Noise revealing in Golub-Kahan bidiagonalization as a mean of regularization in discrete inverse problems

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1 Introduction

We consider an ill-posed linear system

$$Ax \approx b, \qquad A \in \mathbb{R}^{n\times n}, \qquad b = b^{\text{exact}} + b^{\text{noise}} \in \mathbb{R}^n, \tag{1}$$

where A is a nonsingular matrix and b^{noise} is an unknown perturbation of the right-hand side b^{exact}, $\|b^{\text{noise}}\| \ll \|b^{\text{exact}}\|$. Moreover, we assume that the matrix A is a discretized *smoothing* operator with singular values decaying gradually to zero and the vector b^{noise} represents noise (for simplicity, we assume white noise, that is, the noise has flat frequency characteristics). The aim is to approximate the exact solution

$$x^{\text{exact}} \equiv A^{-1} b^{\text{exact}}.$$

Since A has smoothing property, the operator A^{-1} amplifies high-frequencies. For noise significant enough, the discrete Picard condition is violated, which makes the naive solution $x^{\text{naive}} \equiv A^{-1}b$ completely meaningless, and problem (1) has to be regularized. A successful regularization method has to suppress the devastating effect of high-frequency noise while preserving sufficient information from the data. The amount of regularization is usually controlled by a regularization parameter and choice of this parameter represents the most difficult part of solving discrete inverse problems [2]. One can also attempt to eliminate (at least to some extent) the high-frequency part of the noise. Assume, we have an estimate $\tilde{b}^{\text{noise}}$ of the noise vector b^{noise}. Then, a straightforward approach to solve problem (1) is to subtract this estimate from the right-hand side b, and solve the system

$$Ax = b - \tilde{b}^{\text{noise}}. \tag{2}$$

We want system (2) to have better overall properties than the original problem (1). In our case, the aim is to dampen the high frequencies coming from noise. The key part of this approach is to find an estimate $\tilde{b}^{\text{noise}}$. In the following, we will present a cheap parameter-free method for finding such an estimate using Golub-Kahan bidiagonalization [1].

2 Estimating noise via noise propagation in Golub-Kahan bidiagonalization

Golub-Kahan bidiagonalization is an iterative procedure that is widely used in solving large linear systems. Given the initial vectors $w_0 \equiv 0$, $s_1 \equiv b/\beta_1$, $\beta_1 \equiv \|b\| \neq 0$, it computes

$$\begin{aligned} \alpha_k w_k &= A^T s_k - \beta_k w_{k-1}\,, \qquad \|w_k\| = 1\,, \\ \beta_{k+1} s_{k+1} &= A w_k - \alpha_j s_k\,, \quad \|s_{k+1}\| = 1, \end{aligned} \tag{3}$$

until $\alpha_k = 0$ or $\beta_{k+1} = 0$, or until $k = n$. Vectors s_k and w_k form the bases of Krylov subspaces $\mathcal{K}_K(AA^T, b)$ and $\mathcal{K}_K(A^TA, A^Tb)$ respectively.

In hybrid methods (see, e.g., [3, 4]), Golub-Kahan bidiagonalization is used as outer regularization (regularization of the original large problem by projection). Moreover, as shown in [5], due to the orthogonalization, one may also make use of the propagation of the noise through the bidiagonalization process. Since the starting vector s_1 is polluted by white noise, this noise is present in all subsequent left bidiagonalization vectors s_k. As shown in [5], the size of the noise in the vector s_{k+1} can be related to the amplification factor

$$\rho_k^{-1} \equiv \prod_{j=1}^{k} \frac{\alpha_j}{\beta_{j+1}}, \tag{4}$$

where α_j and β_{j+1} are the normalization coefficients from (3). It was also shown in [5] that if A is a discretized smoothing operator, then the factor ρ_k^{-1} has to grow (on average) until it reaches the point where the noise is revealed in the maximal way. This is illustrated in Figures 1 and 2.

