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# Fuzzified linear orderings, fuzzy maxima and minima

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Abstract:

The report collects the definitions and theorems on fuzzy orderings that are prerequisites for the author's forthcoming paper 'Maxima and minima in fuzzified linear orderings'. Most of these results are already known from the literature; in this report they are presented in a systematic manner, and re-proved in a certain format which enables a direct translation into higher-order fuzzy logic MTL-Delta. The theorems can thus be employed in further developments of logic-based fuzzy mathematics formalized in higher-order MTL-Delta, even if presented here in the usual syntax of traditional fuzzy mathematics.

Keywords:
Fuzzy relation; Similarity relation; Fuzzy ordering; Fuzzy maximum; Higher-order fuzzy logic.

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# 1 Introduction

This report collects the results from the theory of similarity relations and fuzzy orderings that are prerequisite for the author's forthcoming paper [3]. The results are presented in a systematic manner to serve as 'extended preliminaries' to [3] for non-experts in fuzzy orderings. The report therefore covers the two topics relevant for [3], namely the correspondence between similarity relations and fuzzy orderings in fuzzified crisp linear orders (Section 3) and the notions of fuzzy bounds, maxima, minima, suprema, and infima of fuzzy sets in fuzzy orderings (Section 4). Two alternative notions of fuzzy maxima and minima (depending on whether lattice conjunction or residuated conjunction is employed in the definition) are discussed, and a justification for the weaker notions employing lattice conjunction is offered; the results proved in [6] for the stronger notions based on residuated conjunction are adapted here for the weaker, lattice conjunction–based notions.

Most of the results presented here are already known from the literature [14, 15, 16, 17, 21]. The secondary aim of this report is to re-prove these known results in a certain format (described in Section 5) that enables a direct translation into the formalism of higher-order fuzzy logic $\mathrm{MTL}_\triangle$. There are two motivations for this enterprise: the general one is the fact that when formalized in higher-order $\mathrm{MTL}_\triangle$, these theorems can be utilized in further developments of logic-based fuzzy mathematics [7, 8, 9]. A more particular motivation is connected with the fact that the results of both the present report and the paper [3] are themselves prerequisites for fuzzy semantics of counterfactual conditionals (preliminarily described in [13]), which is carried out in higher-order fuzzy logic. For the applicability of these theorems to fuzzy counterfactuals we thus need to demonstrate that they are formally provable in higher-order fuzzy logic, which may not be immediately clear from their proofs found in the literature. As explained in Section 5, the format of proofs given here ensures that all of the results are indeed provable in higher-order fuzzy logic. A tertiary aim of this report is thus to demonstrate that formal results of logic-based fuzzy mathematics provable in higher-order fuzzy logic can be presented in the syntax of traditional fuzzy mathematics, which may be more accessible to a broader community of fuzzy mathematicians.

# 2 Preliminaries

In this section we collect some standard definitions and lemmata on fuzzy sets and fuzzy relations which will be needed in the following sections.

## 2.1 Membership degrees

Unless said otherwise, in this report we value all fuzzy sets and fuzzy relations in an arbitrary fixed complete linear MTL-algebra:[2]

**Definition 2.1** ([22])**.** The algebra $\mathbf{L} = \langle L, *, \Rightarrow, \wedge, \vee, 0, 1\rangle$ is a *complete linear* MTL-*algebra* if:

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a linearly ordered complete lattice
- $\langle L, *, \wedge, \vee, 1\rangle$ is an ordered commutative monoid
- $\Rightarrow$ is the residuum of $*$, i.e., $x * y \le z$ iff $x \le y \Rightarrow z$, for all $x, y, z \in L$, where $\le$ is the lattice order generated by $\wedge, \vee$.

[2]For some relaxation of the conditions required of the system of membership degrees see Section 5.

Complete linear MTL-algebras can be characterized as linearly ordered complete (commutative bounded integral) residuated lattices (see, e.g., [23, 11]). The class includes the most commonly used algebras of membership degrees:

**Example 2.2.** Recall that a left-continuous t-norm is a binary operation $*$ on the real unit interval $[0, 1]$ which is commutative, associative, monotone, has the neutral element 1, and is left-continuous in both arguments (see, e.g., [24, 11]).

In particular, the real unit interval equipped with any left-continuous t-norm $*$, its residuum $\Rightarrow$, and the usual order of reals is a complete linear MTL-algebra. (In fact, all MTL-algebras on $[0, 1]$ with the usual order of reals are of this form.)

The following lemma lists some properties of complete linear MTL-algebras, which will be employed in proofs in subsequent sections. Their validity follows easily from theorems of the logic MTL found, e.g., in [22, 11].

**Lemma 2.3.** *The following statements are valid in any linear complete* MTL-*algebra:*

1. $(1 \Rightarrow \alpha) = \alpha$
2. $(\alpha \Rightarrow \beta) = 1$ *iff* $\alpha \leq \beta$
3. $\alpha * (\alpha \Rightarrow \beta) \leq \beta$
4. $\alpha \leq (\beta \Rightarrow \gamma)$ *iff* $\beta \leq (\alpha \Rightarrow \gamma)$
5. *If* $\alpha \leq \beta$ *and* $\alpha' \leq \beta'$, *then* $\alpha \wedge \alpha' \leq \beta \wedge \beta'$
6. $(\alpha \Rightarrow \beta) \wedge (\alpha' \Rightarrow \beta') \leq (\alpha \wedge \alpha' \Rightarrow \beta \wedge \beta')$
7. *If* $\alpha * \beta \leq \chi$ *and* $\alpha' * \beta' \leq \chi'$, *then* $(\alpha \wedge \alpha') * (\beta \wedge \beta') \leq \chi \wedge \chi'$
8. $(\alpha \Rightarrow \beta) \leq (\alpha \vee \gamma \Rightarrow \beta \vee \gamma)$
9. $(\alpha \Rightarrow \beta) \leq ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma))$
10. $\bigwedge_{i \in I} (\alpha_i \Rightarrow \beta_i) \leq \left(\bigwedge_{i \in I} \alpha_i \Rightarrow \bigwedge_{i \in I} \beta_i\right)$

### 2.2 Fuzzy sets and fuzzy relations

Throughout this report, $X$ will denote a fixed crisp domain. A fuzzy set $A$ on $X$ is identified with a mapping $A\colon X \to \mathbf{L}$, and a binary fuzzy relation $R$ on $X$ with a mapping $R\colon X^2 \to \mathbf{L}$, where $\mathbf{L}$ is a complete linear MTL-algebra. Since this report does not deal with fuzzy relations of arities higher than two, we will use the term 'fuzzy relation' always to mean 'binary fuzzy relation'.

