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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:nedoma@cs.cas.cz

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Abstract:

Propagation of elastic waves in fractured media, that is, in broken media with contact boundaries between bodies being in mutual contacts, and under self-gravity field will be discussed. Such problems are e.g. problems of global seismology. For numerical solution the spectral-element method can be used with a great merit.

The paper extends the global seismology results to problems modelling elastic wave propagations in the global Earth with a fractured lithosphere and under a self-gravitation. Numerical solution of the model problem, based on its variational formulation and the spectral-element method, is presented.

Keywords:
Propagation of elastic waves, global model of the Earth, fractured lithosphere, elastic and fluid rheologies, self-gravitating, contact problems, variational and numerical solutions, algorithm.

1 Introduction

Mathematical modelling of elastic wave propagations play an important role in seismological studies. Next, we will introduce a mathematical model of elastic wave propagations through the partly fractured media. In our study the mathematical model of the Earth with broken lithosphere under self-gravitation will be formulated and discussed. The main aim of the paper is to extend results of the global seismology to problems modelling propagation of elastic waves in the Earth with a fractured lithosphere and under a self-gravity field and to outline the numerical method and an idea of algorithm for a computational realization. The model facilitates to understand movements of the lithospheric plates and blocks evoked by the bigger earthquake in the sense of the plate tectonic theory as well as to simulate ensuing aftershocks.

In the global Earth's model the surface of the Earth is a free surface, therefore, in global seismological models prescribed loads (nonzero or zero) are only given. The great merit of mathematical modelling of seismic wave propagation inside the Earth is that there are no absorbing boundaries, because the Earth's surface is a free boundary. On the other hand the effects of self gravitation and rotation on seismic wave propagations, namely for long-period surface waves, need to be taken into the consideration. Hence, the Earth is modelled by a rotating, self-gravitating layered model, in which the elastic wave equations are written for the lithosphere and the mantle, for the outer core and for the inner core. The spectral-element method can be used with a great merit in global seismology problems. Since the aim of the paper is outside the detailed study of the wave propagation inside the Earth from the point of view of the global seismology, we will introduce the main ideas of the global Earth's model with partly fractured lithosphere and then discuss the model and method for its numerical solution. The readers, who are interested in these problems, will find the detailed studies namely in the books regarding problems of the global seismology, inclusive methods of their solutions, e.g. in Dahlen and Tromp (1998), Chaljub (2000), Liu (2006) and in special papers as e.g. Anderson (1987), Rouchi et al. (1996), Seriani (1998), Komatitsch and Tromp (1999), (2002a,b), Komatitsch et al. (2000a,b), (2002), (2003), (2005), Capdeville et al. (2003), Chaljub et al. (2003), Chaljub and Valette (2004). For algorithms see e.g. Canuto et al. (1988), Quarteroni et al. (1988), Dahlen and Tromp (1998), Komatitsch and Tromp (1999). In all problems discussed in these papers and books the contact boundaries between two different parts of the Earth, like the different lithospheric blocks being divided by the deep faults as well as the lithospheric plates and the mantle that being in mutual contacts and along which they can mutually shift, are not assumed. In this paper we will assume the existence of contact boundaries between the neighboring lithospheric plates and the neighboring lithospheric blocks in the broken up lithosphere as well as the contact boundary between the lithosphere and the mantle and the existence of the friction acting at these contact boundaries, because from the plate tectonic point of view the lithospheric plates and blocks mutually collide, and therefore, they can move one to another. These movements can be also evoked by earthquakes. In our study we will assume the existence of the non-penetrability conditions (known as the Signorini type conditions) and the Coulomb friction acting on these contact boundaries. These non-penetrability conditions describe the facts (i) that the relative normal displacement on the common contact part of the contact boundary cannot be larger than the distance d between the bodies (lithospheric plates and blocks) being in mutual contact; (ii) that by contact only compressive normal forces can be transmitted; (iii) that normal forces can be transmitted only if there is contact, that is, if $d = 0$. The existence of the friction at the contact boundaries, which is described by the Coulomb law of friction, is possible only if colliding bodies are in mutual contact, i.e., if $d = 0$. The frictional forces acting on the contact boundaries are, in their absolute value, proportional to the normal stress component, where the coefficient of proportionality is the coefficient of Coulomb friction. Then, due to the tangential and frictional forces acting at the contact boundary, we have two cases: (i) if the absolute value of tangential forces is less than the frictional forces, then the frictional forces preclude the mutual shifts of both bodies being in contact and thus in the collision area stresses are further accumulated; (ii) if the tangential forces are equal in their absolute value to the frictional forces, then there are no forces that can preclude the mutual movements of both elastic bodies. Thus the contact points change their position in the direction opposite to that in which the tangential stress component acts. This is one mechanism of the earthquake origin as well as the mechanism of the possible aftershocks.

2 The model

The Earth is the spherical-like body occupying the domain $\Omega \in \mathbb{R}^3$, where $\overline{\Omega} = \overline{\Omega}^L \cup \overline{\Omega}^M \cup \overline{\Omega}^{OC} \cup \overline{\Omega}^{IC}$, and where Ω^L represents the domain occupied by the lithosphere, Ω^M is the domain occupied by the mantle, Ω^{OC} and Ω^{IC} are the domains occupied by the outer core and the inner core, respectively. The surface of the Earth is denoted by $\partial\Omega$ and represents the free boundary. The interface boundary between the lithosphere and the mantle is denoted by $\Gamma_c^{LM} = \partial\Omega^L \cap \partial\Omega^M$, and the interface boundaries between the mantle and the outer core is denoted by $\Gamma_c^{MOC} = \partial\Omega^M \cap \partial\Omega^{OC}$ and between the outer core and the inner core by $\Gamma_c^{OIC} = \partial\Omega^{OC} \cap \partial\Omega^{IC}$, respectively. The lithosphere Ω^L is assumed to be occupied by a system of bodies of arbitrary shapes $\Omega^{L\iota}$, $\overline{\Omega}^L = \cup_{i=1}^r \overline{\Omega}^{L\iota}$. The contact boundaries Γ_c^{sm} between the neighboring lithospheric blocks $\Omega^{L\iota}$, $\iota = s, m \in [1, r]$, being in common contacts, are defined as $\Gamma_c^{sm} = \partial\Omega^{Ls} \cap \partial\Omega^{Lm}$, $s \neq m$, $s, m \in [1, r]$. Let $I = (0, t_p)$, $t_p > 0$, be a time interval.

Let $\mathbf{n}$ denote the outer normal vector to the boundary, $\mathbf{v}$ be the displacement vector, $\overline{\boldsymbol{\tau}}$ be the stress tensor, $v_n = v_i n_i$, $\mathbf{v}_t = \mathbf{v} - v_n \mathbf{n}$, $\tau_n = \overline{\tau}_{ij} n_j n_i$, $\boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n \mathbf{n}$ be the normal and tangential components of displacement vector $\mathbf{v} = (v_i)$ and stress vector $\boldsymbol{\tau} = (\tau_i)$, $\tau_i = \overline{\tau}_{ij} n_j$, $i, j = 1, 2, 3$. Let us denote by $\mathbf{v}' = \frac{d\mathbf{v}}{dt}$ the velocity vector. Let us denote by $[w]^{sm} = w^s - w^m$ the jump of function w across the contact boundary between neighboring bodies Ω^s and Ω^m.

To formulate the contact and friction conditions, let us introduce at each point of Γ_c^s the vectors $\mathbf{t}_i^s, i = N-1$, $N = 3$, spanning in the corresponding tangential plane. Let $\{\mathbf{n}^s, \mathbf{t}_i^s\}, i = 1, 2$, be an orthogonal basis in $\mathbb{R}^N$ for each point of Γ_c^s. To formulate the non-penetration condition we use a predefined relation between the points of the possible contact zones Γ_c. Therefore, we introduce a smooth mapping $\mathcal{R} : \Gamma_c^s \to \Gamma_c^m$ such that $\mathcal{R}(\Gamma_c^s) \subset \Gamma_c^m$, and we will assume that the mapping $\mathcal{R}$ is well defined and maps any $\mathbf{x} \in \Gamma_c^s$ to the intersection of the normal on Γ_c^s at $\mathbf{x}$ with Γ_c^m. Then $[\mathbf{v}]^{sm} := \mathbf{v}^s(\mathbf{x}) - \mathbf{v}^m(\mathcal{R}(\mathbf{x}))$, $[v_n]^{sm} := [\mathbf{v}]^{sm}.\mathbf{n}^s$ is the jump in normal direction, $[\mathbf{v}_t]^{sm} = (\mathbf{v}^s(\mathbf{x}) - \mathbf{v}^m(\mathcal{R}(\mathbf{x}))) - [\mathbf{v}]^{sm}.\mathbf{n}^s$ and $\tau_n^s = (\mathbf{n}^s)^T \boldsymbol{\tau}^s(\mathbf{x})\mathbf{n}^s = (\mathbf{n}^s)^T \boldsymbol{\tau}^m(\mathcal{R}(\mathbf{x}))\mathbf{n}^s$ is the boundary stress in normal direction on the possible contact part, and moreover, $(\mathbf{t}_i^s)^T \boldsymbol{\tau}^s(\mathbf{x})\mathbf{t}_i^s = (\mathbf{t}_i^s)^T \boldsymbol{\tau}^m(\mathcal{R}(\mathbf{x}))\mathbf{t}_i^s$, $i = N-1$, must be ensured.

