



národní
úložiště
šedé
literatury

A New Proof of the Hansen-Bliek-Rohn Optimality Result

Rohn, Jiří
2014

Dostupný z <http://www.nusl.cz/ntk/nusl-170645>

Dílo je chráněno podle autorského zákona č. 121/2000 Sb.

Tento dokument byl stažen z Národního úložiště šedé literatury (NUŠL).

Datum stažení: 20.05.2024

Další dokumenty můžete najít prostřednictvím vyhledávacího rozhraní [nusl.cz](http://www.nusl.cz) .



Institute of Computer Science
Academy of Sciences of the Czech Republic

A New Proof of the Hansen-Bliek-Rohn Optimality Result

Jiří Rohn

<http://uivtx.cs.cas.cz/~rohn>

Technical report No. V-1212

20.03.2014



Institute of Computer Science
Academy of Sciences of the Czech Republic

A New Proof of the Hansen-Bliek-Rohn Optimality Result

Jiří Rohn¹

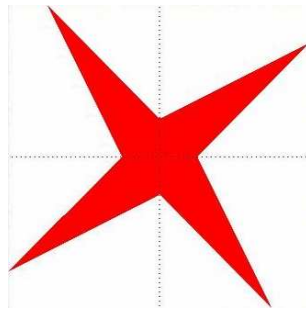
<http://uivtx.cs.cas.cz/~rohn>

Technical report No. V-1212

20.03.2014

Abstract:

We present a new proof of the Hansen-Bliek-Rohn optimality result for interval linear equations with unit midpoint.²



Keywords:

Interval linear equations, unit midpoint, Hansen-Bliek-Rohn optimality result.

¹This work was supported with institutional support RVO:67985807.

²Above: logo of interval computations and related areas (depiction of the solution set of the system $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$, $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$ (Barth and Nuding [1])).

1 Introduction

For a system of interval linear equations $\mathbf{A}x = \mathbf{b}$, where \mathbf{A} is an $n \times n$ interval matrix and \mathbf{b} is an interval n -vector, the interval hull is defined as

$$\mathbf{x}(\mathbf{A}, \mathbf{b}) = \bigcap_{\mathbf{X}(\mathbf{A}, \mathbf{b}) \subseteq [x, y]} [x, y],$$

where

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) = \{x \mid Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\},$$

i.e., as the narrowest interval vector containing the solution set $\mathbf{X}(\mathbf{A}, \mathbf{b})$. Computing the interval hull is NP-hard [11]. Yet it was shown by Hansen [3], Bliiek [2] and Rohn [6] that the hull can be expressed by relatively simple closed-form formulae in case that the system matrix has unit midpoint, i.e., is of the form $\mathbf{A} = [I - \Delta, I + \Delta]$, where I is the unit matrix. However, the proof of this result is by no means straightforward. The formulae not using interval arithmetic were proved in [6], [8] and those formulated in terms of interval arithmetic by Ning and Kearfott [5] (using the result from [6]) and by Neumaier [4].

In this report we present another proof of the optimality result, based on a new characterization of the interval hull (Theorem 1). We give an interval-arithmetic-free version (Theorem 3) and an interval arithmetic version (Theorem 4), both in new formulations aimed at minimizing the number of auxiliary variables.

Notation used: $\text{diag}(M)$ denotes the diagonal of a matrix M , $M_{k\bullet}$ its k th row, T_z is the diagonal matrix with diagonal vector z , $a \circ b$ stands for the Hadamard (entrywise) product of vectors a, b and a/b for their Hadamard division, minimum/maximum of a finite number of vectors is taken entrywise, I is the identity matrix and e is the vector of all ones.

2 Interval hull

We shall later make use of the following characterization of the interval hull.

Theorem 1. *Let $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ be regular. Then for each $z \in \{-1, 1\}^n$ the matrix equation*

$$QA_c - |Q|\Delta T_z = I$$

has a unique solution Q_z and for each right-hand side $\mathbf{b} = [b_c - \delta, b_c + \delta]$ there holds

$$\mathbf{x}(\mathbf{A}, \mathbf{b}) = \left[\min_{z \in \{-1, 1\}^n} (Q_z b_c - |Q_z| \delta), \max_{z \in \{-1, 1\}^n} (Q_z b_c + |Q_z| \delta) \right]. \quad (2.1)$$

Proof. The first part of the theorem is the assertion of [10, Thm. 1], the second one follows from [7, Thm. 2] if we take $Z = \{-1, 1\}^n$ there. \square

3 Matrices Q_z

In this section we show that the matrices Q_z can be expressed explicitly in case of an interval matrix of the form $\mathbf{A} = [I - \Delta, I + \Delta]$. The result, as well as the subsequent ones, is formulated in terms of the matrix

$$M = (I - \Delta)^{-1}.$$

The assumption $M \geq I$ is equivalent to regularity of $[I - \Delta, I + \Delta]$, see [10].