**Convention 2.4.** As usual, a crisp subset $A$ of $X$ is identified with the fuzzy set defined as:

$$Ax = \begin{cases} 1 & \text{for all } x \in A \\ 0 & \text{for all } x \in X \setminus A, \end{cases}$$

and similarly for crisp fuzzy relations on $X$.

We will need several standard definitions of fuzzy set theory:

**Definition 2.5.** We define the following operations with fuzzy sets $A, B$ on a crisp domain $X$, by setting for all $x \in X$:

$$
\begin{aligned}
(A \cap B)x &=_{\mathrm{df}} Ax * Bx && \textit{strong intersection} \\
(A \sqcap B)x &=_{\mathrm{df}} Ax \wedge Bx && \textit{min-intersection} \\
(A \sqcup B)x &=_{\mathrm{df}} Ax \vee Bx && \textit{max-union}
\end{aligned}
$$

Furthermore we define the crisp kernel and support of a fuzzy set $A$ on $X$ as follows:

$$
\begin{aligned}
\operatorname{Ker} A &=_{\mathrm{df}} \{x \in X \mid Ax = 1\} && \textit{kernel} \\
\operatorname{Supp} A &=_{\mathrm{df}} \{x \in X \mid Ax > 0\} && \textit{support}
\end{aligned}
$$

Crisp inclusion is the relation $\sqsubseteq$ between fuzzy sets defined as follows:

$$
A \sqsubseteq B \equiv_{\mathrm{df}} (\forall x \in X)(Ax \leq Bx) \qquad \textit{crisp inclusion}
$$

Graded inclusion is a fuzzy relation on fuzzy sets, which assigns a degree from $\mathbf{L}$ to any two fuzzy sets $A, B$ on $X$ as follows:

$$
A \subseteq B =_{\mathrm{df}} \bigwedge_{x \in X} (Ax \Rightarrow Bx) \qquad \textit{graded inclusion}
$$

Mutatis mutandis (namely, by taking $x \in X^2$ instead of $x \in X$), the above definitions apply to fuzzy relations as well. The following definitions are particular to fuzzy relations. For all $x, y \in X$ we set:

$$
\begin{aligned}
\operatorname{Id} xy &=_{\mathrm{df}} \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} && \textit{identity relation} \\
R^{-1}xy &=_{\mathrm{df}} Ryx && \textit{converse fuzzy relation} \\
(R \circ S)xy &=_{\mathrm{df}} \bigvee_{z \in X} (Rxz * Szy) && \textit{(sup-T) composition}
\end{aligned}
$$

All notions introduced in Definition 2.5 are standard and their properties can be found in any comprehensive monograph on fuzzy set theory (e.g., [24, 15]).[3] Observe that due to Lemma 2.3(2), $A \sqsubseteq B$ iff $(A \subseteq B) = 1$.

## 2.3 Similarity relations and fuzzy orderings

Below we define several standard properties of fuzzy relations that generalize the corresponding properties of crisp relations. The first four properties of Definition 2.6 can be found already in [32]. The property we call here *linearity* is also known as *strong linearity* or *strong completeness* (e.g., [16, 18]), while the name linearity may denote various different properties in the literature (as is the case, e.g., in [32, 18]).

[3] The (perhaps less widely known) notion of graded inclusion has been employed, i.a., in [2, 24, 15, 6, 12]. As reported in [25], it was first considered (for Łukasiewicz logic) by Klaua as early as the 1960's.

**Definition 2.6.** A fuzzy relation $R$ on a crisp domain $X$ is called:

- *Reflexive* if $(\forall x \in X)(Rxx = 1)$; that is, if $\mathrm{Id} \sqsubseteq R$
- *Transitive* if $(\forall x, y, z \in X)(Rxy * Ryz \leq Rxz)$; that is, if $R \circ R \sqsubseteq R$
- *Symmetric* if $(\forall x, y \in X)(Rxy \leq Ryx)$; that is, if $R \sqsubseteq R^{-1}$
- *Antisymmetric* if $(\forall x, y \in X)((Rxy * Ryx > 0) \Rightarrow (x = y))$; that is, if $R \cap R^{-1} \sqsubseteq \mathrm{Id}$
- *Linear* if $(\forall x, y \in X)(Rxy \vee Ryx = 1)$; that is, if $R \sqcup R^{-1} = X^2$.

*Remark.* Note that the properties of transitivity and antisymmetry are defined relative to the strong conjunction $*$. For this reason they are often called $*$-transitivity and $*$-antisymmetry in the literature. Since in this report we always work in a fixed MTL-algebra $\mathbf{L}$ of membership degrees, we omit the parameter $*$ in the names as it is already determined by $\mathbf{L}$. (The same remark applies to the notions of $E$-antisymmetry, similarity, and fuzzy ($E$-)ordering defined below.) For these properties to be well-defined, the MTL-algebra $\mathbf{L}$ of membership degrees has to be specified in advance, including its monoidal operation $*$. The other properties of fuzzy relations introduced in this section, on the other hand, rely just on the lattice ordering of $\mathbf{L}$ (and so, e.g., coincide for all MTL-algebras on $[0, 1]$, regardless of the left-continuous t-norm $*$ used).

The following generalization of reflexivity and antisymmetry replaces the crisp equality in the definition by a fuzzy relation. This move was first proposed in [28] and further developed in [16, 17, 18, 4, 6].

**Definition 2.7.** Let $X$ be a crisp domain and $E$ a fuzzy relation on $X$. A fuzzy relation $R$ on $X$ is called:

- *$E$-reflexive* if $(\forall x, y \in X)(Exy \leq Rxy)$; that is, if $E \sqsubseteq R$
- *$E$-antisymmetric* if $(\forall x, y \in X)(Rxy * Ryx \leq Exy)$; that is, if $R \cap R^{-1} \sqsubseteq E$.

Similarly as in the theory of crisp relations, combinations of the above properties delimit certain important classes of fuzzy relations.

**Definition 2.8.** A fuzzy relation $R$ on a crisp domain $X$ is called:

- *Fuzzy preorder(ing)* if it is reflexive and transitive
- *Similarity* (or *fuzzy equivalence relation*) if it is reflexive, transitive, and symmetric
- *Fuzzy order(ing)* if it is reflexive, transitive, and antisymmetric
- *Fuzzy $E$-order(ing)* if it is $E$-reflexive, transitive, and $E$-antisymmetric, where $E$ is a fuzzy similarity relation on $X$.

Similarity relations and fuzzy orderings have been extensively studied in the literature, see, e.g., [32, 31, 30, 19, 15, 6]. Similarity-based fuzzy orderings, or fuzzy $E$-orders, have been studied, e.g., in [28, 16, 17, 18, 6]. The following properties of similarity relations are standard, too (e.g., [27, 16, 17]). Separated similarities are also known as *unimodal* similarities or *fuzzy equality* relations.

