**A Reduction Theorem for Absolute Value Equations**

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## Academy of Sciences of the Czech Republic


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Jiří Rohn
http://uivtx.cs.cas.cz/~rohn



Technical report No. V-1204

10.02.2014



Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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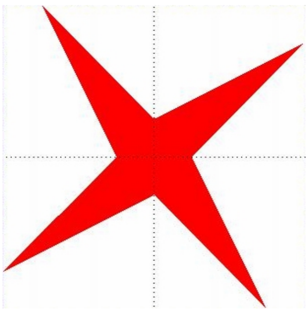
Abstract:

It is shown that under certain assumption an absolute value equation of size $n \times n$ can be reduced to an absolute value equation of size $p \times p$, $p \leq n$, such that both equations are simultaneously solvable or unsolvable and from a solution of the reduced equation a solution of the original equation can be computed by using a single matrix-vector multiplication.[2]



Keywords:
Absolute value equation, size reduction, solvability.

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[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

# 1 Introduction

In this report we show that under certain assumption an absolute value equation

$$Ax + B|x| = b \tag{1.1}$$

with $A, B$ of size $n \times n$ can be transformed into an absolute value equation

$$A'x' + B'|x'| = b'' \tag{1.2}$$

with $A', B'$ of size $p \times p$, where $p$ is the number of negative entries of the vector

$$(A + B)^{-1}b,$$

such that (1.1) is solvable if and only if (1.2) is solvable and from each solution $x'$ of (1.2) a solution $x$ of (1.1) can be computed using a single matrix-vector multiplication. This means that we can do with solving the smaller system (1.2). The method works under assumption of nonnegativity of the matrix

$$N = (A + B)^{-1}A - I.$$

In Section 2 we prove the above-stated assertions and in Section 3 we show that for 1000 randomly generated $100 \times 100$ absolute value equations the average value of the reduction ratio $p/n$ was close to 0.5. This shows that further investigation into this matter might be worth doing.

# 2 The result

For $N \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ and a set of indices $K = \{k_1, \ldots, k_p\}$ with $1 \le k_1 < k_2 < \ldots < k_p \le n$, denote

$$\begin{aligned} N_{KK} &= (N_{k_i k_j})_{i,j=1}^{p} \\ N_{\bullet K} &= (N_{i,k_j})_{i=1,\, j=1}^{n,\, p} \\ x_K &= (x_{k_1}, \ldots, x_{k_p})^T. \end{aligned}$$

Given an absolute value equation

$$Ax + B|x| = b \tag{2.1}$$

with $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, put

$$N = (I + A^{-1}B)^{-1} - I = (A + B)^{-1}A - I \tag{2.2}$$

(assuming implicitly that the inverses exist),

$$b' = (N + I)A^{-1}b = (A + B)^{-1}b, \tag{2.3}$$

$$K = \{\, i \mid b'_i < 0 \,\},$$

and assuming that $K \neq \emptyset$, construct a new absolute value equation

$$(I + N_{KK})x' - N_{KK}|x'| = b'_K, \tag{2.4}$$

where $N_{KK} \in \mathbb{R}^{p\times p}$, $p \le n$. Our basic results concerns the interconnection between solutions of (2.1) and (2.4).

**Theorem 1.** *Let $N$ be nonnegative and let $K \neq \emptyset$. Then we have:*

*(i) if $x$ is a solution of (2.1), then $x' = x_K$ is a solution of (2.4),*

*(ii) if $x'$ is a solution of (2.4), then*

$$x = N_{\bullet K}(|x'| - x') + b'$$

*is a solution of (2.1) satisfying $x_K = x'$.*

*Proof.* (i) Let $x$ solve

$$Ax + B|x| = b.$$

Through a series of rearrangements

$$x + A^{-1}B|x| = A^{-1}b,$$

$$(I + A^{-1}B)x = -A^{-1}B(|x| - x) + A^{-1}b,$$

$$x = -(I + A^{-1}B)^{-1}A^{-1}B(|x| - x) + (I + A^{-1}B)^{-1}A^{-1}b,$$

$$x = -(I + A^{-1}B)^{-1}(A^{-1}B + I - I)(|x| - x) + (I + A^{-1}B)^{-1}A^{-1}b,$$

