}) \in S$.

Hence we have verified that $Bel_{cons} \oplus -Bel_{cons} \neq \bigoplus_{\pi\in\Pi_3} \pi(Bel_{cons})$.

Example 4. A counterexample: a singleton simple support belief function.
Let us show also a counterexample for a representative of the simplest consonant BFs, a singleton simple support BF. Let us take $Bel_{sSSF} = (\frac{1}{4}, 0, 0, 0, 0, 0; \frac{3}{4})$ now. Analogously to the previous example, we obtain corresponding $Pl_{sSSF} = (\frac{4}{4}, \frac{3}{4}, \frac{3}{4}, \frac{4}{4}, \frac{4}{4}, \frac{3}{4}; \frac{4}{4})$ and $Pl\_P_{sSSF} = (\frac{4}{10}, \frac{3}{10}, \frac{3}{10})$, further $h(Bel_{sSSF}) = (\frac{4}{10}, \frac{3}{10}, \frac{3}{10}, 0,0,0;0)$. Thus $-h(Bel_{sSSF}) = (\frac{3}{11}, \frac{4}{11}, \frac{4}{11}, 0,0,0;0)$ (as $(-h(Bel_{sSSF}))(\{\omega_1\}) = \frac{3\cdot 3}{3\cdot 3+3\cdot 4+3\cdot 4} = \frac{9}{33} = \frac{3}{11}$, etc.) and $-Bel_{sSSF} = (-h(Bel_{sSSF}))_0 = (0, 0, 0, 0, 0, \frac{1}{4}; \frac{3}{4})$.

[15] We use 2^n notation, i.e., 8-tuples $(m_{\textcircled{\cap}}(\{\omega_1\}), m_{\textcircled{\cap}}(\{\omega_2\}), m_{\textcircled{\cap}}(\{\omega_3\}), m_{\textcircled{\cap}}(\{\omega_1, \omega_2\}), m_{\textcircled{\cap}}(\{\omega_1, \omega_2\}), m_{\textcircled{\cap}}(\{\omega_2, \omega_3\}); m_{\textcircled{\cap}}(\Omega) \,|\, m_{\textcircled{\cap}}(\emptyset))$ for representation non-normalised intermediate results; $m_{\textcircled{\cap}}(\Omega)$ is separated by semicolon and sum of multiples of conflicting belief masses $m_{\textcircled{\cap}}(\emptyset)$ by '|' there.

Hence, we can easily compute $Bel_{sSSF} \oplus -Bel_{sSSF} = (\frac{1}{4}, 0, 0, 0, 0, 0; \frac{3}{4}) \oplus (0, 0, 0, 0, 0, \frac{1}{4}; \frac{3}{4}) = (\frac{3}{15}, 0, 0, 0, 0, \frac{3}{15}; \frac{9}{15})$. Analogously to the previous example we have $h(Bel_{sSSF} \oplus -Bel_{cons}) = U_3$; unfortunately the result is not symmetric again.
$\bigoplus_{\pi\in\Pi_3} \pi(Bel_{sSSF}) = ((\frac{1}{4}, 0, 0, 0, 0, 0; \frac{3}{4}) \oplus (\frac{1}{4}, 0, 0, 0, 0, 0; \frac{3}{4})) \oplus ((0, \frac{1}{4}, 0, 0, 0, 0; \frac{3}{4}) \oplus (\frac{1}{4}, 0, 0, 0, 0, 0; \frac{3}{4})) \oplus ((0, 0, \frac{1}{4}, 0, 0, 0; \frac{3}{4}) \oplus (0, 0, \frac{1}{4}, 0, 0, 0; \frac{3}{4})) = (\frac{7}{16}, 0, 0, 0, 0, 0; \frac{9}{16}) \oplus (0, \frac{7}{16}, 0, 0, 0, 0; \frac{9}{16}) \oplus (0, 0, \frac{7}{16}, 0, 0, 0; \frac{9}{16}) = (\frac{9\cdot7}{256}, \frac{9\cdot7}{256}, 0, 0, 0, 0; \frac{9\cdot9}{256} | \frac{7\cdot7}{256}) \oplus (0, 0, \frac{7}{16}, 0, 0, 0; \frac{9}{16}) = (\frac{9\cdot7}{207}, \frac{9\cdot7}{207}, 0, 0, 0, 0; \frac{9\cdot9}{207}) \oplus (0, 0, \frac{7}{16}, 0, 0, 0; \frac{9}{16}) = (\frac{9\cdot9\cdot7}{9\cdot9(7+7+7+9)}, \frac{9\cdot9\cdot7}{81\cdot30}, \frac{9\cdot9\cdot7}{81\cdot30}, 0, 0, 0; \frac{9\cdot9\cdot9}{81\cdot30} | \frac{9\cdot7\cdot7+9\cdot7\cdot7}{81\cdot30}) = (\frac{7}{30}, \frac{7}{30}, \frac{7}{30}, 0, 0, 0; \frac{9}{30}) \in S_0$.
Hence we have verified that $Bel_{sSSF} \oplus -Bel_{sSSF} \neq \bigoplus_{\pi\in\Pi_3} \pi(Bel_{sSSF})$.