**Definition 2.9.** Let $\trianglelefteq$ be a crisp ordering on a crisp domain $X$. We say that a similarity relation $E$ on $X$ is:

- *Separated* if $(\forall x, y \in X)((Exy = 1) \Leftrightarrow (x = y))$; that is, if $\mathrm{Ker}(E) = \mathrm{Id}$
- *Compatible with* $\trianglelefteq$ if $(\forall x, y, z \in X)((x \trianglelefteq y \trianglelefteq z) \Rightarrow (Ezx \leq Eyx \wedge Ezy))$.

For convenience, we also define the corresponding properties of fuzzy orderings:

**Definition 2.10.** Let $\trianglelefteq$ be a crisp ordering on a crisp domain $X$. We say that a fuzzy $E$-ordering $L$ on $X$:

- *Extends* $\trianglelefteq$ if $(\forall x, y \in X)((x \trianglelefteq y) \Rightarrow (Lxy = 1))$; that is, if $\trianglelefteq \sqsubseteq L$
- *Fuzzifies* $\trianglelefteq$ if $(\forall x, y \in X)((x \trianglelefteq y) \Leftrightarrow (Lxy = 1))$; that is, if $\trianglelefteq = \mathrm{Ker}(L)$
- Is *compatible with* $\trianglelefteq$ if $(\forall x, y, z \in X)((x \trianglelefteq y \trianglelefteq z) \Rightarrow (Lzx \leq Lyx \wedge Lzy))$.

*Remark.* Clearly, the properties introduced in Definition 2.10 can meaningfully be defined for any fuzzy relation $L$ and w.r.t. any crisp relation $\trianglelefteq$ on $X$, regardless of whether $L$ is a fuzzy $E$-ordering and $\trianglelefteq$ a crisp ordering. (Note, however, that in [16], fuzzification of a crisp relation is a concept different from ours.) By definition, every fuzzy relation fuzzifies its kernel, which is a crisp equivalence relation for a similarity and a crisp ordering for a fuzzy $E$-ordering if $E$ is separated. Separated similarities are thus those similarities which fuzzify the crisp equality Id.

Notice also that since the crisp order $\trianglelefteq$ fuzzified by a fuzzy $E$-ordering $L$ can be recovered from $L$ (as its kernel), the notions related to fuzzified crisp orderings could as well be defined in terms of fuzzy $E$-orderings only. Since, however, considerations on fuzzified orders typically regard the underlying crisp order as primary, we prefer to make our definitions relative to an underlying crisp order (which is then to be specified in advance).

# 3 Correspondence between similarities and similarity-based fuzzy orderings on linearly ordered domains

Further on, let $\trianglelefteq$ be a given crisp linear ordering of a crisp domain $X$.

**Definition 3.1.** Given a fuzzy relation $L$ on $X$, we define its *min-intersective symmetrization* $E_L$, by setting:

$$E_L xy =_{\mathrm{df}} Lxy \wedge Lyx$$

for all $x, y \in X$; that is, $E_L = L \sqcap L^{-1}$.

Conversely, given a fuzzy relation $E$ on $X$, we can define the *fuzzification* $L_E$ of the crisp order $\trianglelefteq$ with $E$:

$$L_E xy \quad =_{\mathrm{df}} \quad (x \trianglelefteq y) \vee Exy$$

for all $x, y \in X$; that is, $L_E = \trianglelefteq \sqcup E$.

It will be seen that in fuzzified linear orderings, the operators $L \mapsto E_L$ and $E \mapsto L_E$ are mutually inverse. Indeed, the theorems given in this section show that the assignments $L \mapsto E_L$ and $E \mapsto L_E$ can be regarded as inverse functors between the $\sqsubseteq$-ordered poset category of all similarity relations compatible with $\trianglelefteq$ on $X$ and the $\sqsubseteq$-ordered poset category of all fuzzy orderings extending and compatible with $\trianglelefteq$, as well as between their respective subcategories delimited by the conditions of separatedness and $\trianglelefteq$-fuzzification.

These results effectively say that it is immaterial in the setting of fuzzified linear orderings whether we start with a fuzzy similarity relation or a similarity-based fuzzy ordering. Even if in many cases it is admittedly more natural to regard the fuzzy similarity as primary and the similarity-based fuzzy ordering as derivative, the theorems show that under certain assumptions, both ways are formally equivalent.

Most of the results of this section are known in the setting of t-norms on the real unit interval [16, 17]; we re-prove them here over any complete linear residuated lattices. The proofs are presented in a specific format, for reasons explained in Section 5.

**Theorem 3.2.** *Let $\trianglelefteq$ be a crisp linear order on a crisp domain $X$ and $L$ a fuzzy preorder on $X$. Then:*

1. *$E_L$ is a similarity relation.*
2. *$L$ is actually a fuzzy $E_L$-ordering.*
3. *If $L$ is linear, then $E = E_L$ is the unique similarity such that $L$ is an $E$-ordering.*
4. *If $L$ fuzzifies $\trianglelefteq$, then $E_L$ is separated.*
5. *If $L$ is compatible with and extends $\trianglelefteq$, then $E_L$ is compatible with $\trianglelefteq$.*

*Proof.*

1. Reflexivity: $E_Lxx = Lxx \wedge Lxx = 1$ by the definition of $E_L$ and the reflexivity of $L$.

   Transitivity: The required condition $(Lxy \wedge Lyx) * (Lyz \wedge Lzy) \leq Lxz \wedge Lzx$ is obtained by Lemma 2.3(7) from the inequalities $Lxy * Lyz \leq Lxz$ and $Lzy * Lyx \leq Lzx$, ensured by the transitivity of $L$.

   Symmetry: Trivial by the definition of $E_L$.

2. Trivially $E_L = L \sqcap L^{-1} \sqsubseteq L$, so $L$ is $E_L$-reflexive; also trivially $L \cap L^{-1} \sqsubseteq L \sqcap L^{-1} = E_L$, so $L$ is $E_L$-antisymmetric (cf. Definition 2.7).

3. Observe that if $L$ is linear, then $L \cap L^{-1} = L \sqcap L^{-1} = E_L$; thus $L$ is $E$-antisymmetric only if $E_L \sqsubseteq E$. Furthermore, $L$ is $E$-reflexive only if $E \sqsubseteq L$; then, however, also $E^{-1} \sqsubseteq L^{-1}$, thus $E \sqcap E^{-1} \sqsubseteq L \sqcap L^{-1} = E_L$. Since $E \sqcap E^{-1} = E$ by the symmetry of $E$, $L$ is therefore $E$-reflexive only if $E \sqsubseteq E_L$. Thus the linear fuzzy preorder $L$ is a fuzzy $E$-ordering iff $E = E_L$.

