A posteriori algebraic error estimation in numerical solution of linear diffusion PDEs

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1 Introduction

The paper [1], see also the references therein, proposes an adaptive method with a posteriori stopping criteria for numerical solution of nonlinear partial differential equations of diffusion type. The main idea in [1] is to distinguish different components of the error, namely the discretization, the linearization, and the algebraic ones, and to design stopping criteria based on balancing these error components. The estimates rely on quasi-equilibrated flux reconstructions and yield a general framework which can be applied to various discretization schemes.

In the present contribution we tightly follow [1] and concentrate specifically on estimating the algebraic part of the error. We show that, with an additional assumption on the flux reconstructions, the algebraic error can be bounded using the algebraic a posteriori error estimator. This justifies the distinction of error components presented in [1]. For simplicity we restrict ourselves to a linear model problem discretized using the conforming finite element method. We show that the flux reconstruction given in [1] can be modified such that the newly introduced assumption is satisfied. We believe that an analogous modification is possible also for other discretization schemes, as well as for the nonlinear setting considered in [1].

2 Model problem and discrete setting

Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a polygonal (polyhedral) domain. We consider the Poisson model problem: find $u : \Omega \to \mathbb{R}$ such that

$$\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $f : \Omega \to \mathbb{R}$ is the source term. Assuming $f \in L^2(\Omega)$, the model problem (1) can be casted into the weak form: find $u \in V \equiv H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in V, \tag{2}$$

where $H_0^1(\Omega)$ denotes the standard Hilbert space of $L^2(\Omega)$ functions whose weak derivatives are in $L^2(\Omega)$ and with trace vanishing on $\partial\Omega$. Owing to (2), the flux $-\nabla u$ is in the space $\mathbf{H}(\mathrm{div}, \Omega)$ spanned by the functions in $[L^2(\Omega)]^d$ with weak divergences in $L^2(\Omega)$.

Let $\mathcal{T}_h$ be a simplicial mesh of Ω. We suppose that the mesh is conforming in the sense that, for two distinct elements of $\mathcal{T}_h$, their intersection is either an empty set or a common l-dimensional face, $0 \leq l \leq d-1$. We denote a generic element of $\mathcal{T}_h$ by K and its diameter by h_K. We denote by $\mathbb{P}_m(K)$ the space of m-th order polynomial functions on an element K and by $\mathbb{P}_m(\mathcal{T}_h)$ the broken polynomial space spanned by $v_h|_K \in \mathbb{P}_m(K)$ for all $K \in \mathcal{T}_h$. Let

$$V_h \equiv H_0^1(\Omega) \cap \mathbb{P}_m(\mathcal{T}_h) = \left\{ v \in H_0^1(\Omega), v|_K \in \mathbb{P}_m(K) \quad \forall K \in \mathcal{T}_h \right\} \tag{3}$$

be the usual finite element space of continuous, piecewise m-th order polynomial functions, $m \geq 1$. The corresponding discrete formulation of problem (2) reads: find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h\,. \tag{4}$$

Let $\psi_j \in V_h,\ j \in \mathcal{C} \equiv \{1, \dots, \dim(V_h)\}$, denote the usual Lagrange basis of V_h. Employing this basis in (4) gives rise to the system of linear algebraic equations

$$\mathsf{AU} = \mathsf{F}\,. \tag{5}$$

At the i-th step, $i = 1, 2, \dots$, of an iterative solver applied to the algebraic system (5), we obtain the approximation $\mathsf{U}^i = [\mathsf{U}^i_j]_{j\in\mathcal{C}}$ to the solution U and the algebraic residual vector $\mathsf{R}^i = [\mathsf{R}^i_j]_{j\in\mathcal{C}}$ such that

$$\mathsf{AU}^i = \mathsf{F} - \mathsf{R}^i\,. \tag{6}$$

Finally, by u_h^i we denote the approximation to the solution u determined by the coefficient vector U^i,

$$u_h^i \equiv \sum_{j\in\mathcal{C}} \mathsf{U}^i_j \psi_j\,. \tag{7}$$

3 Error measure and a posteriori error estimates for total error and for the algebraic error

The (total) error between the exact solution u of the weak formulation (2) and the approximate solution $u_h^i \in V_h$ given by (7) is measured as

$$\|\nabla(u - u_h^i)\| = \sup_{\varphi\in V, \|\nabla\varphi\|=1} \big(\nabla(u - u_h^i), \nabla\varphi\big)\,. \tag{8}$$

The following assumption is the starting point for a posteriori error estimation proposed in [1].

