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**Institute of Computer Science**  
**Academy of Sciences of the Czech Republic**

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Technical report No. V-1206

20.02.2014



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## Explicit Form of Matrices $Q_z$ for an Interval Matrix with Unit Midpoint

Jiří Rohn<sup>1</sup>

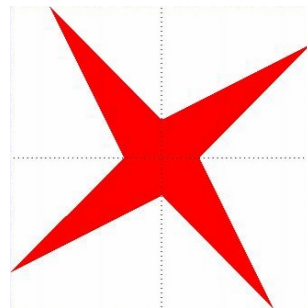
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Abstract:

We give three versions of explicit formulae for matrices  $Q_z$  for an interval matrix with unit midpoint.<sup>2</sup>



Keywords:

Interval matrix,  $Q_z$  matrix, unit midpoint, explicit formula.

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<sup>2</sup>Above: logo of interval computations and related areas (depiction of the solution set of the system  $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$ ,  $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$  (Barth and Nuding [1])).

# 1 Introduction

Matrices  $Q_z$ , first appearing in [5], may be best introduced by way of the following theorem.

**Theorem 1.** *If an  $n \times n$  interval matrix  $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$  is regular,<sup>3</sup> then for each  $z \in \{-1, 1\}^n$  the nonlinear matrix equation*

$$QA_c - |Q|\Delta T_z = I \tag{1.1}$$

*has a unique matrix solution  $Q_z$ .*

**Comment.** Here the absolute value of a matrix is understood entrywise,  $I$  is the identity matrix and  $T_z = \text{diag}(z)$  denotes the diagonal matrix with diagonal vector  $z$ .

*Proof.* Because  $\mathbf{A}$  is regular, its transpose  $\mathbf{A}^T = \{A^T \mid A \in \mathbf{A}\} = [A_c^T - \Delta^T, A_c^T + \Delta^T]$  is also regular, hence by [3, Thm. 5.1, Assertion (A3)] for each  $z \in \{-1, 1\}^n$  the equation

$$A_c^T B - T_z \Delta^T |B| = I \tag{1.2}$$

has a unique matrix solution  $B_z$ . Then

$$B_z^T A_c - |B_z^T| \Delta T_z = I,$$

hence  $Q_z = B_z^T$  solves (1.1), and its uniqueness follows from that of (1.2).  $\square$

A general algorithm **qzmatrix** for computing  $Q_z$  based on the algorithm **absvaleqn** for solving absolute value equations was described in [5]. Neither this nor any other known result gives any clue about the shape of these matrices. In this report we describe an explicit form of matrices  $Q_z$  for interval matrices with unit midpoint, i.e., satisfying  $A_c = I$ . That is, we look for explicit form of the solution of the equation

$$Q - |Q|\Delta T_z = I \tag{1.3}$$

where  $\Delta$  is an arbitrary nonnegative matrix bound only by regularity requirement (see below). In this way we make a step towards our main goal, namely a new proof of the Hansen-Blik-Rohn optimality result [4].

# 2 Matrices $Q_z$

For a nonnegative square matrix  $\Delta$  put

$$M = (I - \Delta)^{-1}.$$

It is known that the following four assertions are equivalent:

- (i)  $[I - \Delta, I + \Delta]$  is regular,
- (ii)  $M \geq I$ ,
- (iii)  $M \geq 0$ ,
- (iv)  $\varrho(\Delta) < 1$

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<sup>3</sup>I.e., each  $A \in \mathbf{A}$  is nonsingular.

(see [6], [2]). Thus any of (ii)-(iv) can be used as a regularity condition. We choose (ii) because we need the fact that each diagonal entry of  $M$  is greater or equal than one. Now we have the following explicit description of solution of the equation (1.3).