4. $\mathrm{Ker}(E_L) = \mathrm{Ker}(L \sqcap L^{-1}) = \mathrm{Ker}(L) \cap \mathrm{Ker}(L^{-1}) = \trianglelefteq \cap \trianglerighteq$ as $L$ fuzzifies $\trianglelefteq$, and $\trianglelefteq \cap \trianglerighteq = \mathrm{Id}$ by the antisymmetry of $\trianglelefteq$.

5. Let $x \trianglelefteq y \trianglelefteq z$. Then $Lzx \leq Lyx$, as $L$ is compatible with $\trianglelefteq$. Since moreover $Lxz = Lxy = 1$ as $L$ extends $\trianglelefteq$, we obtain $Lzx \wedge Lxz \leq Lyx \wedge Lxy$, i.e., $E_Lzx \leq E_Lyx$. Similarly it is proved that $E_Lzx \leq E_Lzy$; thus $E_L$ is compatible with $\trianglelefteq$. □

**Theorem 3.3.** *Let $\trianglelefteq$ be a crisp linear order on a crisp domain $X$ and $E$ a similarity on $X$.*

1. *If $E$ is separated, then $L_E$ fuzzifies $\trianglelefteq$.*
2. *If $E$ is compatible with $\trianglelefteq$, then $L_E$ is a fuzzy $E$-ordering compatible with and extending $\trianglelefteq$.*

*Proof.*

1. $(L_E xy = 1) \Leftrightarrow ((x \trianglelefteq y) \vee (Exy = 1)) \Leftrightarrow ((x \trianglelefteq y) \vee (x = y)) \Leftrightarrow (x \trianglelefteq y)$, respectively by the definition of $L_E$, the separatedness of $E$, and the reflexivity of $\trianglelefteq$.

2. Trivially, $L_E$ is $E$-reflexive (as $E \sqsubseteq E \sqcup \trianglelefteq$) and extends $\trianglelefteq$ (as $\trianglelefteq \sqsubseteq E \sqcup \trianglelefteq$).

   Transitivity: We shall prove $L_E xy * L_E yz \leq L_E xz$ for all $x, y, z \in X$ by taking the following crisp cases (which are exhaustive due to the linearity of $\trianglelefteq$):

   – If $x \trianglelefteq z$, then $L_E xz = 1 \geq L_E xy * L_E yz$.
   – If $z \trianglelefteq y \trianglelefteq x$, then $L_E xy = Exy$, $L_E yz = Eyz$, and $L_E xz = Exz$, so the claim follows from the transitivity of $E$.
   – If $z \trianglelefteq x \trianglelefteq y$, then $L_E xy * L_E yz = Eyz \leq Exz$ as $E$ is compatible with $\trianglelefteq$.
   – If $y \trianglelefteq z \trianglelefteq x$, then $L_E xy * L_E yz = Exy \leq Exz$ as $E$ is compatible with $\trianglelefteq$.

   $E$-antisymmetry: We need to prove $L_E \cap L_E^{-1} \sqsubseteq E$. By the distributivity of $*$ over $\vee$ we obtain: $L_E \cap L_E^{-1} = (E \sqcup \trianglelefteq) \cap (E \sqcup \trianglerighteq) = (E \cap E) \sqcup (E \cap \trianglelefteq) \sqcup (E \cap \trianglerighteq) \sqcup (\trianglelefteq \cap \trianglerighteq)$. Now observe that the first three components are contained in $E$ (as $E \cap R \sqsubseteq E$ for any $R$) and that $(\trianglelefteq \cap \trianglerighteq) = \mathrm{Id} \sqsubseteq E$ by the antisymmetry of $\trianglelefteq$ and the reflexivity of $E$.

   Compatibility with $\trianglelefteq$: If $x \trianglelefteq y \trianglelefteq z$, then $L_E zx = Ezx$, $L_E yx = Eyx$, and $L_E zy = Ezy$, so the claim instantiates the assumption that $E$ is compatible with $\trianglelefteq$. □

**Theorem 3.4.** *Let $E_1, E_2, L_1, L_2$ be fuzzy relations on a crisp domain $X$ equipped with a crisp linear order $\trianglelefteq$, and let $E_L, L_E$ be defined for any $E, L$ as in Definition 3.1. Then:*[4]

1. *$(E_1 \subseteq E_2) \leq (L_{E_1} \subseteq L_{E_2})$; in particular, if $E_1 \sqsubseteq E_2$ then $L_{E_1} \sqsubseteq L_{E_2}$.*

2. *$(L_1 \subseteq L_2) \leq (E_{L_1} \subseteq E_{L_2})$; in particular, if $L_1 \sqsubseteq L_2$ then $E_{L_1} \sqsubseteq E_{L_2}$.*

*Proof.*

1. By definitions and Lemma 2.3(8) we obtain: $(E_1 \subseteq E_2) = \bigwedge_{x,y \in X}(E_1 xy \Rightarrow E_2 xy) \leq \bigwedge_{x,y \in X}\big((E_1 xy \vee (x \trianglelefteq y)) \Rightarrow (E_2 xy \vee (x \trianglelefteq y))\big) = (L_{E_1} \subseteq L_{E_2})$. The non-graded claim $(E_1 \sqsubseteq E_2) \Rightarrow (L_{E_1} \sqsubseteq L_{E_2})$ is an instance of the graded claim for $(E_1 \subseteq E_2) = 1$.

2. By definition, $(L_1 \subseteq L_2) = \bigwedge_{x,y \in X}(L_1 xy \Rightarrow L_2 xy) \leq (L_1 pq \Rightarrow L_2 pq)$, for any $p, q \in X$. Similarly, $(L_1 \subseteq L_2) \leq (L_1 qp \Rightarrow L_2 qp)$. Combining these inequalities by Lemma 2.3(5) and applying Lemma 2.3(6) thus yields: $(L_1 \subseteq L_2) \leq \big((L_1 pq \Rightarrow L_2 pq) \wedge (L_1 qp \Rightarrow L_2 qp)\big) \leq \big((L_1 pq \wedge L_1 qp) \Rightarrow (L_2 pq \wedge L_2 qp)\big)$, for any $p, q \in X$. Consequently, $(L_1 \subseteq L_2) \leq \bigwedge_{p,q \in X}\big((L_1 pq \wedge L_1 qp) \Rightarrow (L_2 pq \wedge L_2 qp)\big) = \big((L_1 \sqcap L_1^{-1}) \subseteq (L_2 \sqcap L_2^{-1})\big) = (E_{L_1} \subseteq E_{L_2})$. The non-graded claim $(L_1 \sqsubseteq L_2) \Rightarrow (E_{L_1} \sqsubseteq E_{L_2})$ is an instance of the graded claim for $(L_1 \subseteq L_2) = 1$. □

**Theorem 3.5.** *Let $E, L$ be fuzzy relations on a crisp domain $X$, and let $E_L, L_E$ be defined for any $E, L$ as above.*

1. *If $E$ is reflexive, then $E_{L_E} = E$.*

2. *If $L$ extends $\trianglelefteq$, then $L_{E_L} = L$.*

[4] See Definition 2.5 for the graded notion of inclusion $\subseteq$.