Assumption 3.1 (Quasi-equilibrated flux reconstructions)**.** *There exist vector-valued functions* $\mathbf{t}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{d}_h^i, \mathbf{a}_h^i \in [L^2(\Omega)]^d$, *and a scalar-valued function* $\rho_h^i \in L^2(\Omega)$ *such that*

1. $\nabla\cdot \mathbf{t}_h^i = f_h - \rho_h^i\,,$
2. $\mathbf{t}_h^i = \mathbf{d}_h^i + \mathbf{a}_h^i\,,$
3. *as the linear solver converges,* $\|\mathbf{a}_h^i\| \to 0\,.$

Here f_h *is a piecewise polynomial approximation of the source term* f *verifying* $(f_h, 1)_K = (f, 1)_K$ *for all* $K \in \mathcal{T}_h$*.*

For any $K \in \mathcal{T}_h$, the Poincaré inequality states that

$$\|\varphi - \varphi_K\|_K \leq C_{\mathrm{P}} h_K \|\nabla\varphi\|_K \qquad \forall\varphi \in H^1(K)\,, \tag{9}$$

where φ_K denotes the mean value of φ in K. Since the simplices K are convex, there holds $C_{\mathrm{P}} = 1/\pi$; see, e.g., [2, 3]. The Friedrichs inequality states that

$$\|\varphi\| \leq h_\Omega \|\nabla\varphi\| \qquad \forall\varphi \in V\,, \tag{10}$$

where h_Ω denotes the diameter of the domain Ω. The following theorem is a simple application of [1, Theorems 3.4 and 3.6] to our model problem. We denote local estimators in the form $\eta^i_{\Box,K}$, where $i = 1, 2, \ldots$ stands for the algebraic iteration step and $K \in \mathcal{T}_h$ for the mesh element. The global versions of these estimators are defined as $\eta^i_\Box \equiv \left\{\sum_{K\in\mathcal{T}_h} (\eta^i_{\Box,K})^2\right\}^{1/2}$.

Theorem 3.2 (Total error a posteriori estimate distinguishing error components)**.** *Let $u \in V$ solve (2), let $u^i_h \in V_h$ be given by (7), and let Assumption 3.1 hold. For any $K \in \mathcal{T}_h$, define respectively the* discretization estimator*, the* algebraic estimator*, the* algebraic remainder*, and the* data oscillation estimator *as*

$$\begin{aligned}
\eta^i_{\mathrm{disc},K} &\equiv \|\nabla u^i_h + \mathbf{d}^i_h\|_K\,, &(11)\\
\eta^i_{\mathrm{alg},K} &\equiv \|\mathbf{a}^i_h\|_K\,, &(12)\\
\eta^i_{\mathrm{rem},K} &\equiv h_\Omega\|\rho^i_h\|_K\,, &(13)\\
\eta^i_{\mathrm{osc},K} &\equiv C_{\mathrm{P}} h_K\|f - f_h\|_K\,. &(14)
\end{aligned}$$

Then

$$\|\nabla(u - u^i_h)\| \le \eta^i_{\mathrm{disc}} + \eta^i_{\mathrm{alg}} + \eta^i_{\mathrm{rem}} + \eta^i_{\mathrm{osc}}\,. \tag{15}$$

In the adaptive algorithm proposed in [1] the flux reconstruction $\mathbf{d}^i_h$ is constructed using the approximate algebraic solution U^i given at the i-th step of algebraic iterative solver. Then one performs $\nu > 0$ additional iteration steps yielding the vector $\mathsf{U}^{i+\nu}$ and the corresponding flux reconstruction $\mathbf{d}^{i+\nu}_h$. The algebraic error flux reconstruction is defined as $\mathbf{a}^i_h \equiv \mathbf{d}^{i+\nu}_h - \mathbf{d}^i_h$. The number ν of the additional iteration steps and the convergence of the algebraic solver are controlled using the (global) criteria