To introduce the wave equation for the global model of the Earth the law of conservation of momentum will be used. The differential form of the law of conservation of momentum is of the following form

$$\rho \frac{d^2\mathbf{v}}{dt^2} = \nabla.\overline{\boldsymbol{\tau}} + \mathbf{f} \qquad \text{in } I \times \Omega, \tag{2.1}$$

where ρ is the density, $\overline{\boldsymbol{\tau}}$ is the stress tensor, $\mathbf{f}$ is the body forces, where $\mathbf{f}$ may be written in terms of the moment tensor $\overline{\mathbf{M}}$ or of the moment-density tensor $\overline{\mathbf{m}}$ (Dahlen and Tromp (1998), Komatitsch et al. (2005)) and where $\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}.\nabla) f$. Since the Earth rotates and the geological structure of the Earth depends with depth, the wave equations will be different in the lithosphere, the upper and lower mantle and in the outer and inner core, and therefore, the effects of self gravitation and rotations on seismic wave propagation inside the Earth′s body can also be taken into the consideration (Cathles III (1975), Dahlen and Tromp (1998)). Then the wave equation in the lithosphere and the upper and lower mantle can be written as

$$\rho\left(\frac{d^2\mathbf{v}}{dt^2} + 2\hat{\boldsymbol{\omega}} \times \frac{\partial \mathbf{v}}{\partial t}\right) = \nabla.\overline{\boldsymbol{\tau}} + \nabla(\rho\mathbf{v}.\mathbf{g}) - \rho\nabla\Phi - \nabla.(\rho\mathbf{v})\,\mathbf{g} + \mathbf{f}, \tag{2.2}$$

where ρ is the density, $\hat{\omega}$ is the Earth′s angular rotation vector, $\mathbf{g}$ is the gradient of the geopotential, i.e., $\nabla\Phi = -\mathbf{g}$, $\overline{\boldsymbol{\tau}}$ is the stress tensor, which can be defined by the generalized Hooke′s law in elasticity or visco-elasticity with short or long memories or by the generalized Hooke′s law in a special nonlinear (thermo-visco-) elasticity, where the deformation energy is the non-linear function of a strain, that is, $W = (c_{ijkl}(\mathbf{x}, \mathbf{v})\, e_{ij}(\mathbf{v})\, e_{kl}(\mathbf{v}))^\lambda$, $\overline{\tau}_{ij} = \partial W / \partial e_{ij}$, and where λ is some parameter, that covered up elastic properties of rocks (if $\lambda = 1$), their hardening (if $\lambda > 0$) and softening (if $\lambda < 0$) properties and/or their partly melted areas characterized by soften rocks. This last rheology can be useful because it can describe the rheology in all places of the Earth. The visco-plastic Bingham rheology can also be used with a great merit (Nedoma (1998), (2006), (2010), (2012)). In this paper we limit ourselves to the elastic rheology for solid parts of the Earth only and to the fluid rheology for the outercore. The earthquake source is simulated by the body force $\mathbf{f}$ given in terms of the moment-density tensor $\overline{\mathbf{m}}$ for a source of finite size, such as a fault plane $\sum_{source}$, that is, $\mathbf{f} = -\overline{\mathbf{m}}\left(\mathbf{x}_s, t\right).\nabla\delta\left(\mathbf{x} - \mathbf{x}_s\right)$ on $\sum_{source}$, or in terms of the moment tensor $\overline{\mathbf{M}}$ for a point source, that is, $\mathbf{f} = -\overline{\mathbf{M}}.\nabla\delta\left(\mathbf{x} - \mathbf{x}_s\right).S\left(t\right)$, where $\mathbf{x}_s$ denotes the locations of the source, $S\left(t\right)$ is the source-time function and $\delta\left(\mathbf{x} - \mathbf{x}_s\right)$ is the Dirac distribution located at $\mathbf{x}_s$. The perturbed gravitational potential Φ is given by the Poisson equation within the Earth and the Laplace equation in the space outside the Earth, respectively, that is,

$$\begin{aligned} \nabla^2\Phi &= -4\pi G\nabla\cdot(\rho\mathbf{v}) \quad \text{within the Earth,} \\ \nabla^2\Phi &= 0 \qquad\qquad\qquad \text{outside the Earth,} \end{aligned} \tag{2.3}$$

where G is the gravitational constant.

The Laplace equation is defined in all of the space, then solving (2.2) in conjunction with the Poisson′s and Laplace's equations bring some numerical problems. Therefore, the so-called Cowling′s approach (see Cowling (1941), Valette (1987), Dahlen and Tromp (1998), Chaljub (2000), Chaljub et al. (2003)) is used, where perturbations Φ in the gravitational potential are ignored while the unperturbed gravitational potential is reduced and the convective term can also be ignored. Then (2.2) leads to

$$\rho\left(\frac{\partial^2\mathbf{v}}{\partial t^2} + 2\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}}{\partial t}\right) = \nabla.\overline{\boldsymbol{\tau}} + \nabla\left(\rho\mathbf{v}.\mathbf{g}\right) - \nabla.\left(\rho\mathbf{v}\right)\mathbf{g} + \mathbf{f}. \tag{2.4}$$

The outer core does not transmit S-waves (Jeanloz (1990), Jacobs (1992)), which is interpreted in such a way that the outer core is strongly plastic close to liquid state, while the inner core is solid. In the outer core the equation of motion is of the form

$$\rho\left(\frac{\partial^2\mathbf{v}}{\partial t^2} + 2\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}}{\partial t}\right) = \nabla\left(k\nabla.\mathbf{v}+\rho\mathbf{v}.\mathbf{g}\right) - \rho\nabla\Phi - \nabla.\left(\rho\mathbf{v}\right)\mathbf{g}, \tag{2.5}$$

where k is the bulk modulus of the fluid. Since before the earthquake event the environment of the outer core is assumed to be in a hydrostatic equilibrium, the equation of motion (2.5) can be of the following form (Komatitsch et al. (2005))

$$\frac{\partial^2\mathbf{v}}{\partial t^2} + 2\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}}{\partial t} = \nabla\left(\rho^{-1}k\nabla.\mathbf{v} + \mathbf{v}.\mathbf{g} - \Phi\right) + \rho^{-1}g^{-2}k\left(\nabla.\mathbf{v}\right)\tilde{N}^2\mathbf{g}, \tag{2.6}$$

where $\tilde{N}^2 = \left(\rho^{-1}\nabla\rho - \rho k^{-1}\mathbf{g}\right).\mathbf{g}$, $g = |\mathbf{g}|$, $\tilde{N}$ is the Brunt-Väisälä frequency (Valette (1986), Dahlen and Tromp (1998), Chaljub and Vallete (2004), Komatitsch et al. (2005)); if $\tilde{N}^2 = 0$ then the outer core is assumed to be stably stratified and isentropic (i.e., of the same entropy), studied e.g. by Komatitsch and Tromp (2002b), who also assumed that $\Phi \equiv 0$, that is, the case without perturbation in gravity, based upon the Cowling approximation. The case if $\tilde{N}^2 \neq 0$ represents the case of a fluid in self-gravity field (Chaljub and Vallete (2004)), where the Cowling approximation is not required. Let us decompose the displacement field $\mathbf{v}$ into the scalar potential φ and a vector field $\boldsymbol{\psi}$ as

$$\mathbf{v} = \nabla\varphi + \boldsymbol{\psi}, \tag{2.7}$$

where φ and $\boldsymbol{\psi}$ remain to be determined, and then substitute them into (2.6). Thus we obtain

$$\begin{aligned} &\nabla\left(\frac{\partial^2\varphi}{\partial t^2}\right) + 2\hat{\boldsymbol{\omega}}\times\nabla\left(\frac{\partial\varphi}{\partial t}\right) + \frac{\partial^2\boldsymbol{\psi}}{\partial t^2} + 2\hat{\boldsymbol{\omega}}\times\frac{\partial\boldsymbol{\psi}}{\partial t} = \\ &= \nabla\left(\rho^{-1}k\nabla^2\varphi + \mathbf{g}.\nabla\varphi + \rho^{-1}k\nabla\cdot\boldsymbol{\psi} + \boldsymbol{g}\cdot\boldsymbol{\psi} - \Phi\right) + \\ &\qquad + \rho^{-1}g^{-2}k\left(\nabla^2\varphi + \nabla.\boldsymbol{\psi}\right)\tilde{N}^2\boldsymbol{g}. \end{aligned} \tag{2.8}$$

Eq. (2.8) can be decomposed into the system corresponding to the scalar potential φ and the vector $\boldsymbol{\psi}$, which are the unknown functions (Komatitsch and Tromp (2002a,b)). Thus φ and $\boldsymbol{\psi}$ satisfy

$$\frac{\partial^2\varphi}{\partial t^2} = \rho^{-1}k\nabla.\left(\nabla\varphi + \boldsymbol{\psi}\right) + \mathbf{g}.\left(\nabla\varphi + \boldsymbol{\psi}\right) - \Phi, \tag{2.9}$$

and

$$\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} + 2\left(\hat{\boldsymbol{\omega}} \times \frac{\partial \boldsymbol{\psi}}{\partial t}\right) =$$
$$= -2\hat{\boldsymbol{\omega}} \times \nabla\left(\frac{\partial \varphi}{\partial t}\right) + g^{-2}\left(\frac{\partial^2 \varphi}{\partial t^2} - \mathbf{g}.\left(\nabla\varphi + \boldsymbol{\psi}\right) + \Phi\right)\tilde{N}^2\mathbf{g}. \tag{2.10}$$

Since the seismic data indicate that the inner core is solid, the equation of motion is the same as in the lithosphere and the mantle, that is,

$$\rho\left(\frac{\partial^2 \mathbf{v}}{\partial t^2} + 2\hat{\boldsymbol{\omega}} \times \frac{\partial \mathbf{v}}{\partial t}\right) = \nabla.\overline{\boldsymbol{\tau}} + \nabla\left(\rho\mathbf{v}.\mathbf{g}\right) - \nabla.\left(\rho\mathbf{v}\right)\mathbf{g}, \tag{2.11}$$

and because any earthquakes are not generated in the inner core, thus $\mathbf{f} = \mathbf{0}$.