*Proof.* 1. $E_{L_E} = (E \sqcup \trianglelefteq) \sqcap (E \sqcup \trianglelefteq)^{-1} = (E \sqcup \trianglelefteq) \sqcap (E \sqcup \trianglerighteq) = (E \sqcap E) \sqcup (E \sqcap \trianglelefteq) \sqcup (E \sqcap \trianglerighteq) \sqcup (\trianglelefteq \sqcap \trianglerighteq)$ by the distributivity of $\sqcap$ over $\sqcup$. Now observe that the first term in the max-union is $E$, the second and third are contained in $E$, and the fourth equals Id (by the antisymmetry of $\trianglelefteq$), which is contained in $E$ (by the reflexivity of $E$). Thus $E_{L_E} = E$.

2. $L_{E_L} = \trianglelefteq \sqcup (L \sqcap L^{-1}) = (\trianglelefteq \sqcup L) \sqcap (\trianglelefteq \sqcup L^{-1})$ by the distributivity of $\sqcup$ over $\sqcap$. Now observe that the first term in the min-intersection equals $L$, since $L$ extends $\trianglelefteq$; and the second equals $X^2$, since $L^{-1}$ extends $\trianglerighteq$ and $\trianglelefteq \sqcup \trianglerighteq = X^2$ as $\trianglelefteq$ is linear. Thus $L_{E_L} = L \sqcap X^2 = L$. □

*Remark.* As seen from their proofs, Theorems 3.4–3.5 actually do not need $\trianglelefteq$ to be a crisp linear order. Only the antisymmetry of $\trianglelefteq$ is used in the proof of Theorem 3.5(1), and the linearity of $\trianglelefteq$ in the proof of Theorem 3.5(2); the proof of Theorem 3.4 makes no requirement on $\trianglelefteq$ whatsoever.

# 4 Maxima and minima in fuzzy orderings

In this section we develop a basic apparatus of maxima and minima in fuzzy orderings, as a prerequisite for the investigation of their special case in fuzzified crisp linear orderings, dealt with in [3].

The notions of minima and maxima can in the fuzzy setting be defined in several ways. We shall use one of the most natural generalizations of the corresponding crisp notions, namely the definitions based on bounds.

**Definition 4.1.** Let $L$ be a fuzzy relation on a crisp domain $X$ and $A$ a fuzzy subset of $X$. The fuzzy sets $A^{\uparrow_L}, A^{\downarrow_L}$ of *upper* and *lower bounds* of $A$ with respect to $L$, or the *upper* and *lower cones* of $A$ in $L$, are defined, respectively, by setting for all $q \in X$:

$$A^{\uparrow_L} q = \bigwedge_{x \in X} (Ax \Rightarrow Lxq), \qquad A^{\downarrow_L} q = \bigwedge_{y \in X} (Ay \Rightarrow Lqy).$$

The definition of bounds can be understood as reinterpreting the classical (crisp set–theoretical) definitions in the fuzzy setting, arrived at by replacing the quantifier $\forall$ with the lattice infimum $\bigwedge$ and the classical implication with the residuum, which is a usual method of fuzzification in formal fuzzy logic (cf. Section 5). The notions introduced in Definition 4.1 are standard in the theory of fuzzy relations (see, e.g., [14, 15, 20, 6]); formally they are special instances of inf-R composition of fuzzy relations, see [12, Section 5] and cf. [1] and [15, Remark 6.16].

**Definition 4.2.** The fuzzy sets $\mathrm{Max}_L\, A, \mathrm{Min}_L\, A$ of the *maxima* and *minima* of $A$ w.r.t. a fuzzy relation $L$ are defined, respectively, as:

$$\mathrm{Max}_L\, A = A \sqcap A^{\uparrow_L}, \qquad \mathrm{Min}_L\, A = A \sqcap A^{\downarrow_L}.$$

The definition of fuzzy maxima and minima is based on the classical concept of selecting those elements in a set which are its upper (or lower) bounds. However, unlike in classical mathematics where the maximum or minimum of a set is unique (if it exists), the fuzzified condition yields in general a *fuzzy set* of minima or maxima. It can also be noticed that even though the notions of cones and maxima or minima are most meaningful for fuzzy preorders, they are actually well-defined for any fuzzy relation $L$.

The notions of fuzzy maxima and minima introduced in Definition 4.2 have already occurred in the literature [20]. A similar notion of fuzzy maxima and minima was employed in [6], differing only in using the strong intersection $\cap$ instead of the min-intersection $\sqcap$ in the defining terms. We will denote these alternative notions by $\mathrm{Max}^*_L A$ and $\mathrm{Min}^*_L A$:

**Definition 4.3.** The fuzzy sets $\mathrm{Max}^*_L A, \mathrm{Min}^*_L A$ of $*$-*based maxima* and *minima* of $A$ in $L$ are defined, respectively, as:

$$\mathrm{Max}^*_L A = A \cap A^{\uparrow_L}, \qquad \mathrm{Min}^*_L A = A \cap A^{\downarrow_L}.$$

Even though both variants of the definitions are formally sound, the following reasons can be given for preferring the min-intersection $\sqcap$ over the $*$-intersection $\cap$ in the defining terms:

- As argued in [10], notions defined as the strong conjunction $\varphi_1 * \ldots * \varphi_k$ of separate conditions $\varphi_1, \ldots, \varphi_k$ (such as the two conditions in our definition of maxima, namely that of being an element of $A$ and that of being an upper bound of $A$) are of limited utility in contraction-free fuzzy logics (i.e., fuzzy logics with non-idempotent conjunction $*$). Instead, a parameterized set of notions $\varphi^{(n_1,\ldots,n_k)} \equiv_{\mathrm{df}} \varphi_1^{n_1} * \cdots * \varphi_k^{n_k}$ with $n_i$-tuple conjunction $\varphi_i^{n_i} = \varphi_i * \cdots * \varphi_i$ ($n_i$ times) of each condition $\varphi_i$ should be considered, as each of the conditions $\varphi_i$ may be needed with varied multiplicity $n_i$ in different graded theorems (see [10, Sect. 7] for details). Thus, the definition employed in [6] is only a fragment of a useful $*$-based notion of maxima or minima in the fuzzy setting.
- There appears to be no strong intuition as to why *both* of the defining conditions (rather than either one of them) should be needed for inferring facts from the graded assumption that an element is a minimum or maximum, and therefore why strong (rather than weak) conjunction should be used in the definition.[5]