$$x = ((I + A^{-1}B)^{-1} - I)(|x| - x) + (I + A^{-1}B)^{-1}A^{-1}b$$

we arrive at an equivalent form

$$x = N(|x| - x) + b'. \tag{2.5}$$

Now, if $i \notin K$, then $b'_i \geq 0$ and nonnegativity of both $N$ and $|x| - x$ in (2.5) implies $x_i \geq 0$, so that $|x_i| - x_i = 0$ and (2.5) can be reduced to the form

$$x = N_{\bullet K}(|x_K| - x_K) + b'.$$

Considering only equations for $i \in K$, we get

$$x_K = N_{KK}(|x_K| - x_K) + b'_K$$

and thus also

$$(I + N_{KK})x_K - N_{KK}|x_K| = b'_K,$$

hence $x' = x_K$ solves (2.4).

(ii) Conversely, let $x'$ solve (2.4). Then

$$x' = N_{KK}(|x'| - x') + b'_K. \tag{2.6}$$

Define $x \in \mathbb{R}^n$ by

$$x = N_{\bullet K}(|x'| - x') + b'. \tag{2.7}$$

Then

$$x_K = N_{KK}(|x'| - x') + b'_K = x' \tag{2.8}$$

by (2.6), and if $i \notin K$, then $b'_i \geq 0$ and from (2.7) we have $x_i \geq 0$, hence $|x_i| - x_i = 0$, which enables us to rearrange (2.7) to the form

$$x = N(|x| - x) + b'. \tag{2.9}$$

Now, using the definitions of $N$ and $b'$ in (2.2) and (2.3), we can transform (2.9) to the form

$$Ax + B|x| = b,$$

hence $x$ solves (2.1) and additionally satisfies $x_K = x'$ by (2.8). $\square$

Notice that the reduced equation (2.4), if written in the form (1.2), satisfies $A' \geq I$, $B' \leq 0$, $A' + B' = I$, and $b'' < 0$.

We have assumed that $K \neq \emptyset$. But if $K = \emptyset$, them $b' \geq 0$ and from (2.5) we conclude that $x \geq 0$ and thus $x = b'$, so that we immediately obtain a solution of (2.1) without necessity of solving (2.4).

Now the idea of reiterating the whole process anew with the reduced system (2.4) certainly comes to reader's mind. Unfortunately, this is no more possible. The reduced right-hand side for (2.4) according to (2.3) is

$$(b'_K)' = ((I + N_{KK}) - N_{KK})^{-1} b'_K = b'_K < 0,$$

hence $K' = K$ and no more reduction can be achieved.

We have this immediate consequence.

**Theorem 2.** *Under assumptions and notation of Theorem 1, (2.1) is solvable if and only if (2.4) is solvable.*

Denote by

$$X(A, B, b) = \{\, x \mid Ax + B|x| = b \,\}$$

the solution set of $Ax + B|x| = b$. We have this interconnection between the solution sets.

**Theorem 3.** *Under assumptions and notation of Theorem 1 there holds*

$$X(I + N_{KK}, -N_{KK}, b'_K) = \{\, x_K \mid x \in X(A, B, b) \,\}.$$

In other words, the solution set of (2.4) consists of the $K$-parts of solutions of (2.1). This again follows immediately from Theorem 1.

Our main result works under assumption of nonnegativity of the matrix $N$. In the last theorem we delineate a class of matrices for which this property holds true.

**Theorem 4.** *If $A^{-1}B \leq 0$ and $\varrho(A^{-1}B) < 1$, then $N \geq 0$.*

*Proof.* Indeed, in this case $N = (I - |A^{-1}B|)^{-1} - I = \sum_{j=1}^{\infty} |A^{-1}B|^j \geq 0$. $\square$