Any of the above counterexamples shows that $-Bel \oplus Bel$ is not equal to $\bigoplus_{\pi\in\Pi_3} \pi(Bel)$ in general.

Lemma 6 *$-Bel \oplus Bel$ is not equal to $\bigoplus_{\pi\in\Pi_3} \pi(Bel)$ in general. Thus there are two different generalisations of homomorphism f to $\mathbf{D}_3$.*

Learning this, a series of new open problems arises, both theoretic algebraic problems and problems related to the decomposition of a BF into conflicting and non-conflicting parts. On the other hand, we can update the diagram of decomposition of a BF *Bel* into its conflicting and non-conflicting part, as it is in Figure 7.1.

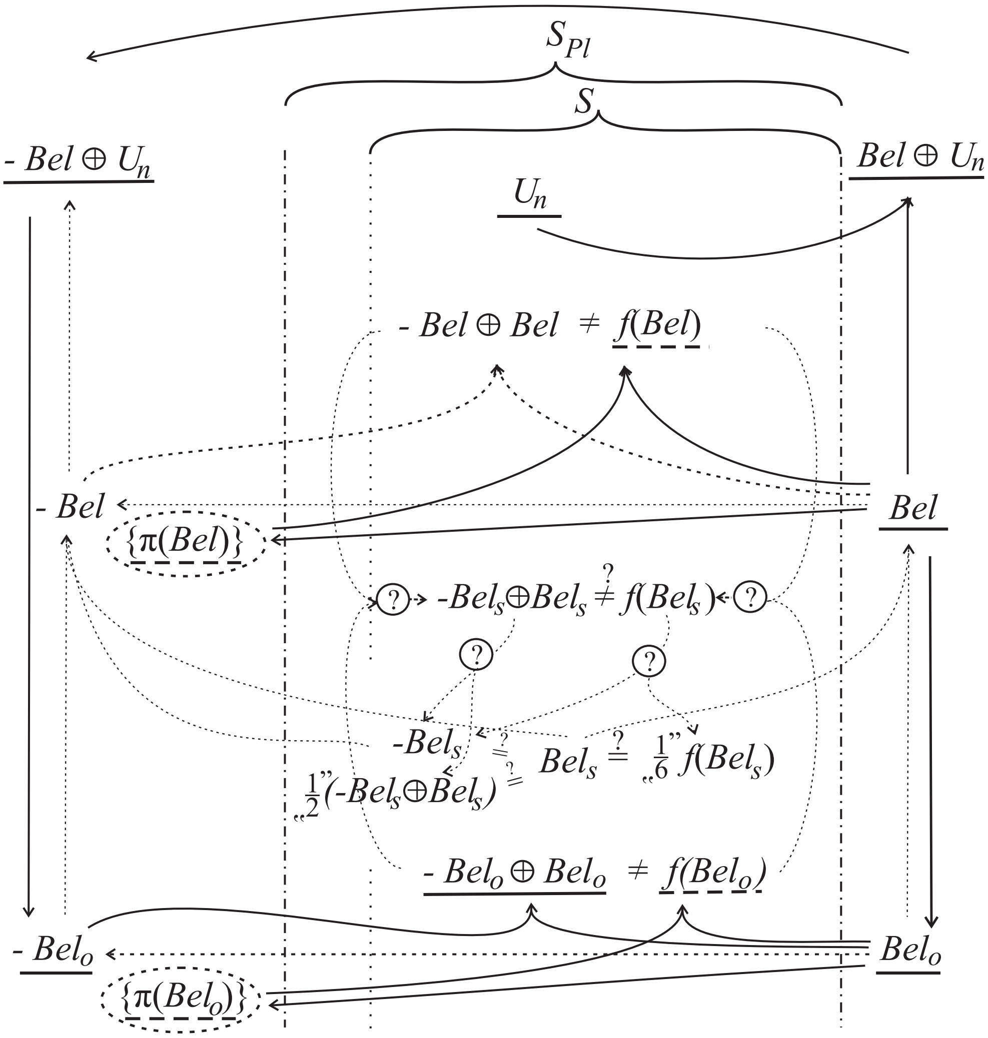

Figure 7.1: Updated schema of decomposition of *Bel*.

Neither $-Bel_0 \oplus Bel_0$ is equal to $f(Bel_0) = \bigoplus_{\pi \in \Pi_3} \pi(Bel_0)$ nor $-Bel \oplus Bel$ is equal to $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel)$ in general. We yet do not know, what is a relationship of these two approaches? What is their relationship to the decomposition of *Bel*. Whether one of them (and which one) can be used for the decomposition of a BF into conflicting and non-conflicting parts. Thus it is more correct to use $Bel \oplus -Bel \neq \bigoplus_{\pi \in \Pi_3} \pi(Bel)$ instead of the original $Bel \oplus -Bel \stackrel{?}{=} f(Bel)$ and $Bel \oplus -Bel \neq f(Bel)$ (analogously for Bel_0) as it is in Figure 7.2.