$$\begin{aligned}
\eta^i_{\mathrm{rem}} &\le \gamma_{\mathrm{rem}} \max\left\{\eta^i_{\mathrm{disc}}, \eta^i_{\mathrm{alg}}\right\}, &(16)\\
\eta^i_{\mathrm{alg}} &\le \gamma_{\mathrm{alg}}\, \eta^i_{\mathrm{disc}}\,, &(17)
\end{aligned}$$

or using the elementwise equivalents

$$\begin{aligned}
\eta^i_{\mathrm{rem},K} &\le \gamma_{\mathrm{rem},K} \max\left\{\eta^i_{\mathrm{disc},K}, \eta^i_{\mathrm{alg},K}\right\}, & &(18)\\
\eta^i_{\mathrm{alg},K} &\le \gamma_{\mathrm{alg},K}\, \eta^i_{\mathrm{disc},K}\,, & \forall K \in \mathcal{T}_h\,. &(19)
\end{aligned}$$

Here $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}$ (respectively $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}$) are the user-given weights (typically of order 0.1). The criteria (16)–(17) are sufficient to establish the global efficiency of the total error estimator; the *local* criteria (18)–(19) assure the *local* efficiency; see [1, Section 5].

Elaborating on the results from [1], our goal is to bound also the algebraic error

$$\|\nabla(u_h - u^i_h)\| = \sup_{\varphi_h\in V_h, \|\nabla\varphi_h\|=1} \left(\nabla(u_h - u^i_h), \nabla\varphi_h\right),$$

where u_h is the (unknown) solution of the discrete formulation (4) and $u^i_h \in V_h$ is an approximation to u_h as given by (7). We introduce for this purpose an additional assumption on the flux reconstruction.

Assumption 3.3 (Quasi-equilibration of $\mathbf{d}^i_h$)**.** *The function $\mathbf{d}^i_h$ satisfies $\mathbf{d}^i_h \in \mathbf{H}(\mathrm{div}, \Omega)$ and there exists a scalar-valued function $r^i_h \in L^2(\Omega)$ such that*

$$\begin{aligned}
\nabla\cdot \mathbf{d}^i_h &= f_h - r^i_h\,, &(20)\\
(r^i_h, \psi_j) &= \mathsf{R}^i_j \qquad \forall j \in \mathcal{C}\,. &(21)
\end{aligned}$$

Assuming (20) and setting $\mathbf{a}_h^i = \mathbf{d}_h^{i+\nu} - \mathbf{d}_h^i$ as above, Assumption 3.1 is satisfied with $\rho_h^i \equiv r_h^{i+\nu}$.

Theorem 3.4 (Algebraic error a posteriori estimate)**.** *Let u_h be the solution of* (4) *and $u_h^i \in V_h$ be given by* (7)*. Let $\eta_{\mathrm{alg}}^i, \eta_{\mathrm{rem}}^i$ be defined respectively by* (12) *and* (13)*. Let Assumption 3.3 hold. Then*

$$\|\nabla(u_h - u_h^i)\| \leq \eta_{\mathrm{alg}}^i + \eta_{\mathrm{rem}}^i \,. \tag{22}$$

Therefore, using the criteria (16) or (18), the algebraic estimator η_{alg}^i provides an upper bound on the algebraic error. The efficiency of this estimator is a subject of further study — the techniques used for the proof of global and local efficiency of the total error estimator (see [1, Section 5]) are not applicable in this case.

4 Flux reconstructions

The paper [1] presents flux reconstruction in various discretization schemes that fulfill Assumption 3.1 and the first part (20) of Assumption 3.3. In this contribution we restrict ourselves to the conforming finite element method. We show that we can easily modify the flux reconstruction from [1] such that the relation (21) required for proving the bound (22) is also satisfied. The flux reconstruction is sought in the Raviart–Thomas–Nédélec finite element space and it is constructed using (mutually independent) local homogeneous Neumann mixed finite element problems posed on patches around mesh vertices.

5 Conclusion

Following [1] we presented a posteriori error estimate for the total error that distinguishes its different components. The estimate yields a guaranteed upper bound on the total error. Additionally, we showed that the parts of the estimate denoted as algebraic estimator and algebraic reminder provide an upper bound on the algebraic error. This justifies the distinction of error components and the stopping criteria presented in [1]. We applied the general framework from [1] to a linear problem and the conforming finite element discretization. The application for other discretization schemes and nonlinear problems and the efficiency of the estimate are subjects of further study.

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