## 3 Examples

We have incorporated the described reduction method into the MATLAB file **absvaleqnred.m**. If $N \geq 0$, then reduction is performed and the resulting absolute value equation is solved by the **absvaleqn.m** file (for its description see [2], [3]), otherwise the unreduced absolute value equation is solved by the same file.

```
function [Ap,Bp,bp,K,xp,x]=absvaleqnred(A,B,b) % AVE via REDuction
Ap=[]; Bp=[]; bp=[]; K=[]; xp=[]; x=[];
n=length(b);
N=inv(A+B)*A-eye(n,n);
if ~all(all(N>=0))
    x=absvaleqn(A,B,b); return
end
bpp=inv(A+B)*b;
K=find(bpp<0);
if isempty(K)
    x=bpp; return
end
NKK=N(K,K);
Ap=eye(size(NKK))+NKK;
Bp=-NKK;
bp=bpp(K);
xp=absvaleqn(Ap,Bp,bp);
if isempty(xp), return, end
x=N(1:n,K)*(abs(xp)-xp)+bpp;
```

For example, solving the problem with the data

```
A =
   -0.1281    0.6661   -0.7544   -0.2097    0.9153    0.7977    0.0295
    0.6425   -0.0353    0.0753    0.8638    0.3687    0.6106    0.2361
    0.3294   -0.9756   -0.0242    0.0571    0.3553   -0.2312   -0.3262
    0.2911    0.7952   -0.0436    0.4250   -0.3719    0.7087    0.1880
    0.6880   -0.4864    0.0025   -0.5859    0.8378    0.9116    0.6197
    0.8989    0.1423   -0.5385   -0.5601   -0.7465   -0.4859   -0.7328
   -0.1766   -0.7195    0.2596    0.8765    0.8840    0.0993   -0.6118
B =
   -0.0260   -0.0142   -0.0042   -0.0188   -0.0158    0.0036   -0.0128
   -0.0232   -0.0362   -0.0348   -0.0287   -0.0500   -0.0237   -0.0348
    0.0189    0.0153    0.0123    0.0041    0.0035    0.0023    0.0130
   -0.0273   -0.0380   -0.0294   -0.0135   -0.0254   -0.0152   -0.0261
   -0.0199   -0.0141   -0.0100   -0.0304   -0.0373   -0.0223   -0.0237
    0.0117    0.0052    0.0102    0.0297    0.0402    0.0113    0.0441
    0.0086    0.0007    0.0034   -0.0111   -0.0213    0.0022   -0.0136
b =
    0.5581
    0.6435
   -0.2816
    0.5931
   -0.5970
    0.5860
   -0.2015
```

by `[Ap,Bp,bp,K,xp,x]=absvaleqnred(A,B,b)` leads to the output

```
Ap =
    1.0066    0.0113
    0.0050    1.0092
Bp =
   -0.0066   -0.0113
   -0.0050   -0.0092
bp =
   -0.8370
   -0.5263
K =
     3
     6
xp =
   -0.8149
   -0.5088
x =
    0.5680
    0.6779
   -0.8149
    0.7621
    0.1841
   -0.5088
    0.2860
```

where `Ap, Bp, bp` are the data of the reduced system and `xp` is its solution ('p' stands for 'prime'). Hence the size of the problem has been reduced from $7 \times 7$ to $2 \times 2$. The solution $x$ of the original system has then been computed by (2.7). Notice that $x_K = xp$, as predicted by the theory.

We call the number $r = p/n$=`length(K)/length(b)` the reduction ratio of the problem. To assess its average value, we wrote the file **redrataver.m** which solves $m$ absolute value equations of size $n \times n$ whose data are generated randomly on the basis of Theorem 4 by the subfunction **averandata.m**, computes for each problem its reduction ratio and at the end outputs the average value of all $m$ reduction ratios.

```
function r=redrataver(m,n) % REDuction RATio AVERage
r=0;
for i=1:m
    [A,B,b]=averandata(i,n);
    [Ap,Bp,bp,K,xp,x]=absvaleqnred(A,B,b);
    r=r+length(K)/n;
end
r=r/m;
function [A,B,b]=averandata(i,n) % AVE RANdom DATA
rand('state',i);
A=2*rand(n,n)-1;
C=-rand(n,n);
C=(rand/ro(C))*C;
```

```
B=A*C;
b=2*rand(n,1)-1;
function ro=ro(A) % spectral radius
ro=max(abs(eig(A)));
```

We have run the file for $m = 1000$, $n = 100$ (i.e., 1000 problems of size $100 \times 100$):

```
>> tic, r=redrataver(1000,100), toc
r =
    0.4989
Elapsed time is 106.348758 seconds.
```

As we can see, the average reduction ratio, at least for this set of test problems, was about 50%.

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