The $\wedge$-based definitions of maxima and minima are logically weaker than the $*$-based ones (as $\varphi \wedge \psi \geq \varphi * \psi$). Nevertheless, it turns out that the properties proved in [6, Sect. 5] for $\mathrm{Max}^*_L$ and $\mathrm{Min}^*_L$ hold for the $\wedge$-based notions $\mathrm{Max}_L$ and $\mathrm{Min}_L$ as well, as shown by the following Theorem 4.6. Since the theorems for the $\wedge$-based notions are stronger than those of [6] (as their assumptions with the $\wedge$-based maxima and minima are weaker), the $*$-based results of [6] need be proved anew for the $\wedge$-based definitions. We shall only prove the variants for maxima. The analogous theorems for minima are obtained as corollaries by the following lemma, obvious by expansion of definitions:

**Lemma 4.4.** *For any fuzzy relation $L$ and fuzzy set $A$, $A^{\uparrow_L} = A^{\downarrow_{L^{-1}}}$ and $A^{\downarrow_L} = A^{\uparrow_{L^{-1}}}$. Consequently,* $\mathrm{Min}_L A = \mathrm{Max}_{L^{-1}} A$ *and* $\mathrm{Max}_L A = \mathrm{Min}_{L^{-1}} A$ *(and analogously for* $\mathrm{Max}^*$ *and* $\mathrm{Min}^*$*).*

[5]Cf. the discussion of the inferential rôles of weak and strong conjunction in [5, Sect. 4], esp. at the end of Remark 8 and in Example 9. Admittedly, our definition implies

$$(\forall x \in X)\big((Ax \leq (\mathrm{Max}_L A)x) \vee (A^{\uparrow_L} x \leq (\mathrm{Max}_L A)x)\big), \tag{4.1}$$

which fuzzifies the classical-logic formula

$$(\forall x \in X)\big(((x \in A) \Rightarrow (x \in \mathrm{Max}_L A)) \vee ((x \in A^{\uparrow_L}) \Rightarrow (x \in \mathrm{Max}_L A))\big). \tag{4.2}$$

Even though (4.2) is true as well for crisp sets, it is hardly intuitive: its validity for crisp sets can be viewed as related to the paradoxes of material implication in classical logic. On the other hand, the fuzzy concept of residual implication *is* a material one (rather than, e.g., relevant or strict), and so the '$\Rightarrow$' in fuzzified (4.2) should *not* be understood along the lines of "$x \in A$ *justifies* $x \in \mathrm{Max}_L A$" or similar, which would produce the implausible reading of the definition. Consequently, the present author's condemnation of the use of $\wedge$ in such contexts, expressed in [5, Example 9], was probably too hurried.

We shall also need the following lemmata on cones, which generalize well-known properties of crisp cones. Lemma 4.5(1) expresses the graded fact that the elements of a set indeed follow the elements of its lower cone and precede the elements of its upper cone in the fuzzy ordering. Lemmata 4.5(2)–(4) have been proved in [14, Lem. 3] and [6, Th. 5.8]; for the sake of completeness and clarity, we re-prove them here in our notation.

**Lemma 4.5.** *Let $A, B$ be fuzzy subsets of a crisp domain $X$ and $L$ a fuzzy relation on $X$. Then:*

1. $A^{\downarrow_L} x * Ay \leq Lxy$ *and* $Ax * A^{\uparrow_L} y \leq Lxy$*, for all* $x, y \in X$.
2. $(A \subseteq B) \leq (B^{\uparrow_L} \subseteq A^{\uparrow_L})$ *and* $(A \subseteq B) \leq (B^{\downarrow_L} \subseteq A^{\downarrow_L})$.
   *In particular, if* $A \sqsubseteq B$*, then* $B^{\uparrow_L} \sqsubseteq A^{\uparrow_L}$ *and* $B^{\downarrow_L} \sqsubseteq A^{\downarrow_L}$.
3. $A \sqsubseteq (A^{\uparrow_L})^{\downarrow_L}$ *and* $A \sqsubseteq (A^{\downarrow_L})^{\uparrow_L}$.
4. $A^{\uparrow_L} = ((A^{\uparrow_L})^{\downarrow_L})^{\uparrow_L}$ *and* $A^{\downarrow_L} = ((A^{\downarrow_L})^{\uparrow_L})^{\downarrow_L}$.

*Proof.*

1. By definition, $A^{\downarrow_L} x = \bigwedge_{z \in X}(Az \Rightarrow Lxz) \leq (Ay \Rightarrow Lxy)$ for any $x, y \in X$. Thus $A^{\downarrow_L} x * Ay \leq (Ay \Rightarrow Lxy) * Ay \leq Lxy$, by the monotony of $*$ and Lemma 2.3(3). The dual claim for upper cones follows from Lemma 4.4.

2. By Lemma 2.3(9), $(Ax \Rightarrow Bx) \leq ((Bx \Rightarrow Lxy) \Rightarrow (Ax \Rightarrow Lxy))$ for all $x, y \in X$. Thus,

$$\bigwedge_{x \in X}(Ax \Rightarrow Bx) \leq \bigwedge_{x \in X}((Bx \Rightarrow Lxy) \Rightarrow (Ax \Rightarrow Lxy)) \leq \bigwedge_{x \in X}(Bx \Rightarrow Lxy) \Rightarrow \bigwedge_{x \in X}(Ax \Rightarrow Lxy)$$

   by Lemma 2.3(10), for all $y \in X$; hence also

$$\bigwedge_{x \in X}(Ax \Rightarrow Bx) \leq \bigwedge_{y \in X}\Big(\bigwedge_{x \in X}(Bx \Rightarrow Lxy) \Rightarrow \bigwedge_{x \in X}(Ax \Rightarrow Lxy))\Big),$$

   i.e., $(A \subseteq B) \leq (B^{\uparrow_L} \subseteq A^{\uparrow_L})$. The dual claim follows by Lemma 4.4 and the non-graded claims instantiate the graded ones for $(A \subseteq B) = 1$.

3. For each $a \in X$, from $\bigwedge_{y \in X}(Ay \Rightarrow Lyx) \leq (Aa \Rightarrow Lax)$ we obtain by Lemma 2.3(4):

$$Aa \leq \Big(\bigwedge_{y \in X}(Ay \Rightarrow Lyx)\Big) \Rightarrow Lax$$

   for any $x \in X$; hence

$$Aa \leq \bigwedge_{x \in X}\Big(\Big(\bigwedge_{y \in X}(Ay \Rightarrow Lyx)\Big) \Rightarrow Lax\Big)$$

   for each $a \in X$, i.e., $A \sqsubseteq (A^{\uparrow_L})^{\downarrow_L}$. The dual claim follows by Lemma 4.4.