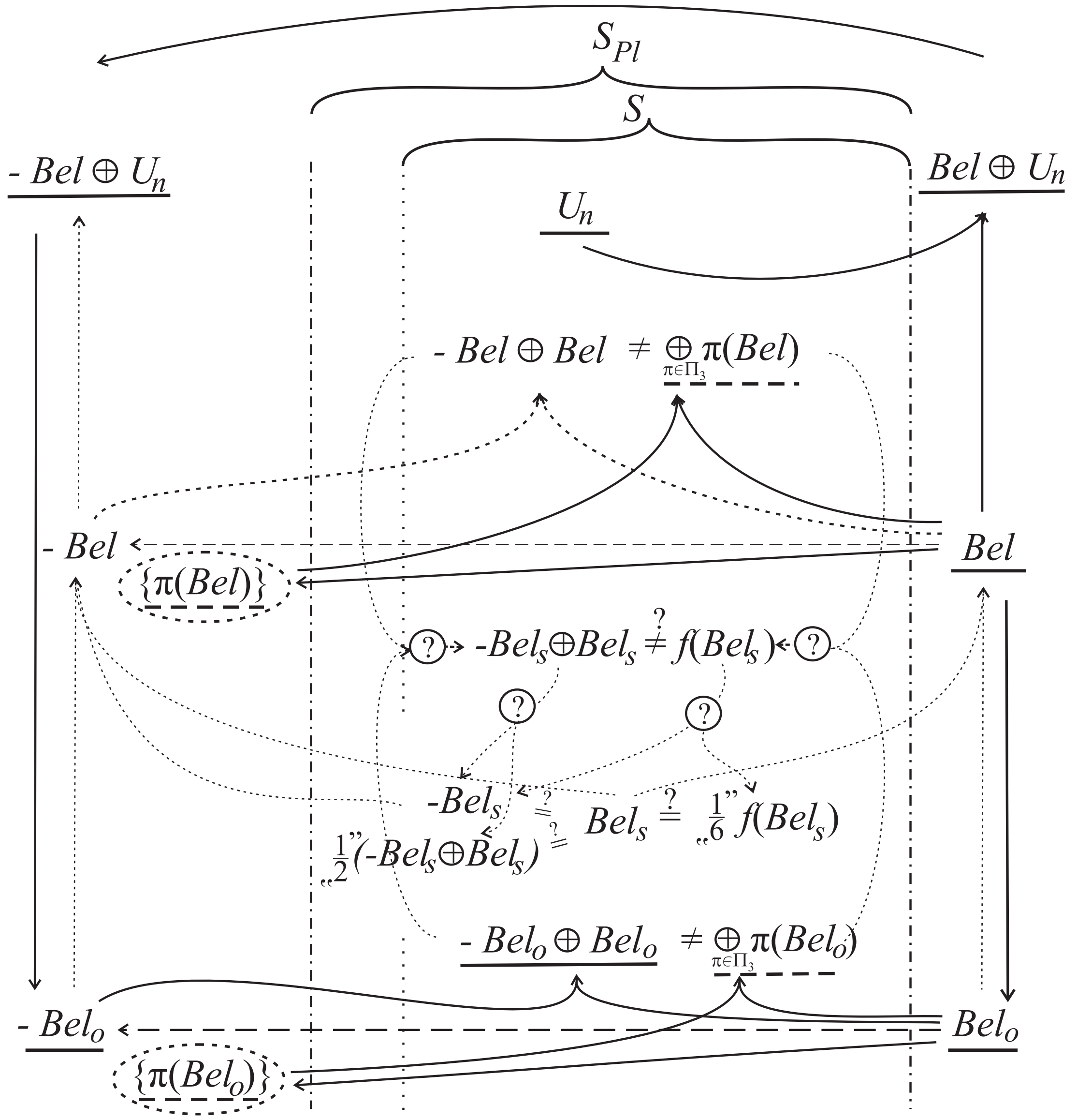

Figure 7.2: Updated and corrected schema of decomposition of *Bel*.

8 Updated Open Problems

There are three main general open problems coming from this study:

- Elaboration of algebraic analysis. Besides problems from [13] and [14] related to generalisation of Dempster's semigroup to a general finite frame of discernment, there has arisen a special importance of an algebraic analysis of sugbroup S_{Pl} (indecisive BFs).

- New question are related to two approaches to generalisation of f:
  (i) homomorphism f defined by $f(Bel) = -Bel \oplus Bel$, respecting '$-$' on G_3,
  (ii) the presented new approach which is based on permutations of elements of the frame of discernment.
  What are the properties of this two different generalisations of homomorphism f; where these generalisation mutually coincide and where not (supposing to find a full generalisation of the classic way of definition); what is their relationship? What is a relationship of these generalisations to conflicting part of a belief function and to decomposition of a BF into its conflicting and non-conflicting parts?

- Principal question of the study: verification of Hypothesis 1; otherwise a specification of sets (or of subalgebras) of BFs which are decomposable into $Bel_0 \oplus Bel_S$ and which are not.

9 Conclusion

New approach to understanding operation '$-$' and homomorphism f from $\mathbf{D}_0$ (a transposition of elements instead of some operation related to group 'minus' of G, G_3) is introduced in this study.

The first complete generalisation of Hájek-Valdés important homomorphism f is presented. It was observed, that this generalisation differs from the previous partial generalisation using partially generalised operation $-$ (defined only for consonant and Bayesian BFs), thus a series of new open problems has arisen. Specification of several classes of BFs (on Ω_3) which are decomposable into $Bel_0 \oplus Bel_S$, and several other partial results were obtained.

The presented results improve general understanding of conflicts of BFs and of the entire nature of belief functions. These results can be also used as one of the mile-stones to further study of conflicts between belief functions. Correct understanding of conflicts may consequently improve a combination of conflicting belief functions.