Theorem 2. *Let $M \geq I$. Then for each $z \in \{-1, 1\}^n$ the matrix Q_z is given rowwise by*

$$(Q_z)_{k\bullet} = \begin{cases} M_{k\bullet} T_z & \text{if } z_k = 1, \\ \nu_k(M_{k1}, \dots, -M_{kk}, \dots, M_{kn}) T_z & \text{if } z_k = -1, \end{cases} \quad (3.1)$$

where

$$\nu_k = \frac{1}{2M_{kk}-1} \quad (k = 1, \dots, n).$$

Proof. The expression for $z_k = 1$ is contained in [10, Thm. 2]. The formula for $z_k = -1$ was given in the same theorem as

$$(Q_z)_{k\bullet} = ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T) T_z,$$

where

$$\mu_k = \frac{2M_{kk}}{2M_{kk}-1} \quad (k = 1, \dots, n).$$

Considering the fact that

$$\begin{aligned} (\mu_k - 1)M_{k\bullet} - \mu_k e_k^T &= (\mu_k - 1)(M_{k1}, \dots, M_{kk} - \frac{\mu_k}{\mu_k - 1}, \dots, M_{kn}) \\ &= (\mu_k - 1)(M_{k1}, \dots, -M_{kk}, \dots, M_{kn}) \\ &= \nu_k(M_{k1}, \dots, -M_{kk}, \dots, M_{kn}), \end{aligned}$$

we arrive at the desired result. \square

4 Optimality result

The Hansen-Bliek-Rohn optimality result gives an explicit formula for the interval hull of an interval linear system of the form

$$\mathbf{I}x = \mathbf{b},$$

where $\mathbf{I} = [I - \Delta, I + \Delta]$.

Theorem 3. *Let $M \geq I$. Then for each right-hand side $\mathbf{b} = [b_c - \delta, b_c + \delta]$, denoting $d = \text{diag}(M)$, $x_* = d \circ b_c$ and $x^* = M(|b_c| + \delta)$, we have*

$$\mathbf{x}(\mathbf{I}, \mathbf{b}) = [\min\{\underline{x}, \underline{x}/(2d - e)\}, \max\{\tilde{x}, \tilde{x}/(2d - e)\}], \quad (4.1)$$

where

$$\begin{aligned} \underline{x} &= x_* - (x^* - |x_*|), \\ \tilde{x} &= x_* + (x^* - |x_*|). \end{aligned}$$

Comment. In (4.1) we use (twice) the Hadamard division of vectors.

Proof. Denote $[\underline{x}, \bar{x}] = \mathbf{x}(\mathbf{A}, \mathbf{b})$. Let $k \in \{1, \dots, n\}$. We shall first derive a formula for \bar{x}_k . From (2.1) we have

$$\bar{x}_k = \max_{z \in \{-1, 1\}^n} (Q_z b_c + |Q_z| \delta)_k = \max_{z \in \{-1, 1\}^n} ((Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet} \delta),$$

so that according to (3.1) for each $z \in \{-1, 1\}^n$ we must consider two cases: $z_k = 1$ and $z_k = -1$.

If $z_k = 1$, then by Theorem 2

$$\begin{aligned} (Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet} \delta &= M_{k\bullet} T_z b_c + M_{k\bullet} \delta \\ &= \sum_{j \neq k} M_{kj} z_j (b_c)_j + M_{kk} (b_c)_k + M_{k\bullet} \delta \\ &\leq \sum_{j \neq k} M_{kj} |(b_c)_j| + M_{kk} (b_c)_k + M_{k\bullet} \delta. \end{aligned}$$

Introducing the vector $\bar{z}(k) \in \{-1, 1\}^n$ by

$$\bar{z}(k)_j = \begin{cases} 1 & \text{if } j = k, \\ 1 & \text{if } j \neq k \text{ and } (b_c)_j \geq 0, \\ -1 & \text{if } j \neq k \text{ and } (b_c)_j < 0 \end{cases} \quad (j = 1, \dots, n),$$

we can write

$$\sum_{j \neq k} M_{kj} |(b_c)_j| + M_{kk} (b_c)_k + M_{k\bullet} \delta = M_{k\bullet} T_{\bar{z}(k)} b_c + M_{k\bullet} \delta = (Q_{\bar{z}(k)})_{k\bullet} b_c + |Q_{\bar{z}(k)}|_{k\bullet} \delta,$$

hence for each $z \in \{-1, 1\}^n$ with $z_k = 1$ we have

$$(Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet} \delta \leq (Q_{\bar{z}(k)})_{k\bullet} b_c + |Q_{\bar{z}(k)}|_{k\bullet} \delta,$$

and the upper bound is obviously attained.