**Theorem 2.** *Let  $M \geq I$ . Then for each  $z \in \{-1, 1\}^n$  the matrix  $Q_z$  is given rowwise by*

$$(Q_z)_{k\bullet} = \begin{cases} M_{k\bullet}T_z & \text{if } z_k = 1, \\ ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T)T_z & \text{if } z_k = -1, \end{cases} \quad (2.1)$$

where

$$\mu_k = \frac{2M_{kk}}{2M_{kk}-1} \quad (k = 1, \dots, n).$$

**Comment.** These formulae are not easy-to-derive ones. But once found, our task is greatly simplified because we are left with checking that  $Q_z$  given by (2.1) satisfies (1.3).  $M_{k\bullet}$  denotes the  $k$ th row of  $M$ , and  $e_k$  stands for the  $k$ th column of the identity matrix  $I$ .

*Proof.* Given a  $z \in \{-1, 1\}^n$ , define a matrix  $Q$  by

$$Q_{k\bullet} = \begin{cases} M_{k\bullet}T_z & \text{if } z_k = 1, \\ ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T)T_z & \text{if } z_k = -1 \end{cases} \quad (k = 1, \dots, n). \quad (2.2)$$

We shall prove that  $Q$  solves (1.3), which under the regularity assumption  $M \geq I$  will mean that  $Q = Q_z$ . Let us note that this assumption implies that  $2M_{kk} - 1 \geq 1$  and  $\mu_k > 1$  for each  $k$ , and  $M\Delta = M - I$ . Take an arbitrary  $k \in \{1, \dots, n\}$ . Now we have either  $z_k = 1$ , or  $z_k = -1$ .

If  $z_k = 1$ , then

$$|Q|_{k\bullet} = M_{k\bullet}, \quad (2.3)$$

hence

$$(Q - |Q|\Delta T_z)_{k\bullet} = M_{k\bullet}T_z - M_{k\bullet}\Delta T_z = M_{k\bullet}T_z - (M_{k\bullet} - e_k^T)T_z = e_k^T T_z = z_k I_{k\bullet} = I_{k\bullet},$$

so that

$$(Q - |Q|\Delta T_z)_{k\bullet} = I_{k\bullet}. \quad (2.4)$$

If  $z_k = -1$ , then (2.2) implies that

$$|Q_{kj}| = |(\mu_k - 1)M_{kj}z_j| = (\mu_k - 1)M_{kj} \quad (2.5)$$

for each  $j \neq k$ , and since

$$Q_{kk} = ((\mu_k - 1)M_{kk} - \mu_k)z_k = \left( \frac{M_{kk}}{2M_{kk} - 1} - \frac{2M_{kk}}{2M_{kk} - 1} \right) z_k = -\frac{M_{kk}}{2M_{kk} - 1} z_k,$$

we have

$$|Q_{kk}| = \frac{M_{kk}}{2M_{kk} - 1} = (\mu_k - 1)M_{kk}$$

which together with (2.5) gives

$$|Q|_{k\bullet} = (\mu_k - 1)M_{k\bullet}. \quad (2.6)$$

Now,

$$\begin{aligned}
(Q - |Q|\Delta T_z)_{k\bullet} &= ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T)T_z - (\mu_k - 1)M_{k\bullet}\Delta T_z \\
&= ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T)T_z - (\mu_k - 1)(M_{k\bullet} - e_k^T)T_z \\
&= -e_k^T T_z = -z_k I_{k\bullet} = I_{k\bullet},
\end{aligned}$$

hence (2.4) again holds. In this way we have shown that  $Q$  solves (1.3), and uniqueness of its solution implies that  $Q = Q_z$ . This proves that  $Q_z$  is given by (2.1).  $\square$

As a by-product of the proof we obtain an explicit description of the matrix  $|Q_z|$ .

**Theorem 3.** *Let  $M \geq I$ . Then for each  $z \in \{-1, 1\}^n$  the matrix  $|Q_z|$  is given rowwise by*

$$|Q_z|_{k\bullet} = \begin{cases} M_{k\bullet} & \text{if } z_k = 1, \\ (\mu_k - 1)M_{k\bullet} & \text{if } z_k = -1 \end{cases} \quad (k = 1, \dots, n).$$

*Proof.* These are simply the equations (2.3) and (2.6).  $\square$

Next we show that both  $Q_z$  and  $|Q_z|$  can be given by compact one-line formulae that, however, may be seen less transparent than the former ones.