4. The claim is a direct corollary of claims 2 and 3. □

Now we are ready to generalize the results of [6, Th. 5.12] for $\wedge$-based minima and maxima. Theorem 4.6(1) is a graded version of the classical property of monotonicity of maxima with respect to inclusion of subsets: if $A$ is a subset of $B$, $x$ a maximum of $A$, and $y$ a maximum of $B$ in $L$, then $x$ precedes $y$ in $L$. Theorem 4.6(2) expresses the graded uniqueness of maxima with respect to the min-symmetrization $E_L$ of $L$: if $x$ and $y$ are maxima of $A$ in $L$, then they are equivalent in $E_L$. Theorem 4.6(3) states an analogous property of $E$-uniqueness of maxima in linear $E$-antisymmetric fuzzy relations (e.g., in linear fuzzy $E$-orderings).

**Theorem 4.6** (cf. [6] for Max*)**.** *Let $A, B$ be fuzzy subsets of $X$, $L$ a fuzzy relation on $X$, and $E$ a similarity on $X$. Then the following properties hold for all $x, y \in X$:*

1. $(A \subseteq B) * (\mathrm{Max}_L\, A)x * (\mathrm{Max}_L\, B)y \leq Lxy$
2. $(\mathrm{Max}_L\, A)x * (\mathrm{Max}_L\, A)y \leq E_L xy$
3. *If $L$ is linear and $E$-antisymmetric, then* $(\mathrm{Max}_L\, A)x * (\mathrm{Max}_L\, A)y \leq Exy$.

*Proof.*

1. By Lemma 4.5(2), $(A \subseteq B) \leq (B^{\uparrow_L} \subseteq A^{\uparrow_L}) = \bigwedge_{y \in X}(B^{\uparrow_L}y \Rightarrow A^{\uparrow_L}y) \leq (B^{\uparrow_L}y \Rightarrow A^{\uparrow_L}y)$. Moreover, by definition, $(\mathrm{Max}\, A)x \leq Ax$ and $(\mathrm{Max}\, B)y \leq B^{\uparrow_L}y$. Combining these inequalities, by the monotonicity of $*$ we obtain:

   $(A \subseteq B) * (\mathrm{Max}_L\, A)x * (\mathrm{Max}_L\, B)y \leq (B^{\uparrow_L}y \Rightarrow A^{\uparrow_L}y) * Ax * B^{\uparrow_L}y \leq Ax * A^{\uparrow_L}y \leq Lxy,$

   where the second inequality uses the instance $B^{\uparrow_L}y * (B^{\uparrow_L}y \Rightarrow A^{\uparrow_L}y) \leq A^{\uparrow_L}y$ of Lemma 2.3(3) and the third uses Lemma 4.5(1).

2. By definition and Lemma 4.5(1), $(\mathrm{Max}_L\, A)x*(\mathrm{Max}_L\, A)y \leq Ax*A^{\uparrow_L}y \leq Lxy$. Similarly, $(\mathrm{Max}_L\, A)x * (\mathrm{Max}_L\, A)y \leq A^{\uparrow_L}x * Ay \leq Lyx$. Thus $(\mathrm{Max}_L\, A)x * (\mathrm{Max}_L\, A)y \leq Lxy \wedge Lyx = E_L xy$.

3. By claim 2, $(\mathrm{Max}_L\, A)x*(\mathrm{Max}_L\, A)y \leq E_L xy$. Moreover by the proof of Theorem 3.2(3), if $L$ is linear and $E$-antisymmetric then $E_L \sqsubseteq E$; thus $E_L xy \leq Exy$. □

**Corollary 4.7.** *In the setting of Theorem 4.6, the following properties of fuzzy minima hold:*

1. $(B \subseteq A) * (\mathrm{Min}_L\, A)x * (\mathrm{Min}_L\, B)y \leq Lxy$
2. $(\mathrm{Min}_L\, A)x * (\mathrm{Min}_L\, A)y \leq E_L xy$
3. *If $L$ is linear and $E$-antisymmetric, then* $(\mathrm{Min}_L\, A)x * (\mathrm{Min}_L\, A)y \leq Exy$.

*Proof.* The claims follow directly from Theorem 4.6 by the duality of Lemma 4.4. □

The related notions of suprema and infima of fuzzy sets in fuzzy orderings can be defined, respectively, as the minima and maxima of the upper and lower cones. This definition formalizes, in the fuzzy setting, the classical notion of the supremum as the least upper bound and the infimum as the greatest lower bound. Like fuzzy minima and maxima, the fuzzified suprema and infima are fuzzy sets rather then single elements. This notion of fuzzy suprema and infima is standard in the literature on fuzzy lattices (e.g., [15, Ch. 4], [20]).

**Definition 4.8.** Let $L$ be a fuzzy relation on a crisp domain $X$ and $A$ a fuzzy subset of $X$. Then the fuzzy sets of ($\wedge$-based) *suprema* and *infima* are defined, respectively, as follows:

$$\mathrm{Sup}_L\, A = \mathrm{Min}_L(A^{\uparrow_L}), \qquad \mathrm{Inf}_L\, A = \mathrm{Max}_L(A^{\downarrow_L}).$$

In [6, Sect. 5], the analogous notions of fuzzy suprema and infima have been defined by means of $*$-based minima and maxima:

**Definition 4.9** ([6]). Let $L$ be a fuzzy relation on a crisp domain $X$ and $A$ a fuzzy subset of $X$. Then the fuzzy sets of *$*$-based suprema* and *infima* are defined, respectively, as follows:

$$\mathrm{Sup}^*_L A = \mathrm{Min}^*_L(A^{\uparrow_L}), \qquad \mathrm{Inf}^*_L A = \mathrm{Max}^*_L(A^{\downarrow_L}).$$

Several properties of $*$-based suprema and infima have been proved in [6, Sect. 5]. It turns out that all of these properties can be proved for $\wedge$-based suprema and infima as well, as shown in Theorem 4.10 below. Since the duality of Lemma 4.4 extends to suprema and infima by Theorem 4.10(1), we will only prove the versions for suprema. Theorem 4.10(5) and Theorem 4.10(5) are known (see [21, Prop. 2.4(v)] and [15, Lem. 4.54], respectively); in our exposition they are direct consequences of previous results on fuzzy minima.