If $z_k = -1$, then, again by Theorem 2,

$$\begin{aligned} (Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet} \delta &= \nu_k (M_{k1}, \dots, -M_{kk}, \dots, M_{kn}) T_z b_c + \nu_k M_{k\bullet} \delta \\ &= \nu_k \sum_{j \neq k} M_{kj} z_j (b_c)_j + \nu_k M_{kk} (b_c)_k + \nu_k M_{k\bullet} \delta \\ &\leq \nu_k \sum_{j \neq k} M_{kj} |(b_c)_j| + \nu_k M_{kk} (b_c)_k + \nu_k M_{k\bullet} \delta \\ &= \nu_k (M_{k1}, \dots, -M_{kk}, \dots, M_{kn}) T_{\underline{z}(k)} b_c + \nu_k M_{k\bullet} \delta \\ &= (Q_{\underline{z}(k)})_{k\bullet} b_c + |Q_{\underline{z}(k)}|_{k\bullet} \delta \end{aligned}$$

where we have employed the vector $\underline{z}(k)$ given by

$$\underline{z}(k)_j = \begin{cases} -1 & \text{if } j = k, \\ 1 & \text{if } j \neq k \text{ and } (b_c)_j \geq 0, \\ -1 & \text{if } j \neq k \text{ and } (b_c)_j < 0 \end{cases} \quad (j = 1, \dots, n),$$

hence for each $z \in \{-1, 1\}^n$ with $z_k = -1$ we have

$$(Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet} \delta \leq (Q_{\underline{z}(k)})_{k\bullet} b_c + |Q_{\underline{z}(k)}|_{k\bullet} \delta,$$

and the upper bound is again obviously attained. In this way we have proved the formula

$$\bar{x}_k = \max\{(Q_{\bar{z}(k)})_{k\bullet} b_c + |Q_{\bar{z}(k)}|_{k\bullet} \delta, (Q_{\underline{z}(k)})_{k\bullet} b_c + |Q_{\underline{z}(k)}|_{k\bullet} \delta\}.$$

Now,

$$\begin{aligned}
(Q_{\bar{z}(k)})_{k\bullet}b_c + |Q_{\bar{z}(k)}|_{k\bullet}\delta &= \sum_{j \neq k} M_{kj}|(b_c)_j| + M_{kk}(b_c)_k + M_{k\bullet}\delta \\
&= M_{k\bullet}(|b_c| + \delta) + M_{kk}((b_c)_k - |b_c|_k) \\
&= (x_* + x^* - |x_*|)_k \\
&= \tilde{x}_k
\end{aligned}$$

and similarly

$$\begin{aligned}
(Q_{\underline{z}(k)})_{k\bullet}b_c + |Q_{\underline{z}(k)}|_{k\bullet}\delta &= \nu_k \sum_{j \neq k} M_{kj}|(b_c)_j| + \nu_k M_{kk}(b_c)_k + \nu_k M_{k\bullet}\delta \\
&= \nu_k (M_{k\bullet}(|b_c| + \delta) + M_{kk}((b_c)_k - |b_c|_k)) \\
&= \nu_k (x_* + x^* - |x_*|)_k \\
&= \nu_k \tilde{x}_k
\end{aligned}$$

which together gives

$$\bar{x}_k = \max\{\tilde{x}_k, \nu_k \tilde{x}_k\}.$$

Since

$$\nu_k \tilde{x}_k = \tilde{x}_k / (2M_{kk} - 1),$$

we finally obtain

$$\bar{x} = \max\{\tilde{x}, \tilde{x}/(2d - e)\},$$

where we have used the Hadamard (entrywise) division of vectors.

To prove the formula for \underline{x} , consider the system $\mathbf{I}\mathbf{x} = -\mathbf{b}$, where $\mathbf{I} = [I - \Delta, I + \Delta]$ as before and $-\mathbf{b} = \{-b \mid b \in \mathbf{b}\} = [-b_c - \delta, -b_c + \delta]$. Then $\mathbf{X}(\mathbf{I}, -\mathbf{b}) = -\mathbf{X}(\mathbf{I}, \mathbf{b})$, hence $\mathbf{x}(\mathbf{I}, -\mathbf{b}) = [-\bar{x}, -\underline{x}]$. Now we can apply the previously derived formula for the upper bound of the interval hull:

$$-\underline{x} = \max\{-d \circ b_c + M(|b_c| + \delta) - |d \circ b_c|, (-d \circ b_c + M(|b_c| + \delta) - |d \circ b_c|)/(2d - e)\},$$

hence

$$\begin{aligned}
\underline{x} &= \min\{d \circ b_c - M(|b_c| + \delta) + |d \circ b_c|, (d \circ b_c - M(|b_c| + \delta) + |d \circ b_c|)/(2d - e)\} \\
&= \min\{x_* - x^* + |x_*|, (x_* - x^* + |x_*|)/(2d - e)\} \\
&= \min\{\underline{x}, \underline{x}/(2d - e)\}.
\end{aligned}$$