Now we need to determine the boundary, interface and initial conditions. Since the surface of the Earth $\partial\Omega$ is the free boundary, therefore, the traction forces $\boldsymbol{\tau}.\boldsymbol{n}$ vanish, that is,

$$\boldsymbol{\tau}.\mathbf{n} = 0 \qquad \text{on } I \times \partial\Omega, \tag{2.12}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

Remark 1 *In a generalized case the resulting movements of water in the oceans and seas are also assumed.*

At the interface boundary Γ_c^{LM} between the lithosphere and the mantle we will assume the non-penetration contact conditions together with (or without) the Coulomb law of friction, that is, on $\Gamma_c^{LM}(t) = I \times \Gamma_c^{LM}$,

$$\begin{aligned}
&\text{(i) the non} - \text{penetration contact conditions}\\
&[v_n]^{LM} \le d_n^{LM},\ \tau_n^L = \tau_n^M \equiv \tau_n^{LM} \le 0,\ \left([v_n]^{LM} - d_n^{LM}\right)\tau_n^{LM} = 0,\\
&\text{(ii) the Coulomb law of friction}\\
&[\mathbf{v}_t']^{LM} = \mathbf{0} \Longrightarrow \left|\boldsymbol{\tau}_t^{LM}\right| \le \mathcal{F}_c^{LM}(0)\left|\tau_n^{LM}\right|,\\
&[\mathbf{v}_t']^{LM} \ne \mathbf{0} \Longrightarrow \boldsymbol{\tau}_t^{LM} = -\mathcal{F}_c^{LM}\left([\mathbf{v}_t']^{LM}\right)\left|\tau_n^{LM}\right|\frac{[\mathbf{v}_t']^{LM}}{\left|[\mathbf{v}_t']^{LM}\right|},
\end{aligned} \tag{2.13}$$

are prescribed, as well as on the contact boundaries between lithospheric plates and blocks, divided by the deep faults and being in mutual contacts, the non-penetration contact conditions together with (or without) the Coulomb law of friction, that is, on $\Gamma_c(t) = I \times \cup_{s,m}\Gamma_c^{sm}$,

$$\begin{aligned}
&\text{(i) the non} - \text{penetration contact conditions}\\
&[v_n]^{sm} \le d_n^{sm},\ \tau_n^s = \tau_n^m \equiv \tau_n^{sm} \le 0,\ \left([v_n]^{sm} - d_n^{sm}\right)\tau_n^{sm} = 0,\\
&\text{(ii) the Coulomb law of friction}\\
&[\mathbf{v}_t']^{sm} = \mathbf{0} \Longrightarrow \left|\boldsymbol{\tau}_t^{sm}\right| \le \mathcal{F}_c^{sm}(0)\left|\tau_n^{sm}\right|,\\
&[\mathbf{v}_t']^{sm} \ne \mathbf{0} \Longrightarrow \boldsymbol{\tau}_t^{sm} = -\mathcal{F}_c^{sm}\left([\mathbf{v}_t']^{sm}\right)\left|\tau_n^{sm}\right|\frac{[\mathbf{v}_t']^{sm}}{\left|[\mathbf{v}_t']^{sm}\right|},
\end{aligned} \tag{2.14}$$

are prescribed, where d_n^{LM} and d_n^{sm} are gaps between the lithosphere and the mantle and/or between the contact boundaries between lithospheric plates (subduction and collision zones) and/or between blocks Ω^{Ls} and Ω^{Lm} if exist, $\mathcal{F}_c^{LM}$ and $\mathcal{F}_c^{sm}$ are coefficients of friction that in general depend on the tangential components of the displacement-velocity vector $\mathbf{v}' = \frac{d\mathbf{v}}{dt}$ as well as on the material properties (Nečas et al. (1980), Haslinger et al. (1996), Nedoma (1998), Eck et al. (2005)). The lithospheric plates and blocks start to mutually move if the tangential force is much more bigger than the normal force $|\tau_n|$. The proportionality coefficient $\mathcal{F}_c^{(1)}$ is called coefficient of friction. The friction creates a tangential force $|\boldsymbol{\tau}_t|$ opposite to the moving velocity with magnitude again proportional to the normal force, $|\boldsymbol{\tau}_t| = \mathcal{F}_c^{(2)} |\tau_n|$. The proportionality coefficients $\mathcal{F}_c^{(1)}$ and $\mathcal{F}_c^{(2)}$ in general are different. For simplicity we will assume that $\mathcal{F}_c^{(1)} = \mathcal{F}_c^{(2)} \equiv \mathcal{F}_c$, and moreover, we will assume that the coefficient of friction $\mathcal{F}_c$ depends on the sliding displacement $\mathbf{v}_t$, while in the general case it depends on the sliding displacement-velocity $\mathbf{v}'_t$. For the existence result the admissible coefficient of friction is given in dependence of the constants in the special trace estimates (Nečas et al. (1980), Haslinger et al. (1996), Eck et al. (2005) and the references presented here). But it is particularly important for applications to know their precise values. For more details and for special estimates of the friction coefficients see Nečas et al. (1980), Haslinger et al. (1996), Nedoma (1987), (1998), Eck et al. (2005), therefore, for special cases

$$\|\mathcal{F}_c^{LM}\|_{L^\infty(\Gamma_c^{LM})} \left(\text{ or } \|\mathcal{F}_c^{sm}\|_{L^\infty(\Gamma_c^{sm})} \right) < \left(\frac{3}{4}\frac{c_0}{c_1}\right)^{1/2} < \left(\frac{3-4\nu}{4-4\nu}\right)^{1/2},$$

$$\|\mathcal{F}_c^{LM}\|_{L^\infty(\Gamma_c^{LM})} \left(\text{ or } \|\mathcal{F}_c^{sm}\|_{L^\infty(\Gamma_c^{sm})} \right) < \left(\frac{c_s}{c_p}\right)^{1/2},$$

where $c_0, c_1 > 0$ are some constants (i.e., coefficients of ellipticity c_0 and upper bounds c_1 e.g. of the Hooke tensor), ν is the Poisson's ratio and c_s and c_p are velocities of S and P waves (Nedoma (1987), (1998)). For the case without Coulomb friction the Coulombian friction coefficient $\mathcal{F}_c^{sm}(.) \equiv 0$, that can be used because the contact zones between geological blocks and lithospheric plates being in mutual contacts are partially melted. The above estimates can be presumed also in our case, because we assume the elastic rheology.

Remark 2 *According to the results presented in the above mentioned references, the dynamic multibody contact problems are in general open problems, some results are known for the static and quasistatic contact problems. It was shown that the admissible coefficient of friction for the existence results are given in dependence of the constants in the special trace estimates (e.g. based on the shift technique, see Eck et al. (2005), Section 1.7 and Section 3.3). The existence of these constants is known, but it is important for applications to derive their precise values depending on material properties of the rocks. Unfortunately, these estimates were derived for special problems only. It is possible to derive optimal lower bounds for these constants only. But it must be pointed out that at present the existence results were proved for special problems only. Therefore, the above mentioned estimates are introduced for some orientation only.*

The non-penetration conditions can be approximated as

$$\tau_n = -\frac{1}{\varepsilon}\left([v_n]^{LM} - d_n^{LM}\right)_+, \quad \varepsilon > 0, \quad \varepsilon \to 0, \qquad \text{on } \Gamma_c^{LM}$$

$$\tau_n = -\frac{1}{\varepsilon}\left([v_n]^{sm} - d_n^{sm}\right)_+, \varepsilon > 0, \quad \varepsilon \to 0, \qquad \text{on } \Gamma_c^{sm} \tag{2.15}$$

which means that the support of the lithosphere is assumed to be not as perfectly rigid and smooth, but infinitely resistent to compression, perfectly yielding in tension and shear and adhesively "stuck" on the lithosphere. Thus it leads to the idea of regarding it as a limit of supports perfectly yielding in tension and shear whose resistance to compression subsequently increases. The symbol $(z)_+ = \max\{0, z\}$ denotes the non-negative part of z. The non-penetration contact conditions between the neighboring lithospheric plates and the neighboring lithospheric blocks being in mutual contacts are approximated by the similar way as above.

Let us assume that the Coulombian law of friction in every time level is approximated by its value g_c^{LM} from the previous time level, i.e., $g_c^{LM} \equiv \mathcal{F}_c^{LM}|\tau_n^{LM}(\mathbf{v}, \mathbf{v}')|)\ (t - \Delta t)$, in the case of the contact between the lithosphere and the mantle, and in the case of fractured lithosphere it is approximated by its value g_c^{sm} from the previous time level, i.e., $g_c^{sm} \equiv \mathcal{F}_c^{sm}|\tau_n^{sm}(\mathbf{v}, \mathbf{v}')|)\ (t - \Delta t)$. Thus g_c^{LM} is a non-negative functions and has a meaning of a given friction limit (or a given friction bound, representing the magnitude of the limiting friction traction at which slip originates), and where $-g_c^{LM}$

has a meaning of a given frictional force, and Δt is a time element. Similarly, g_c^{sm} are non-negative function and have a meaning of given friction limits (or given friction bounds, representing the magnitudes of the limiting friction tractions at which slips at contacts between lithospheric blocks originate), and where $-g_c^{sm}$ have a meaning of given frictional forces in the case of fractured lithosphere. Thus the problem investigated will be approximated by another problem in which in every time level we will solve the dynamic contact problem with the given friction. In this case we speak about the Tresca model of friction. Therefore, it is evident that for a variational formulation and for numerical computations the so-called Tresca model of friction can be used with a great merit.

