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# A Hybrid Method for Solving Absolute Value Equations 

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## Abstract:

We present a hybrid method for solving an absolute value equation of the form $x+B|x|=b$ with $\varrho(|B|)<1$. It first uses the iterative process $x^{i+1}=-B\left|x^{i}\right|+b$ performed until certain condition is met, then the unique solution $x^{*}$ of the equation is found by solving a single system of linear equations. The method is shown to work whenever all entries of $x^{*}$ are nonzero. ${ }^{\square}$


Keywords:
Absolute value equation, iterative method, hybrid method.

[^0]
## 1 Introduction

In [罒, the authors proposed an iterative method for solving an absolute value equation of the form

$$
\begin{equation*}
x+B|x|=b \tag{1.1}
\end{equation*}
$$

They showed that if

$$
\begin{equation*}
\varrho(|B|)<1, \tag{1.2}
\end{equation*}
$$

then the sequence $\left\{x^{i}\right\}_{i=0}^{\infty}$ generated by

$$
\begin{equation*}
x^{0}=b \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
x^{i+1}=-B\left|x^{i}\right|+b \quad(i=0,1, \ldots) \tag{1.4}
\end{equation*}
$$

tends to the unique solution $x^{*}$ of the equation (떼) and, moreover, that there holds

$$
\begin{equation*}
\left|x^{*}-x^{i+1}\right| \leq N\left|x^{i+1}-x^{i}\right| \tag{1.5}
\end{equation*}
$$

for each $i \geq 0$, where

$$
\begin{equation*}
N=(I-|B|)^{-1}-I . \tag{1.6}
\end{equation*}
$$

The condition ( $[-2$ ) is equivalent to $N \geq 0$ (Horn and Johnson [ [ ] ).
In this note we show that under mild assumption (inequality ( nate generation of the sequence $\left\{x^{i}\right\}$ after a finite number of steps and use the information gathered in the last generated iteration to find the unique solution $x^{*}$ by solving a single system of linear equations. This is what we call the hybrid method.

We use the following notation. Inequalities and absolute value are taken entrywise; "o" denotes the Hadamard (entrywise) product of vectors, $\operatorname{diag}(z)$ denotes the diagonal matrix with diagonal vector $z$ and for $x \in \mathbb{R}^{n}$, the sign vector of $x$ is defined by $(\operatorname{sgn}(x))_{i}=1$ if $x_{i} \geq 0$ and $(\operatorname{sgn}(x))_{i}=-1$ otherwise $(i=1, \ldots, n)$. Notice that $|x|=\operatorname{diag}(\operatorname{sgn}(x)) x$ for each $x \in \mathbb{R}^{n} . \varrho(A)$ stands for the spectral radius of $A$ and $I$ is the identity matrix.

## 2 The hybrid method

We shall need the following auxiliary result.
Theorem 1. If $x, y \in \mathbb{R}^{n}$ satisfy

$$
\begin{equation*}
|x-y|<|y|, \tag{2.1}
\end{equation*}
$$

then

$$
\begin{equation*}
0<x \circ y<2 y \circ y . \tag{2.2}
\end{equation*}
$$

Proof. For each $i$, ( $\mathbb{L}$ ) implies $y_{i} \neq 0$, and we have

$$
\left|x_{i} y_{i}-y_{i}^{2}\right|=\left|x_{i}-y_{i}\right|\left|y_{i}\right|<\left|y_{i}\right|^{2}=y_{i}^{2}
$$

hence

$$
-y_{i}^{2}<x_{i} y_{i}-y_{i}^{2}<y_{i}^{2}
$$

and

$$
0<x_{i} y_{i}<2 y_{i}^{2}
$$

which amounts to ( 2 ).
Now the main idea behind the hybrid method is contained in the following theorem.
Theorem 2. Let $\varrho(|B|)<1$ and let the sequence $\left\{x^{i}\right\}$ generated by (I.3), (1.4) satisfy

$$
\begin{equation*}
N\left|x^{i+1}-x^{i}\right|<\left|x^{i+1}\right| \tag{2.3}
\end{equation*}
$$

for some $i$, where $N$ is as in (L.6). Then the unique solution $x^{*}$ of (I.لत) is given by

$$
\begin{equation*}
x^{*}=\left(I+B \operatorname{diag}\left(\operatorname{sgn}\left(x^{i+1}\right)\right)\right)^{-1} b \tag{2.4}
\end{equation*}
$$