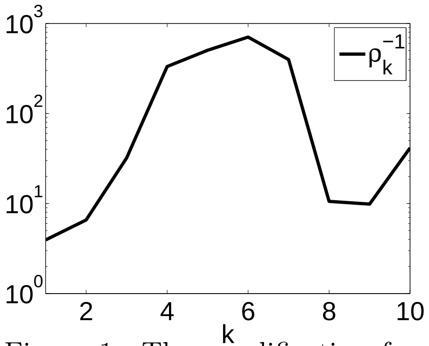

Figure 1: The amplification factor ρ_k^{-1} for problem `shaw(400)` from [6] with relative noise level 10^{-3}.

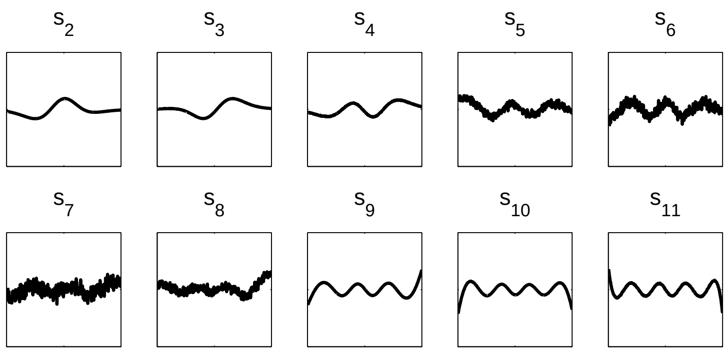

Figure 2: Corresponding left bidiagonalization vectors $s_k, k = 2, \ldots, 11$.

We see that for ρ_k^{-1} maximal, the bidiagonalization vector s_{k+1} may be fully dominated by the noise. This observation forms the basis of the proposed method for finding an estimate $\tilde{b}^{\text{noise}}$.

Let $\hat{k}+1$, where $\hat{k} \equiv \operatorname*{argmax}_k \rho_k^{-1}$, be the iteration of maximal noise revealing (in our example presented above, $\hat{k}+1 = 7$). Then, one may approximate the noise vector by the (properly scaled) left bidiagonalization vector $s_{\hat{k}+1}$. In [7] it was shown that the resulting right-hand side $b - \tilde{b}^{\text{noise}}$ lies in the span of smooth vectors (the troublesome high-frequencies coming from the noise are subtracted) and therefore the method has a regularization effect, as illustrated in Figure 3.

Despite being computationally undemanding, this method is, as shown in [7], competitive with standard methods for solving inverse problems such as truncated SVD or Tikhonov [4]. The method still needs to be tested on real-world examples and it has to be investigated, how to solve system (2) efficiently, or whether rounding errors and consecutive loss of orthogonality may harm the method significantly.

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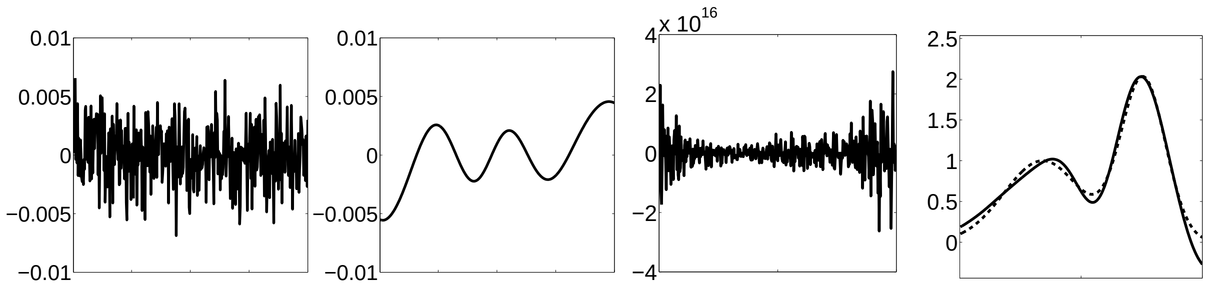

(a) b^{noise} (b) $b^{\text{noise}} - \tilde{b}^{\text{noise}}$ (c) $A^{-1}b$ (d) $A^{-1}(b - \tilde{b}^{\text{noise}})$

Figure 3: Regularizing effect of the proposed method – problem from Figure 1. Left to right: (a) original noise, (b) noise with reduced high-frequency part, (c) naive solution, (d) inverse operator applied to thr new right-hand side together with exact solution x^{exact} (dashed line).

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