At the interface boundary between the mantle and the outer core Γ_c^{MOC} the normal component of displacement v_n is continuous and the normal stress component (traction) τ_n at the bottom of the mantle is equal to the traction p_n at the top of the outercore, where the outercore is assumed to be the fluid, p denotes the perturbed pressure in the fluid. Thus

$$\mathbf{v}.\mathbf{n}_{|(\partial\Omega^M)} = \mathbf{v}.\mathbf{n}_{|(\partial\Omega^{OC})}, \quad \boldsymbol{\tau}.\mathbf{n}_{|(\partial\Omega^M)} = -p\mathbf{n}_{|(\partial\Omega^{OC})} \qquad \text{on } \Gamma_c^{MOC}. \tag{2.16}$$

At the boundary between the mantle and the outer core Γ_c^{MOC} we need to exchange pressure p between the solid mantle and the fluid outer core. Since $\mathbf{v} = \nabla\varphi + \boldsymbol{\psi}$ and using Eq. (2.9) we find

$$p = -k\nabla.\mathbf{v} = -\rho\left[\frac{\partial^2\varphi}{\partial t^2} - \mathbf{g}.\left(\nabla\varphi + \boldsymbol{\psi}\right) + \Phi\right]. \tag{2.17}$$

At the interface Γ_c^{OIC} between the outer core and the inner core the interface conditions are similar to that of (2.16), that is,

$$\mathbf{v}.\mathbf{n}_{|\partial\Omega^{OC}} = \mathbf{v}.\mathbf{n}_{|\partial\Omega^{IC}}, \; p.\mathbf{n}_{|\partial\Omega^{OC}} = \boldsymbol{\tau}.\mathbf{n}_{|\partial\Omega^{IC}} \qquad \text{on } \Gamma_c^{OIC}. \tag{2.18}$$

Moreover, under the assumption that the medium is initialy at rest, the initial conditions are

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}'(\mathbf{x},0) = \mathbf{0}, \;\; \mathbf{x} \in \Omega^s, \; \varphi(\mathbf{x},0) = \varphi'(\mathbf{x},0) = 0, \; \boldsymbol{\psi}(\mathbf{x},0) = \boldsymbol{\psi}'(\mathbf{x},0) = \mathbf{0}, \;\; \mathbf{x} \in \Omega^{OC}, \tag{2.19}$$

are given (Chaljub et al. (2003)).

To complete the set of boundary and initial conditions, the conditions for the Eulerian perturbation of the gravitational potential (PGP, known also as the mass redistribution potential (MRP)) Φ at the interface boundaries are as follows: the PGP must be continuous across these boundaries, that is, $[\Phi] = 0$, and the normal derivative of Φ have a jump $[\nabla\Phi.\mathbf{n}] = -4\pi G\left[\rho\mathbf{v}.\mathbf{n}\right]$, where $[.] \equiv [.]_-^+$ denotes the jump operator across the interface boundary, defined in accordance with the unit normal vector $\mathbf{n}$, pointing from the "-" side to the "+" side, say between the mantle and the outer core as well as between the outer core and the inner core, that is,

$$[\Phi] = 0, \; [\nabla\Phi.\mathbf{n}] = -4\pi G\left[\rho\mathbf{v}.\mathbf{n}\right] \quad \text{on } \Gamma_c^{LM} \text{ and/or on } \Gamma_c^{MOC} \cup \Gamma_c^{OIC}. \tag{2.20}$$

Under the assumption that there is no perturbation of the gravitational potential in the initial configuration, thus $\Phi\left(\mathbf{x},0\right) = 0$.

Remark 3 *Since the Earth's crust in some of its parts is covered by a water in the oceans and seas, therefore, the effects of moving waters in oceans and seas may be also taken into consideration. On the other hand, the problem may be simplified of some effects as rotation or self-gravity field can be omitted (see e.g. Komatitsch and Tromp (2002b), Dahlen and Tromp (1998), Komatitsch et al (2005)).*

3 Variational formulation of the problem

In order to suggest the spectral-element method, we firstly introduce the variational formulation of the problem.

To formulate this generalized model problem variationally then due to (2.13) and (2.14) the problem leads to solve the hyperbolic variational inequality problem. But it must be noted that to prove the existence of the solution of this above mentioned generalized problem as well as its simplified versions in their continuous formulations are at the present open problems.

It is known that the Eulerian perturbation of the gravitational potential (PGP) Φ satisfies the Laplace equation outside the Earth's surface $\partial\Omega$, and that it tends to zero for distances $\mathbf{r}$ tend to infinity (Dahlen and Tromp (1998)). Let r, Θ, ϕ be the spherical coordinates. Due to the asperities of the Earth's surface $\partial\Omega$, we cannot study the harmonic behavior of Φ. Therefore, we firstly construct a spherical ball Σ of radius "a" containing the Earth with asperities, that is, $\Omega \subset \Sigma$, and let $\partial\Sigma$ be its boundary. Let Φ^{int} denote the PGP interior to Σ, and let its expansion onto the orthonormal basis of real spherical harmonics $\mathbb{Y}_l^m$ be as follows

$$\Phi^{int}(a,\Theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm}^{int}(a)\, \mathbb{Y}_l^m(\Theta,\phi)\,; \quad \Phi_{lm}^{int}(a) = \int_{\partial\Sigma} \Phi^{int} \mathbb{Y}_l^m ds,$$

where $\mathbb{Y}_l^m(.)$ is the complex spherical harmonics of degree l and of order m (Dahlen and Tromp (1998)).

Let us denote by Φ^{ext} a potential satisfying the Laplace equation outside a ball Σ and vanishing at infinity. Thus, to extend Φ^{int} continuously to a potential Φ^{ext}, we have

$$\Phi^{ext}(r,\Theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{a}{r}\right)^{l+1} \Phi_{lm}^{int}(a)\, \mathbb{Y}_l^m(\Theta,\phi)\,,\ r \geq a. \tag{3.1}$$

Differentiating Eq. (3.1) with respect to r, then for the normal derivative of Φ^{ext} we have

$$\nabla\Phi^{ext}.\mathbf{n}(a,\Theta,\phi) = -\frac{1}{a}\sum_{l=0}^{\infty}(l+1)\sum_{m=-l}^{l} \Phi_{lm}^{int}(a)\, \mathbb{Y}_l^m(\Theta,\phi)\,, \tag{3.2}$$

which relates the normal derivative of the potential to the potential itself, representing the so-called Dirichlet-to-Neumann (DtN) operator on the spherical boundary. The conditions that the normal derivative of a given field is proportional to the field at the surface are known as the Robin boundary conditions.

Because we approximated the non-penetration conditions by (2.15a,b) then the hyperbolic variational inequality problem leads to solve the hyperbolic variational equation problem. It can be shown that the corresponding (penalty) functionals

$$P^{LM}(\mathbf{v}) = \frac{1}{2}\int_{\Gamma_c^{LM}} \left(\left([v_n]^{LM} - d_n^{LM}\right)_+\right)^2 ds,$$
$$P^{sm}(\mathbf{v}) = \frac{1}{2}\int_{\Gamma_c^{sm}} \left(\left([v_n]^{sm} - d_n^{sm}\right)_+\right)^2 ds$$

have the Gâteaux differentials, that are Lipschitz continuous, and that $P^{LM}(\mathbf{v})$ and $P^{sm}(\mathbf{v})$ are monotone functionals. About such functionals we speak as penalty functionals or penalties and about corresponding problems we speak as penalized problems.

The conditions of Coulomb friction (2.13)-(2.14) in the variational formulations lead to non-differentiable functionals. The smoothing of the frictional functionals are done by replacing the non-differentiable norms $|.|$ in the frictional terms by differentiable and convex approximations. For details see later.

The solution $\mathbf{v}$ of the problem in the solid part of the Earth is searched in the space of kinematically admissible displacements

$$V(t) = \left\{\mathbf{w}(\mathbf{x},t) \in H^{1,N}(\Omega^s) : \Omega^s \times I \to \mathbb{R}^N, N=3\right\},$$

where $H^{1,N}(\Omega^s) = \left[H^1(\Omega^s)\right]^N$, where $H^1(\Omega^s)$ is the Sobolev space of square-integrable functions with square-integrable generalized first derivatives, where $\Omega^s = \Omega^L \cup \Omega^M \cup \Omega^{IC}$, and $\Omega^L = \cup_{\iota=1}^r \Omega^{L\iota}$. Further, let

$$V = \left\{\mathbf{w} \in H^{1,N}(\Omega^s) : \Omega^s \to \mathbb{R}^N, N=3\right\},$$

be the space of test functions. In the fluid part of the Earth the solution will be searched in the spaces

$$V_{fs}^{OC}(t) = \left\{w(\mathbf{x},t) \in H^{1,1}\left(\Omega^{OC}\right) : \Omega^{OC} \times I \to \mathbb{R}^N, N=3\right\}$$

for the admissible scalar potential and

$$V_{fv}^{OC}(t) = \left\{\mathbf{w}(\mathbf{x},t) \in H^{1,N}\left(\Omega^{OC}\right) : \Omega^{OC} \times I \to \mathbb{R}, N=3\right\}$$

for the admissible vector potential and

$$\begin{aligned} V_{fs}^{OC} &= \left\{w \in H^{1,1}\left(\Omega^{OC}\right) : \Omega^{OC} \to \mathbb{R}, N=3\right\}, \\ V_{fv}^{OC} &= \left\{\mathbf{w} \in H^{1,N}\left(\Omega^{OC}\right) : \Omega^{OC} \to \mathbb{R}^N, N=3\right\} \end{aligned}$$

for the spaces of test functions.

In the case of the Eulerian perturbation of the gravitational potential (PGP) (i.e., of the mass redistribution potential (MRP)) the solution will be searched in the space of admissible potential

$$V_p = \left\{\psi \in H^{1,1}(\Omega) : \Omega \to \mathbb{R}^N, N=3\right\},$$

because applying the Dirichlet-to-Neumann DtN operator that is equivalent to use a Robin condition.