Proof. From (L. 4 ) and (2.3) we have

$$
\left|x^{*}-x^{i+1}\right| \leq N\left|x^{i+1}-x^{i}\right|<\left|x^{i+1}\right|,
$$

hence $x^{*} \circ x^{i+1}>0$ by Theorem $\mathbb{D}$ which means that both $x^{*}$ and $x^{i+1}$ belong to the interior of the same orthant of $\mathbb{R}^{n}$. Thus $\operatorname{sgn}\left(x^{*}\right)=\operatorname{sgn}\left(x^{i+1}\right)$ and consequently $\left|x^{*}\right|=$ $\operatorname{diag}\left(\operatorname{sgn}\left(x^{*}\right)\right) x^{*}=\operatorname{diag}\left(\operatorname{sgn}\left(x^{i+1}\right)\right) x^{*}$. Since $x^{*}$ solves

$$
x^{*}+B\left|x^{*}\right|=b,
$$

it also solves

$$
x^{*}+B \operatorname{diag}\left(\operatorname{sgn}\left(x^{i+1}\right)\right) x^{*}=b
$$

hence $x^{*}$ is given by the explicit formula ( $\left.\mathbb{L}, \mathbb{Z}\right)$. Invertibility of $I+B \operatorname{diag}\left(\operatorname{sgn}\left(x^{i+1}\right)\right)$ is guaranteed by the assumption ( $\square$ ).

Finally we show a necessary and sufficient condition for the hybrid method to work. Notice that $\left|x^{*}\right|>0$ is equivalent to $x_{i}^{*} \neq 0$ for each $i$.

Theorem 3. Let $\varrho(|B|)<1$. Then the sequence $\left\{x^{i}\right\}$ generated by (L. ${ }^{\text {(1.4), satisfies }}$

$$
\begin{equation*}
N\left|x^{i+1}-x^{i}\right|<\left|x^{i+1}\right| \tag{2.5}
\end{equation*}
$$

for some $i$ if and only if $\left|x^{*}\right|>0$.
Proof. Let $\left|x^{*}\right|>0$. Since $x_{i} \rightarrow x^{*}$, we have that

$$
\lim _{i \rightarrow \infty}\left(\left|x^{i+1}\right|-N\left|x^{i+1}-x^{i}\right|\right)=\left|x^{*}\right|>0
$$

hence by the definition of limit there exists an $i_{0}$ such that

$$
\left|x^{i+1}\right|-N\left|x^{i+1}-x^{i}\right|>0
$$

holds even for each $i \geq i_{0}$. Conversely, if (2.0) holds for some $i$, then as in the proof of Theorem we obtain

$$
\left|x^{*}-x^{i+1}\right|<\left|x^{i+1}\right|
$$

which means that $\left|x^{*}\right|>0$ since $x_{j}^{*}=0$ for some $j$ would imply $\left|x_{j}^{i+1}\right|<\left|x_{j}^{i+1}\right|$, a contradiction.

## Bibliography

[1] W. Barth and E. Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing, 12 (1974), pp. 117-125. प
[2] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[3] J. Rohn, V. Hooshyarbakhsh, and R. Farhadsefat, An iterative method for solving absolute value equations and sufficient conditions for unique solvability, Optimization Letters, 8 (2014), pp. 35-44.


[^0]:    ${ }^{1}$ This work was supported with institutional support RVO:67985807.
    ${ }^{2}$ Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4] x_{1}+[-2,1] x_{2}=[-2,2],[-1,2] x_{1}+[2,4] x_{2}=[-2,2]$ (Barth and Nuding [畂)).