**Theorem 4.10** (cf. [6] for $\mathrm{Sup}^*$). *Let $L$ be a fuzzy relation on a crisp domain $X$ and $A$ a fuzzy subset of $X$. Then:*

1. $\mathrm{Sup}_L A = \mathrm{Inf}_{L^{-1}} A$ *and* $\mathrm{Inf}_L A = \mathrm{Sup}_{L^{-1}} A$
2. $\mathrm{Sup}_L A = A^{\uparrow_L} \sqcap (A^{\uparrow_L})^{\downarrow_L}$
3. $\mathrm{Sup}_L A = \mathrm{Inf}_L(A^{\uparrow_L})$
4. $(B \subseteq A) * (\mathrm{Sup}_L A)x * (\mathrm{Sup}_L B)y \leq Lxy$
5. $(\mathrm{Sup}_L A)x * (\mathrm{Sup}_L A)y \leq E_L xy$
6. *If $L$ is linear and $E$-antisymmetric, then* $(\mathrm{Sup}_L A)x * (\mathrm{Sup}_L A)y \leq Exy$.

*Proof.*

1. By Lemma 4.4, $\mathrm{Sup}_{L^{-1}} A = \mathrm{Min}_{L^{-1}}(A^{\uparrow_{L^{-1}}}) = \mathrm{Max}_L(A^{\downarrow_L}) = \mathrm{Inf}_L A$, and analogously for the second claim.
2. The claim holds trivially by the definition of Sup.
3. By the dual of claim 2, Lemma 4.5(4), and claim 2,
$$\mathrm{Inf}_L(A^{\uparrow_L}) = (A^{\uparrow_L})^{\downarrow_L} \sqcap ((A^{\uparrow_L})^{\downarrow_L})^{\uparrow_L} = (A^{\uparrow_L})^{\downarrow_L} \sqcap A^{\uparrow_L} = \mathrm{Sup}_L A.$$

The claims 4–6 are proved by applying Corollary 4.7 to $A^{\uparrow_L}$ and $B^{\uparrow_L}$. □

# 5 Formalization in higher-order fuzzy logic

In this report we have deliberately used the usual notation of traditional fuzzy mathematics, in order to make the results more accessible to a broader community of fuzzy mathematicians. Nevertheless, all of the definitions, theorems, and proofs have been intentionally formulated in such a way that they can easily be translated into the higher-order fuzzy logic $\mathrm{MTL}_\triangle$ [11, Sect. 5.5.2].[6] The informal proofs given in this report are thus actually disguised $\mathrm{MTL}_\triangle$-proofs, written in a notational variant of Henkin-style higher-order $\mathrm{MTL}_\triangle$. The translation between the two notations is as follows:

[6] Higher-order fuzzy logic refers here to its form known as Fuzzy Class Theory [7], i.e., Russell-style simple fuzzy type theory, formalized (Henkin-style) in first-order fuzzy logic $\mathrm{MTL}_\triangle$. Formalization in higher-order fuzzy logic in the form of Novák's (Church-style) Fuzzy Type Theory [29] should present no difficulties, either, as the two theories share essentially the same expressive and deductive power.

- We use $*$ and $\Rightarrow$ for the $\mathrm{MTL}_\Delta$-connectives & and $\to$
- We write $\varphi \le \psi$ and $\varphi = \psi$ instead of the $\mathrm{MTL}_\Delta$-formulae $\Delta(\varphi \to \psi)$ and $\Delta(\varphi \leftrightarrow \psi)$
- We write $\varphi = 1$ instead of the $\mathrm{MTL}_\Delta$-formula $\Delta\varphi$ and $\varphi > 0$ instead of $\neg\Delta\neg\varphi$
- We use $\bigwedge$ and $\bigvee$ for $\forall$ and $\exists$; in particular, $\bigwedge_{\varphi(x)} \psi(x)$ stands for $(\forall x)(\varphi(x) \to \psi(x))$ and $\bigvee_{\varphi(x)} \psi(x)$ for $(\exists x)(\varphi(x) \,\&\, \psi(x))$.

A careful reader familiar with higher-order fuzzy logic $\mathrm{MTL}_\Delta$ can verify that under this translation, all proof steps in the present report follow sound rules of inference in higher-order $\mathrm{MTL}_\Delta$ (some of these inference rules are collected in Lemma 2.3). Consequently, all of the results presented in this report are also provable in higher-order $\mathrm{MTL}_\Delta$, and are (under the indicated translation) part of the development of formal fuzzy mathematics in the framework of higher-order fuzzy logic [7, 8, 9]. This, beside the completeness of exposition, was another reason for giving explicit proofs of some known results of Sections 3 and 4, in a form easily translatable into higher-order $\mathrm{MTL}_\Delta$.

Let us illustrate the translation by giving one of the proofs in higher-order $\mathrm{MTL}_\Delta$ for comparison:

*Proof of Lemma 4.5(2).* By suffixing we have $(Ax \to Bx) \to ((Bx \to Lxy) \to (Ax \to Lxy))$, whence we obtain by generalization on $x$ and distribution of the quantifier:

$$(\forall x)(Ax \to Bx) \to (\forall x)((Bx \to Lxy) \to (Ax \to Lxy)) \\ \to ((\forall x)(Bx \to Lxy) \to (\forall x)(Ax \to Lxy)).$$

Generalization on $y$ and a quantifier shift then yields:

$$(\forall x)(Ax \to Bx) \to (\forall y)((\forall x)(Bx \to Lxy) \to (\forall x)(Ax \to Lxy)),$$

i.e., $(A \subseteq B) \to (B^{\uparrow_L} \subseteq A^{\uparrow_L})$. The dual claim follows by Lemma 4.4 and the non-graded claim by $\Delta$-necessitation and distribution of the $\Delta$ over the implication. □

It can be observed that despite the difference in notation and terminology, the steps in the proof of Lemma 4.5(2) as given in Section 4 directly correspond to those in higher-order $\mathrm{MTL}_\Delta$. A similar correspondence is ensured by the chosen format of the proofs for all theorems presented in this report.

While the translation into higher-order $\mathrm{MTL}_\Delta$ is straightforward (or requires just obvious adjustments), two metamathematical remarks are worth making regarding the employed algebra of truth degrees. First, although we assumed (see Section 2.1) **L** to be a *lattice-complete* linear MTL-algebra, the translation into higher-order $\mathrm{MTL}_\Delta$ shows that the results are in fact valid as well in safe models (see [26, 11]) over lattice-incomplete linear MTL-algebras. Moreover, since the axiom ($\forall$3) of first-order $\mathrm{MTL}_\Delta$ (see [22, 11]) has never been used in the derivation of the results, the theorems are actually valid in safe models over *all* (not only linear) MTL-algebras of membership degrees. The formalization in $\mathrm{MTL}_\Delta$ thus in effect slightly generalizes the results presented in the previous sections.

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