To formulate the corresponding variational problem, we firstly multiply Eq. (2.4) by an arbitrary test vector function $\mathbf{w} \in V$, integrate by parts over $\Omega^L \cup \Omega^M$ with $\Omega^L = \cup_{\iota=1}^r \Omega^{L\iota}$, use the stress-free condition on the Earth's surface and the contact conditions on Γ_c^{LM} and Γ_c^{sm}, then we obtain

$$\begin{aligned} &\int_{\Omega^L \cup \Omega^M} \rho \mathbf{w}.\frac{\partial^2 \mathbf{v}}{\partial t^2} d\mathbf{x} + \int_{\Omega^L \cup \Omega^M} 2\rho \mathbf{w}.\left(\hat{\boldsymbol{\omega}} \times \frac{\partial \mathbf{v}}{\partial t}\right) d\mathbf{x} = \\ &= -\int_{\Omega^L \cup \Omega^M} \nabla \mathbf{w} : \left(\overline{\boldsymbol{\tau}} + \overline{\mathbf{G}}\right) d\mathbf{x} + \int_{\sum_{source}} \overline{\mathbf{m}}(\mathbf{x}_s, t) : \nabla \mathbf{w}(\mathbf{x}_s) ds + \\ &+\frac{1}{\varepsilon}\int_{\Gamma_v^{LM}} w_n \left([v_n]^{LM} - d_n^{LM}\right)_+ ds + \frac{1}{\varepsilon}\int_{\cup_{s,m}\Gamma_c^{sm}} w_n \left([v_n]^{sm} - d_n^{sm}\right)_+ ds + \\ &+ j^{LM}(\mathbf{w}) + j(\mathbf{w}) - \int_{\Omega^L \cup \Omega^M} \rho \mathbf{v}.\overline{\mathbf{H}}.\mathbf{w} d\mathbf{x} + \int_{\Gamma_c^{MOC}} p\mathbf{n}.\mathbf{w} ds, \end{aligned} \tag{3.3}$$

where the second term denotes the Coriolis' term, $\mathbf{x}_s$ denotes the co-ordinates of the elastic wave source, and where $\overline{\mathbf{G}}$ and $\overline{\mathbf{H}}$ are the second-order tensors defined by

$$\overline{\mathbf{G}} = \rho\left[\mathbf{v}\mathbf{g} - (\mathbf{v}.\mathbf{g})\overline{\mathbf{I}}\right], \overline{\mathbf{H}} = \nabla \mathbf{g}, \tag{3.4}$$

where $\overline{\mathbf{G}}$ is the non-symmetric tensor, $\overline{\mathbf{I}}$ the identity tensor and since the gravitational acceleration $\mathbf{g}$ is the gradient of a geopotential Φ, i.e., $\mathbf{g} = -\nabla\Phi$, thus $\overline{\mathbf{H}}$ is a symmetric second-order tensor. In the case, if frictions on the contact boundaries are assumed to be operated, then the functionals $j^{LM}(\mathbf{w})$ and $j(\mathbf{w})$, defined by

$$
\begin{aligned}
j^{LM}(\mathbf{w}) &= \int_{\Gamma_c^{LM}} \mathcal{F}_c^{LM}\left([\mathbf{v}_t]^{LM}\right)\left|\tau_n^{LM}(\mathbf{v},\mathbf{v}')\right| . \left|[\mathbf{w}_t]^{LM}\right| ds, \\
j(\mathbf{w}) &= \sum_{s,m} j^{sm}(\mathbf{w}), s,m \in [1,...,r], where \\
j^{sm}(\mathbf{w}) &= \int_{\Gamma_c^{sm}} \mathcal{F}_c^{sm}\left([\mathbf{v}_t]^{sm}\right)\left|\tau_n^{sm}(\mathbf{v},\mathbf{v}')\right| . \left|[\mathbf{w}_t]^{sm}\right| ds \\
&\text{for the generalized Coulombian friction case and} \\
j^{LM}(\mathbf{w}) &= \int_{\Gamma_c^{LM}} g_c^{LM}\left|[\mathbf{w}_t]^{LM}\right| ds \equiv \left\langle g_c^{LM}, \left|[\mathbf{w}_t]^{LM}\right|\right\rangle_{\Gamma_c^{LM}}, \\
j(\mathbf{w}) &= \sum_{s,m} j^{sm}(\mathbf{w}), s,m \in [1,...,r], where \\
j^{sm}(\mathbf{w}) &= \int_{\Gamma_c^{sm}} g_c^{sm}\left|[\mathbf{w}_t]^{sm}\right| ds \equiv \left\langle g_c^{sm}, \left|[\mathbf{w}_t]^{sm}\right|\right\rangle_{\Gamma_c^{sm}}, \\
g_c^{LM} &\equiv \mathcal{F}_c^{LM}\left([\mathbf{v}_t]^{LM}\right)\left|\tau_n^{LM}(\mathbf{v},\mathbf{v}')\right|(t-\triangle t), \\
g_c^{sm} &\equiv \mathcal{F}_c^{sm}\left([\mathbf{v}_t]^{sm}\right)\left|\tau_n^{sm}(\mathbf{v},\mathbf{v}')\right|(t-\triangle t), \\
&\text{for the Tresca model of friction,}
\end{aligned}
\tag{3.5}
$$

are introduced, where the coefficients of friction $\mathcal{F}_c$ will be assumed to be a function of sliding displacements $\mathbf{v}_t$ (in a general case friction coefficients $\mathcal{F}_c$ depend on the sliding velocities $\mathbf{v}_t'$), $\langle .,. \rangle_\Gamma$ denotes the $L^2(\Gamma)$-scalar product, and $\triangle t$ denotes the time element. Both these functionals are non-smooth. The smoothing of these functionals are done by replacing the non-differentiable norms $|.| \equiv \Psi$ in these friction functionals by a smooth and convex approximation $\Psi_\delta(.), \delta > 0$, where δ is a small regularization parameter. The function $\Psi_\delta : \mathbb{R}^3 \to [0,+\infty)$ is a convex C^1-function having its minimum at $\mathbf{x} = \mathbf{0}$ and satisfying the approximation property $|\Psi_\delta(\mathbf{x}) - |\mathbf{x}|| \le \delta$ and $\nabla\Psi_\delta(0) = 0$ (see Eck et al. (2005)), thus $j(\mathbf{w}) \sim j_\delta(\mathbf{w})$, where $j_\delta(\mathbf{w})$ is a convex regularization of $j(\mathbf{w})$. Thus

$$
\begin{aligned}
j^{LM}(\mathbf{w}) &\simeq \int_{\Gamma_c^{LM}} \mathcal{F}_c^{LM}\left([\mathbf{v}_t]^{LM}\right)\frac{1}{\varepsilon}\left([v_n]^{LM} - d_n^{LM}\right)_+ \nabla\Psi_\delta\left([\mathbf{v}_t]^{LM}\right) . [\mathbf{w}_t]^{LM} ds, \\
j(\mathbf{w}) &= \sum_{s,m} j^{sm}(\mathbf{w}), s,m \in [1,...,r], \ where \\
j^{sm}(\mathbf{w}) &\simeq \int_{\Gamma_c^{sm}} \mathcal{F}_c^{sm}\left([\mathbf{v}_t]^{sm}\right)\frac{1}{\varepsilon}\left([v_n]^{sm} - d_n^{sm}\right)_+ \nabla\Psi_\delta\left([\mathbf{v}_t]^{sm}\right) . [\mathbf{w}_t]^{sm} ds, \\
&\text{for the generalized Coulombian friction case and} \\
j^{LM}(\mathbf{w}) &= \int_{\Gamma_c^{LM}} g_c^{LM}\left|[\mathbf{w}_t]^{LM}\right| ds \simeq \int_{\Gamma_c^{LM}} g_c^{LM}\nabla\Psi_\delta\left([\mathbf{w}_t]^{LM}\right) ds, \\
j(\mathbf{w}) &= \sum_{s,m} j^{sm}(\mathbf{w}), s,m \in [1,...,r], \ where \\
j^{sm}(\mathbf{w}) &= \int_{\Gamma_c^{sm}} g_c^{sm}\left|[\mathbf{w}_t]^{sm}\right| ds \simeq \int_{\Gamma_c^{sm}} g_c^{sm}\nabla\Psi_\delta\left([\mathbf{w}_t]^{sm}\right) ds, \\
&\text{for the Tresca model of friction, where } \mathrm{g_c^{LM}} \text{ and } \mathrm{g_c^{sm}} \text{ were defined above.}
\end{aligned}
\tag{3.6}
$$

Remark 4 *Since the functional $j(\mathbf{w})$ is not Gâteaux differentiable, therefore, we have to consider the function $\Psi_\delta : \mathbb{R} \to \mathbb{R}$ defined by*

$$
\Psi_\delta(x) = \sqrt{x^2 + \delta^2} - \delta \, ,
$$

which regularizes the function $x \to |x|$. The function Ψ_δ is twice differentiable and the following inequality

$$
|\Psi_\delta(\mathbf{x}) - |\mathbf{x}|| \le \delta, \ \forall x \in \mathbb{R}, \ \delta > 0,
$$

holds. Moreover, the function (Eck et al. (2005))

$$\Psi_\delta(x) : x \mapsto \left\{ \begin{array}{ll} |x|, & |x| \geq \delta, \\ -\frac{|x|^4}{8\delta^3} + \frac{3|x|^2}{4\delta} + \frac{3}{8}\delta, & |x| < \delta, \end{array} \right\}$$

or other functions of the above desired properties can be used.

Remark 5 *For non-symmetric tensors $\overline{\mathbf{A}} = (A_{ij})$ and $\overline{\mathbf{B}} = (B_{ij})$ the double dot product is defined by $\overline{\mathbf{A}} : \overline{\mathbf{B}} = A_{ij}B_{ij}$.*

The variational formulation corresponding to the fluid outer core will be derived by using Eqs (2.9) and (2.10), that will be multiplied by arbitrary test functions $w \in V_s^{OC}$ or $\mathbf{w} \in V^{OC}$, respectively, then integrate by parts over Ω^{OC} and using corresponding interface conditions (2.16), we obtain

$$\begin{aligned} \int_{\Omega^{OC}} \rho k^{-1} w \frac{\partial^2 \varphi}{\partial t^2} d\mathbf{x} = -\int_{\Omega^{OC}} (\nabla w) \, . \, (\nabla \varphi + \boldsymbol{\psi}) \, d\mathbf{x} + \\ + \int_{\Omega^{OC}} \rho k^{-1} w \left[\mathbf{g} . (\nabla \varphi + \boldsymbol{\psi}) - \Phi \right] d\mathbf{x} + \int_{\Gamma_c^{MOC}} w \mathbf{n}.\mathbf{v} ds - \int_{\Gamma_c^{OIC}} w \mathbf{n}.\mathbf{v} ds \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \int_{\Omega^{OC}} \mathbf{w}.\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} d\mathbf{x} = \\ = -2 \int_{\Omega^{OC}} \mathbf{w}. \left[\hat{\boldsymbol{\omega}} \times \left(\frac{\partial \boldsymbol{\psi}}{\partial t} + \nabla \left(\frac{\partial \varphi}{\partial t} \right) \right) \right] d\mathbf{x} - \int_{\Omega^{OC}} \rho^{-1} g^{-2} p \tilde{N}^2 \mathbf{w}.\mathbf{g} d\mathbf{x}, \end{aligned} \tag{3.8}$$

where (2.17) was used.