This proves that

$$\mathbf{x}(\mathbf{I}, \mathbf{b}) = [\min\{\underline{x}, \underline{x}/(2d - e)\}, \max\{\bar{x}, \bar{x}/(2d - e)\}].$$

□

Using the interval arithmetic, we can bring the result to yet simpler form.

Theorem 4. *Let $M \geq I$. Denoting $d = \text{diag}(M)$, $x_* = d \circ b_c$ and $x^* = M(|b_c| + \delta)$, we have*

$$\mathbf{x}(\mathbf{I}, \mathbf{b}) = \frac{\langle x_*, x^* - |x_*| \rangle}{\langle d, d - e \rangle}. \quad (4.2)$$

Comment. In (4.2) we use the Hadamard (entrywise) division of interval vectors and their midpoint-radius representation, i.e., $\langle a, b \rangle = [a - b, a + b]$.

Proof. Because $\underline{x} \leq \tilde{x}$ and $\nu > 0$, we can write (4.1) as

$$\mathbf{x}(\mathbf{I}, \mathbf{b}) = [\min\{\underline{x}/e, \underline{x}/(2d - e), \tilde{x}/e, \tilde{x}/(2d - e)\}, \max\{\underline{x}/e, \underline{x}/(2d - e), \tilde{x}/e, \tilde{x}/(2d - e)\}],$$

which is the Hadamard division performed in interval arithmetic:

$$\mathbf{x}(\mathbf{I}, \mathbf{b}) = \frac{[\underline{x}, \tilde{x}]}{[e, 2d - e]}. \quad (4.3)$$

Since

$$[\underline{x}, \tilde{x}] = [x_* - (x^* - |x_*|), x_* + (x^* - |x_*|)] = \langle x_*, x^* - |x_*| \rangle$$

and

$$[e, 2d - e] = \langle d, d - e \rangle,$$

(4.3) implies (4.2). □

The Hansen-Blik-Rohn *optimality result* should not be misunderstood for the Hansen-Blik-Rohn *enclosure*, see [9].

Bibliography

- [1] W. Barth and E. Nuding, *Optimale Lösung von Intervallgleichungssystemen*, Computing, 12 (1974), pp. 117–125. [1](#)
- [2] C. Blied, *Computer Methods for Design Automation*, PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, July 1992. [1](#)
- [3] E. R. Hansen, *Bounding the solution of interval linear equations*, SIAM Journal on Numerical Analysis, 29 (1992), pp. 1493–1503. [1](#)
- [4] A. Neumaier, *A simple derivation of the Hansen-Blied-Rohn-Ning-Kearfott enclosure for linear interval equations*, Reliable Computing, 5 (1999), pp. 131–136. [1](#)
- [5] S. Ning and R. B. Kearfott, *A comparison of some methods for solving linear interval equations*, SIAM Journal on Numerical Analysis, 34 (1997), pp. 1289–1305. [1](#)
- [6] J. Rohn, *Cheap and tight bounds: The recent result by E. Hansen can be made more efficient*, Interval Computations, 4 (1993), pp. 13–21. [1](#)
- [7] J. Rohn, *An algorithm for computing the hull of the solution set of interval linear equations*, Linear Algebra and Its Applications, 435 (2011), pp. 193–201. [1](#)
- [8] J. Rohn, *The Hansen-Blied optimality result as a consequence of the general theory*, Technical Report 1104, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, February 2011. <http://uivtx.cs.cas.cz/~rohn/publist/hansen.pdf>. [1](#)
- [9] J. Rohn, *Compact form of the Hansen-Blied-Rohn enclosure*, Technical Report 1157, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, March 2012. <http://uivtx.cs.cas.cz/~rohn/publist/compactforms.pdf>. [5](#)
- [10] J. Rohn, *Explicit form of matrices Q_z for an interval matrix with unit midpoint*, Technical Report 1206, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, February 2014. <http://uivtx.cs.cas.cz/~rohn/publist/report1206.pdf>. [1](#), [2](#)
- [11] J. Rohn and V. Kreinovich, *Computing exact componentwise bounds on solutions of linear systems with interval data is NP-hard*, SIAM Journal on Matrix Analysis and Applications, 16 (1995), pp. 415–420. [1](#)