**Theorem 4.** *Let  $M \geq I$ . Then for each  $z \in \{-1, 1\}^n$  the matrix  $Q_z$  and its absolute value are given by*

$$Q_z = \max\{T_z M, T_z((I - T_\mu)M + T_\mu)\}T_z \quad (2.7)$$

and

$$|Q_z| = \max\{T_z M, T_z(I - T_\mu)M\}, \quad (2.8)$$

where

$$\mu = \frac{2\text{diag}(M)}{2\text{diag}(M) - e}. \quad (2.9)$$

**Comment.** Observe closely the equation (2.7): first the entrywise maximum of two matrices is taken, then the result is postmultiplied by  $T_z$ . In (2.9) we use the Hadamard division of vectors so that

$$\mu_k = \frac{2M_{kk}}{2M_{kk} - 1} \quad (k = 1, \dots, n)$$

as before ( $e$  is the vector of all ones).

*Proof.* For given  $z \in \{-1, 1\}^n$  set

$$Q = \max\{T_z M, T_z((I - T_\mu)M + T_\mu)\}T_z$$

and consider the difference

$$T_z M - T_z((I - T_\mu)M + T_\mu) = T_z T_\mu(M - I).$$

Because of  $\mu > 0$  and  $M \geq I$  we have  $T_\mu(M - I) \geq 0$ , hence for each  $k$  there holds

$$(T_z M)_{k\bullet} \geq (T_z((I - T_\mu)M + T_\mu))_{k\bullet}$$

if  $z_k = 1$  and

$$(T_z M)_{k\bullet} \leq (T_z((I - T_\mu)M + T_\mu))_{k\bullet}$$

if  $z_k = -1$ . Thus

$$Q_{k\bullet} = (T_z M)_{k\bullet} T_z = M_{k\bullet} T_z = (Q_z)_{k\bullet}$$

if  $z_k = 1$  and

$$Q_{k\bullet} = (T_z((I - T_\mu)M + T_\mu))_{k\bullet} T_z = ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T) T_z = (Q_z)_{k\bullet}$$

if  $z_k = -1$ , both by Theorem 2, hence  $Q = Q_z$ .

Similarly, for the matrix defined by

$$Q_a = \max\{T_z M, T_z(I - T_\mu)M\}$$

we have

$$T_z M - T_z(I - T_\mu)M = T_z T_\mu M$$

where  $T_\mu M \geq 0$ , hence

$$(Q_a)_{k\bullet} = M_{k\bullet} = |Q_z|_{k\bullet}$$

if  $z_k = 1$  and

$$(Q_a)_{k\bullet} = (\mu_k - 1)M_{k\bullet} = |Q_z|_{k\bullet}$$

if  $z_k = -1$ , both by Theorem 3, which means that  $Q_a = |Q_z|$ . □

Finally, we bring about the utmost simplification.

**Theorem 5.** *Let  $M \geq I$ . Then for each  $z \in \{-1, 1\}^n$  the matrix  $Q_z$  and its absolute value are given by*

$$Q_z = (D_z + T_z)MT_z - D_z T_z$$

and

$$|Q_z| = (D_z + T_z)M,$$

where

$$D_z = \frac{1}{2}(I - T_z)T_\mu$$

and

$$\mu = \frac{2\text{diag}(M)}{2\text{diag}(M) - e}.$$

*Proof.* For a  $z \in \{-1, 1\}^n$ ,  $D_z$  is a diagonal matrix such that  $(D_z)_{kk} = 0$  if  $z_k = 1$  and  $(D_z)_{kk} = \mu_k$  otherwise, therefore the matrix  $Q$  defined by

$$Q = (D_z + T_z)MT_z - D_z T_z$$

satisfies

$$Q_{k\bullet} = M_{k\bullet} T_z$$

if  $z_k = 1$  and

$$Q_{k\bullet} = (\mu_k - 1)M_{k\bullet} T_z - \mu_k e_k^T T_z$$

if  $z_k = -1$ , hence  $Q = Q_z$  by Theorem 2. The proof for  $|Q_z|$  follows the same line. □

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