Remark 6 *The problem can be simplified setting $\Phi = 0$, $\tilde{N} = 0$ and ignoring the rotation and self-gravitation (see Komatitsch et al. (2005)).*

The variational formulation corresponding to the inner core is similar to that of the lithosphere-mantle case and it is as follows

$$\begin{aligned} \int_{\Omega^{IC}} \rho \mathbf{w}.\frac{\partial^2 \mathbf{v}}{\partial t^2} d\mathbf{x} + \int_{\Omega^{IC}} 2\rho \mathbf{w}. \left(\hat{\boldsymbol{\omega}} \times \frac{\partial \mathbf{v}}{\partial t} \right) d\mathbf{x} = -\int_{\Omega^{IC}} \nabla \mathbf{w} : \left(\overline{\boldsymbol{\tau}} + \overline{\mathbf{G}} \right) d\mathbf{x} - \\ - \int_{\Omega^{IC}} \rho \mathbf{v}.\overline{\boldsymbol{H}}.\mathbf{w} d\mathbf{x} - \int_{\Gamma_c^{OIC}} p \mathbf{n}.\mathbf{w} ds. \end{aligned} \tag{3.9}$$

The coupling between the inner core and the outer core is given through the surface integrals over the interface boundary Γ_c^{OIC} in Eqs (3.9) and (3.7) as well as continuity in traction and continuity of the normal component of displacement and velocity.

To formulate the problem (2.3) with (2.20) variationally, the so-called Dirichlet-to-Neumann (DtN) operator on the spherical boundary will be introduced. Eq. (2.3a) is valid within the solid parts of the Earth, in the fluid part of the Earth Eq. (2.3a) using (2.7) leads to

$$\nabla^2 \Phi = -4\pi G \nabla . (\rho \nabla \varphi + \rho \boldsymbol{\psi}) . \tag{3.10}$$

Multiplying (2.3) with (3.10) by $\tilde{\varphi} \in V_p$, integrating over Ω by parts, then

$$\begin{aligned} \int_\Omega \nabla \Phi . \nabla \tilde{\varphi} d\mathbf{x} - \int_{\partial \Omega} \nabla \Phi . \mathbf{n} \tilde{\varphi} ds = \\ = -4\pi G \left[\int_{\Omega^L \cup \Omega^M \cup \Omega^{IC}} \rho \mathbf{v}.\nabla \tilde{\varphi} d\mathbf{x} - \int_{\partial \Omega} \rho \mathbf{v}.\mathbf{n} \tilde{\varphi} ds + \int_{\Omega^{OC}} \rho (\nabla \varphi + \boldsymbol{\psi}) . \nabla \tilde{\varphi} d\mathbf{x} \right]. \end{aligned} \tag{3.11}$$

It can be shown that the jump condition (2.20b) across the solid-fluid interfaces is naturally taken into account in (3.11). Therefore, the potential decomposition (2.7) in the fluid is defined in displacements and not in velocities. Note that the term $\int_{\partial\Omega} \nabla \Phi . \mathbf{n} \tilde{\varphi} ds$ is unknown.

We know that Φ satisfies the Laplace equation outside the Earth's surface $\partial\Omega$ and that it tends to zero for distances $\mathbf{r}$ tends to infinity (Dahlen and Tromp (1998)). It can be shown that applying the DtN operator the obtained problem is a well-posed problem.

Then taking into account the asperities on the Earth's surface and the jump condition (2.20b) across the spherical boundaries then we have the following variational formulation of the Poisson-Laplace equation (Chaljub and Valette (2004))

$$\begin{aligned} &\int_{\Sigma} \nabla\Phi.\nabla\tilde{\varphi}d\mathbf{x} - \int_{\partial\Sigma} \nabla\Phi^{ext}.\mathbf{n}\tilde{\varphi}ds = \\ &= -4\pi G\left[\int_{\Omega^L\cup\Omega^M\cup\Omega^{IC}} \rho\mathbf{v}.\nabla\tilde{\varphi}d\mathbf{x} + \int_{\Omega^{OC}} \rho\left(\nabla\varphi + \boldsymbol{\psi}\right).\nabla\tilde{\varphi}d\mathbf{x}\right], \end{aligned} \tag{3.12}$$

where Σ is a spherical ball with the radius $''a''$ defined above and where

$$\int_{\partial\Sigma} \nabla\Phi^{ext}.\mathbf{n}\tilde{\varphi}ds = -\frac{1}{a}\sum_{l=0}^{\infty}(l+1)\sum_{m=-l}^{l}\Phi_{lm}^{int}(a)\,\tilde{\varphi}_l^m(a)\,.$$

Since the effect of self-gravitating onto propagation of seismic waves is small, therefore, it can be omitted.

4 Numerical solution

Let $\overline{I} = [0, t_p]$ be a time interval. Let $m > 0$ be an integer, then $\triangle t = t_p/m$, $t_i = i\triangle t$, $i = 0, ..., m$, $\mathbf{v}_h(t_i)$ is the value of $\mathbf{v}_h$ at time $t = t_i$. Let the domain $\Omega \subset \mathbb{R}^3$, with the boundary $\partial\Omega$, be approximated by Ω_h with the boundary $\partial\Omega_h$, where $\overline{\Omega}_h = \overline{\Omega}_h^L \cup \overline{\Omega}_h^M \cup \overline{\Omega}_h^{OC} \cup \overline{\Omega}_h^{IC}$, be a polyhedral domain, that is, Ω_h is an open bounded connected subset such that $\overline{\Omega}_h$ is the union of a finite number of polyhedra (hexahedra),

$$\overline{\Omega}_h = \cup_{T_{hi}\in\mathcal{T}_h} T_{hi},\ i = 1, ..., M,$$

such that two arbitrary neighboring tetrahedra (hexahedra) have either no common point or have a common vertex, and a common edge or a common face. Such a partition of region Ω_h will be called the *division* and it will be denoted by $\mathcal{T}_h$, where $h = \max_{1\le i\le M}(\text{diam } T_{hi})$ is the largest edge of the division $\mathcal{T}_h$. Let ϑ_h denote the size of a minimal angle in the division $\mathcal{T}_h$, defined as the minimal of all angles between the faces and between the edges of all tetrahedra (hexahedra) of $\mathcal{T}_h$. Then the system $\{\mathcal{T}_h\}$, $h \to 0$, is called regular if there exists a positive number $\vartheta_0 > 0$ such that $\min_{h\to 0}\vartheta_h \ge \vartheta_0$. In the sequel we will assume that each element $T_{hi} \in \mathcal{T}_h$ can be obtained as $T_{hi} = Tr\,(T_{h0})$, where T_{h0} is a reference polyhedron (cube) and Tr is a suitable invertible affine map. The used mesh can be also composed by hexahedral elements and honors the main discontinuities that are observed inside the Earth. Then a reference element T_{h0} in this case is the cube.

For a solution the h-version or hp-version of finite element methods, the mortar approach or the spectral-element method can be used. Next, we introduce the main idea of the spectral-element method frequently used in seismology.

In seismology a spectral-element method frequently uses Lagrange polynomials of degree 4 to 10 for the interpolation of functions and Gauss-Lobatto–Legendre (GLL) quadrature, because the GLL quadrature together with the Lagrange interpolants give an exactly diagonal mass matrix. In this choice each spectral element contains a grid of $(n+1)^3$ Gauss-Lobatto-Legendre points, and each edge of an element contains a grid of $(n+1)^2$ Gauss-Lobatto-Legendre points, that is, e.g. for $n = 2$ the spectral element contains 27 GLL points and the edge contains 8 GLL points.

The $n+1$ Lagrange polynomials of degree n are defined in terms of $n+1$ nodal points $-1 \le \xi_\alpha \le 1$, $\alpha = 0, ..., n$, via the standard Lagrange interpolation condition $\theta_\alpha^n(\xi_\beta) = \delta_{\alpha\beta}$, where δ is the Kronecker delta, by

$$\theta_\alpha^n(\xi) = \sqcap_{\substack{0\le\beta\le n\\ \beta\neq\alpha}} \frac{(\xi - \xi_\beta)}{(\xi_\alpha - \xi_\beta)}, \quad \alpha = 0, ..., n. \tag{4.1}$$

In a spectral-element method (SEM), the nodal points $\xi_{\alpha,\alpha}$, $\alpha = 0, ..., n$, in (4.1) are taken as the $n+1$ Gauss-Lobatto-Legendre points, which are the roots of $\left(1-\xi^2\right) P'_n(\xi) = 0$, where P'_n denotes the derivatives of the Legendre polynomial of degree n (see Canuto et al. (1988), Šolín (2006)).

Then functions f on an element are interpolated by using triple products of Lagrange polynomials by

$$f(\mathbf{x}(\xi,\eta,\zeta)) = \sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma} f^{\alpha\beta\gamma}\theta_\alpha(\xi)\,\theta_\beta(\eta)\,\theta_\gamma(\zeta), \tag{4.2}$$

where $f^{\alpha\beta\gamma} = f(\mathbf{x}(\xi_\alpha,\eta_\beta,\zeta_\gamma))$ represents the value of the function f at the Gauss-Lobatto-Legendre points $\mathbf{x}(\xi_\alpha,\eta_\beta,\zeta_\gamma)$ and where for simplicity we omitted the index n of the Lagrange polynomial $\theta^n_\alpha(\xi)$. The gradient of the function f of (4.2), that is, $\nabla f = \sum_{i=1}^{3} \mathbf{e}_{xi}\frac{\partial f}{\partial x_i}$, evaluated at the Gauss-Lobatto-Legendre point $\mathbf{x}(\xi_\alpha,\eta_\beta,\zeta_\gamma)$, is then as follows

$$\begin{aligned}\nabla f(\mathbf{x}(\xi_{\alpha'},\eta_{\beta'},\zeta_{\gamma'})) = \sum_{i=1}^{3}\mathbf{e}_{xi}[(\partial_i\xi)^{\alpha'\beta'\gamma'}\sum_{\alpha=0}^{\eta_\alpha} f^{\alpha\beta'\gamma'}\theta'_\alpha(\xi_{\alpha'}) + \\ +(\partial_i\eta)^{\alpha'\beta'\gamma'}\sum_{\beta=0}^{\eta_\beta} f^{\alpha'\beta\gamma'}\theta'_\beta(\eta_{\beta'}) + (\partial_i\zeta)^{\alpha'\beta'\gamma'}\sum_{\gamma=0}^{\eta_\gamma} f^{\alpha'\beta'\gamma}\theta'_\gamma(\zeta_{\gamma'})],\end{aligned} \tag{4.3}$$

where $\mathbf{e}_{xi}$, $i = 1,2,3$, are unit vectors in the directions of increasing x_i, $i = 1,2,3$, respectively, and ∂_i, $i = 1,2,3$, denote partial derivatives in these directions, and a prime ("$'$") denotes derivatives of the Lagrange polynomials (i.e., $\theta'_\alpha(\xi)$, etc.). The matrix $\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{x}}$ will be obtained by inverting the matrix $\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}}$, as this inverse exists (Canuto et al. (1988)).

To approximate the variational (weak) formulation we firstly give the expressions for the displacement vector $\mathbf{v}_h$ and the test functions $\mathbf{w}_h$ by the Lagrange polynomials, thus

$$\begin{aligned}\mathbf{v}_h(\mathbf{x}(\xi,\eta,\zeta),t) \simeq \sum_{i=1}^{3}\mathbf{e}_{xi}\sum_{\sigma,\tau,\nu=0}^{\eta_\sigma,\eta_\tau,\eta_\nu} v^{\sigma\tau\nu}_{hi}(t)\,\theta_\sigma(\xi)\,\theta_\tau(\eta)\,\theta_\nu(\zeta), \\ \mathbf{w}_h(\mathbf{x}(\xi,\eta,\zeta)) \simeq \sum_{i=1}^{3}\mathbf{e}_{xi}\sum_{\alpha,\beta,\gamma=0}^{\eta_\sigma,\eta_\tau,\eta_\nu} w^{\alpha\beta\gamma}_{hi}\theta_\alpha(\xi)\,\theta_\beta(\eta)\,\theta_\gamma(\zeta).\end{aligned} \tag{4.4}$$

Based on a Gauss-Lobato-Legendre (GLL) integration rule, because it leads to a diagonal mass matrix when used together with the GLL interpolation points, we have

$$\int_{T_{hi}} f(\mathbf{x})\,d\mathbf{x} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\mathbf{x}(\boldsymbol{\xi}))\,J_{T_{hi}}(\boldsymbol{\xi})\,d\boldsymbol{\xi} \simeq \sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma} \omega_\alpha\omega_\beta\omega_\gamma J^{\alpha\beta\gamma}_{T_{hi}} f^{\alpha\beta\gamma}, \tag{4.5}$$

where $\boldsymbol{\xi} = (\xi,\eta,\zeta)$, $-1 \le \xi \le 1$, $-1 \le \eta \le 1$, $-1 \le \zeta \le 1$, $d\mathbf{x} = J_{T_{hi}} d\boldsymbol{\xi}$, $J_{T_{hi}} = \left|\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}}\right| = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(\xi,\eta,\zeta)}\right|$ and where $J^{\alpha\beta\gamma}_{T_{hi}} = J_{T_{hi}}(\xi_\alpha,\eta_\beta,\zeta_\gamma)$ is the Jacobian, $\omega_\alpha > 0$, $\omega_\beta > 0$, $\omega_\gamma > 0$, $\alpha,\beta,\gamma = 0,...,n$, denote the weights of the GLL quadrature associated with ξ,η and ζ, and moreover,

$$\int_{\partial T_{hi}} f(\mathbf{x})\,ds = \int_{-1}^{1}\int_{-1}^{1} f(\mathbf{x}(\xi,\eta))\,J_{\partial T_{hi}}(\xi,\eta)\,d\xi d\eta \simeq \sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta} \omega_\alpha\omega_\beta J^{\alpha\beta}_{\partial T_{hi}} f^{\alpha\beta}, \tag{4.6}$$

where ∂T_{hi} denotes a surface of the element T_{hi}, located on the interface (solid-fluid) boundaries and $J^{\alpha\beta}_{\partial T_{hi}} = J_{\partial T_{hi}}(\xi_\alpha,\eta_\beta)$ is the surface Jacobian, as the normal $\mathbf{n}$ to a boundary ∂T_{hi} of the element T_{hi} is given by

$$\mathbf{n} = \frac{1}{J_{\partial T_{hi}}}\left(\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}} \times \frac{\partial\mathbf{x}}{\partial\boldsymbol{\eta}}\right), \quad J_{\partial T_{hi}} = \left\|\frac{\partial\mathbf{x}}{\partial\boldsymbol{\xi}} \times \frac{\partial\mathbf{x}}{\partial\boldsymbol{\eta}}\right\|.$$

Furthermore, $ds = J_{\partial T_{hi}} d\xi d\eta$ and $d\mathbf{x} = J_{T_{hi}} d\boldsymbol{\xi}$, where $J_{\partial T_{hi}}$ and $J_{T_{hi}}$ are the surface Jacobian and the Jacobian of the used mappings Tr.

Then, we will derive the individual terms in the variational formulations (3.3)- (3.11), based on the SEM-Galerkin method, where we use the same basis functions for approximations of the displacement and the test vectors, and the Gauss-Lobato-Legendre quadrature. The first term on the left-hand side (3.3) using (4.4) gives

$$\begin{gathered}\int_{\Omega_h^L\cup\Omega_h^M}\rho_h\mathbf{w}_h\cdot\frac{\partial^2\mathbf{v}_h}{\partial t^2}d\mathbf{x}\simeq\sum_{T_{hi}}\int_{T_{hi}}\rho_h\mathbf{w}_h\cdot\frac{\partial^2\mathbf{v}_h}{\partial t^2}d\mathbf{x},\\ \int_{T_{hi}}\rho_h\mathbf{w}_h\cdot\frac{\partial^2\mathbf{v}_h}{\partial t^2}d\mathbf{x}=\\ =\int_{-1}^1\int_{-1}^1\int_{-1}^1\rho_h\left(\mathbf{x}\left(\boldsymbol{\xi}\right)\right)\mathbf{w}_h\left(\mathbf{x}\left(\boldsymbol{\xi}\right)\right).\frac{\partial^2\mathbf{v}_h\left(\mathbf{x}\left(\boldsymbol{\xi}\right),t\right)}{\partial t^2}J_{T_{hi}}\left(\boldsymbol{\xi}\right)d\boldsymbol{\xi}\simeq\\ \simeq\sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma}\omega_\alpha\omega_\beta\omega_\gamma J_{T_{hi}}^{\alpha\beta\gamma}\rho_h^{\alpha\beta\gamma}\sum_{i=1}^3 w_{hi}^{\alpha\beta\gamma}\frac{\partial^2 v_{hi}^{\alpha\beta\gamma}\left(t\right)}{\partial t^2},\end{gathered}\tag{4.7}$$

where $\rho_h^{\alpha\beta\gamma}=\rho_h\left(\mathbf{x}\left(\xi_\alpha,\eta_\beta,\zeta_\gamma\right)\right).$ Since the variational formulation hold for any test function $\mathbf{w}$, we can put factors $w_{h1}^{\alpha\beta\gamma}=w_{h2}^{\alpha\beta\gamma}=w_{h3}^{\alpha\beta\gamma}=0$. We see that the acceleration component $\frac{\partial^2 v_{hi}(t)}{\partial t^2}$ at each nodal point $(\xi_\alpha,\eta_\beta,\zeta_\gamma)$ is multiplied by the factor $\omega_\alpha\omega_\beta\omega_\gamma J_{T_{hi}}^{\alpha\beta\gamma}\rho_h^{\alpha\beta\gamma}$, where $J_{T_{hi}}^{\alpha\beta\gamma}$ is the value of the Jacobian at the nodal point $(\xi_\alpha,\eta_\beta,\zeta_\gamma)$, where $\omega_\alpha>0$, $\omega_\beta>0$, $\omega_\gamma>0$, $\alpha,\beta,\gamma=0,....,n$, are the weights of the Gauss-Lobatto-Legendre quadrature (Canuto et al. (1988)). Thus the mass matrix is diagonal.

For the Coriolis' term in (3.3)-(3.9) then we have

$$\begin{gathered}\int_{\Omega_h^L\cup\Omega_h^M}2\rho_h\mathbf{w}_h\cdot\left(\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}_h}{\partial t}\right)d\mathbf{x}\simeq\sum_{T_{hi}}\int_{T_{hi}}2\rho_h\mathbf{w}_h\cdot\left(\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}_h}{\partial t}\right)d\mathbf{x},\\ \int_{T_{hi}}2\rho_h\mathbf{w}_h\cdot\left(\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}_h}{\partial t}\right)d\mathbf{x}=\\ =\int_{-1}^1\int_{-1}^1\int_{-1}^1 2\rho_h\left(\mathbf{x}\left(\boldsymbol{\xi}\right)\right)\mathbf{w}_h\left(\mathbf{x}\left(\boldsymbol{\xi}\right)\right).\left(\hat{\boldsymbol{\omega}}\times\frac{\partial\mathbf{v}_h\left(\mathbf{x}\left(\boldsymbol{\xi}\right)\right)}{\partial t}\right)J_{T_{hi}}\left(\boldsymbol{\xi}\right)d\,\boldsymbol{\xi}\simeq\\ \simeq 2\hat{\omega}\sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma}\omega_\alpha\omega_\beta\omega_\gamma J_{T_{hi}}^{\alpha\beta\gamma}\rho_h^{\alpha\beta\gamma}\sum_{i,j=1}^3 w_{hi}^{\alpha\beta\gamma}\epsilon_{i3j}\frac{\partial v_{hj}^{\alpha\beta\gamma}}{\partial t},\end{gathered}\tag{4.8}$$

where ϵ_{ijk} is the alternating tensor of the third order. The effect of this term can be neglected, because the effect of rotation of the Earth on the wave propagation through the Earth is very small, as it was mentioned above.

For the first term of the right-hand side of (3.3)-(3.9), which represents the stiffness matrix for the lithosphere-mantle part, we find

$$\begin{gathered}\int_{\Omega_h^L\cup\Omega_h^M}\nabla\mathbf{w}_h:\overline{\boldsymbol{\tau}}_h d\mathbf{x}\simeq\sum_{T_{hi}}\int_{T_{hi}}\nabla\mathbf{w}_h:\overline{\boldsymbol{\tau}}_h d\mathbf{x},\\ \int_{T_{hi}}\nabla\mathbf{w}_h:\overline{\boldsymbol{\tau}}_h d\mathbf{x}=\int_{-1}^1\int_{-1}^1\int_{-1}^1\left[\sum_{i,k=1}^3 F_{ik}\frac{\partial w_{hi}}{\partial\xi_k}\right]J_{T_{hi}}d\boldsymbol{\xi}\simeq\\ \simeq\sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma}\sum_{i=1}^3 w_{hi}^{\alpha\beta\gamma}\omega_\beta\omega_\gamma\sum_{\alpha'=0}^{\eta_{\alpha'}}\omega_{\alpha'}J_{T_{hi}}^{\alpha'\beta\gamma}F_{i1}^{\alpha'\beta\gamma}\theta'_\alpha\left(\xi_{\alpha'}\right)+\\ +\omega_\alpha\omega_\gamma\sum_{\beta'=0}^{\eta_{\beta'}}\omega_{\beta'}J_{T_{hi}}^{\alpha\beta'\gamma}F_{i2}^{\alpha\beta'\gamma}\theta'_\beta\left(\eta_{\beta'}\right)+\omega_\alpha\omega_\beta\sum_{\gamma'=0}^{\eta_{\gamma'}}\omega_{\gamma'}J_{T_{hi}}^{\alpha\beta\gamma'}F_{i3}^{\alpha\beta\gamma'}\theta'_\gamma\left(\zeta_{\gamma'}\right),\end{gathered}\tag{4.9}$$

where we put $F_{ik}=\sum_{i,j=1}^3\overline{\tau}_{ij}\frac{\partial\xi_k}{\partial x_j}$, $F_{ik}^{\sigma\tau\nu}=F_{ik}\left(\mathbf{x}\left(\xi_\sigma,\eta_\tau,\zeta_\nu\right)\right)$ represents the value of F_{ik} at the GLL point $\mathbf{x}\left(\xi_\sigma,\eta_\tau,\zeta_\nu\right).$

In the solid parts of the Earth when the term of self-gravitation is taken into consideration, then the tensor $\overline{\boldsymbol{\tau}}_h$ is replaced by $\overline{\boldsymbol{\tau}}_h + \overline{\boldsymbol{G}}_h$, where $\overline{\mathbf{G}}_h = \rho_h \left[\mathbf{v}_h \mathbf{g}_h - (\mathbf{v}_h.\mathbf{g}_h)\,\overline{\mathbf{I}}\right]$. The stress tensor $\overline{\boldsymbol{\tau}}_h$ will be expressed at the GLL points as

$$\overline{\boldsymbol{\tau}}_h\left(\mathbf{x}\left(\xi_\alpha,\eta_\beta,\zeta_\gamma\right),t\right) = \overline{\mathbf{c}}\left(\mathbf{x}\left(\xi_\alpha,\eta_\beta,\zeta_\gamma\right)\right) : \nabla\mathbf{v}_h\left(\mathbf{x}\left(\xi_\alpha,\eta_\beta,\zeta_\gamma\right),t\right), \tag{4.10}$$

where the gradient of displacement vector $\nabla\mathbf{v}_h\left(\mathbf{x}\left(\xi_\alpha,\eta_\beta,\zeta_\gamma\right),t\right)$ at the GLL points follows from (4.4a) upon differentiation (see e.g. Komatitsch and Tromp (2002)).

The wave source term in (3.3) will be approximated as follows

$$\begin{aligned}
\int_{\Sigma^h_{SOURCE}} \overline{\mathbf{m}}_h\left(\mathbf{x}_s,t\right) : \nabla\mathbf{w}_h\left(\mathbf{x}_s\right) ds \simeq \sum_{\partial T_{hi}} \int_{\partial T_{hi}} \overline{\mathbf{m}}_h\left(\mathbf{x}_s,t\right) : \nabla\mathbf{w}_h\left(\mathbf{x}_s\right) ds, \\
\int_{\partial T_{hi}} \overline{\mathbf{m}}_h\left(\mathbf{x}_s,t\right) : \nabla\mathbf{w}_h\left(\mathbf{x}_s\right) ds = \\
= \int_{-1}^{1}\int_{-1}^{1} \overline{\mathbf{m}}_h\left(\mathbf{x}\left(\xi_{\alpha_s},\eta_{\beta_s},\zeta_{\gamma_s}\right),t\right) : \nabla\mathbf{w}_h\left(\mathbf{x}\left(\xi_{\alpha_s},\eta_{\beta_s},\zeta_{\gamma_s}\right)\right) J_{\partial T_{hi}}\left(\xi,\eta\right) d\xi d\eta \simeq \\
\simeq \sum_{i=1}^{3} w_{hi}^{\alpha\beta\gamma}[\omega_\beta \sum_{\alpha_s=0}^{\eta_\alpha} \omega_{\alpha_s} J_{\partial T_{hi}}^{\alpha_s\beta} g_{i1}^{\alpha_s\beta\gamma} \theta'_\alpha\left(\xi_{\alpha_s}\right) + \\
+\omega_\alpha \sum_{\beta_s=0}^{\eta_\beta} \omega_{\beta_s} J_{\partial T_{hi}}^{\alpha\beta_s} g_{i2}^{\alpha\beta_s\gamma} \theta'_\beta\left(\eta_{\beta_s}\right) + \omega_\alpha\omega_\beta J_{\partial T_{hi}}^{\alpha\beta} g_{i3}^{\alpha\beta\gamma_s} \theta'_\gamma\left(\zeta_{\gamma_s}\right)],
\end{aligned} \tag{4.11}$$

where $g_{ik} = \sum_{j=1}^{3} \overline{m}_{hij} \frac{\partial\xi_k}{\partial x_j}$, $J_{\partial T_{hi}}$ is the surface Jacobian of the transformation.

Remark 7 *The wave source term can be also assumed to be evoked by a point source, where* $\mathbf{f} = -\overline{\mathbf{M}}.\nabla\delta\left(\mathbf{x}-\mathbf{x}_s\right).(t)$*, for details see Dahlen and Tromp (1998).*

Let $\left[v_{hn}^{\alpha\beta}\right]^{LM} \simeq \left[\sum_{i=1}^{3} v_{hi}^{\alpha\beta} n_i^{\alpha\beta}\right]^{LM}$, $\left[v_{hn}^{\alpha\beta}\right]^{sm} \simeq \left[\sum_{i=1}^{3} v_{hi}^{\alpha\beta} n_i^{\alpha\beta}\right]^{sm}$ denote the approximations of the jumps in the normal directions and since $\mathbf{v}_{ht} = \mathbf{v}_h - v_{hn}\mathbf{n}$, then we can determine $\left[\mathbf{v}_{ht}\right]^{LM}$ and $\left[\mathbf{v}_{ht}\right]^{sm}$.

Then the penalty terms in (3.3) will be approximated as follows

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\Gamma_{ch}^{LM}} w_{hn}.\left(\left[v_{hn}\right]^{LM} - d_n^{LM}\right)_+ ds \simeq \frac{1}{\varepsilon}\sum_{\partial T_{hi}}\int_{\partial T_{hi}} w_{hn}.\left(\left[v_{hn}\right]^{LM} - d_n^{LM}\right)_+ ds, \\
\frac{1}{\varepsilon}\int_{\partial T_{hi}} w_{hn}.\left(\left[v_{hn}\right]^{LM} - d_n^{LM}\right)_+ ds = \\
= \frac{1}{\varepsilon}\int_{-1}^{1}\int_{-1}^{1} w_{hn}\left(\mathbf{x}\left(\xi,\eta\right)\right)\left(\left[v_{hn}\left(\mathbf{x}\left(\xi,\eta\right)\right)\right]^{LM} - d_n^{LM}\right)_+ J_{\partial T_{hi}}\left(\xi,\eta\right) d\xi d\eta \simeq \\
\simeq \frac{1}{\varepsilon}\sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta} \omega_\alpha\omega_\beta J_{\partial T_{hi}}^{\alpha\beta}\left(\left[v_{hn}^{\alpha\beta}\right]^{LM} - \left(d_n^{\alpha\beta}\right)^{LM}\right)_+ \sum_{i=1}^{3} w_{hi}^{\alpha\beta} n_i^{\alpha\beta}, \\
\frac{1}{\varepsilon}\int_{\Gamma_{ch}^{sm}} w_{hn}.\left(\left[v_{hn}\right]^{sm} - d_n^{sm}\right)_+ ds \simeq \frac{1}{\varepsilon}\sum_{\partial T_{hi}}\int_{\partial T_{hi}} w_{hn}.\left(\left[v_{hn}\right]^{sm} - d_n^{sm}\right)_+ ds, \\
\frac{1}{\varepsilon}\int_{\partial T_{hi}} w_{hn}.\left(\left[v_{hn}\right]^{sm} - d_n^{sm}\right)_+ ds = \\
= \frac{1}{\varepsilon}\int_{-1}^{1}\int_{-1}^{1} w_{hn}\left(\mathbf{x}\left(\xi,\eta\right)\right)\left(\left[v_{hn}\left(\mathbf{x}\left(\xi,\eta\right)\right)\right]^{sm} - d_n^{sm}\right)_+ J_{\partial T_{hi}}\left(\xi,\eta\right) d\xi d\eta \simeq \\
\simeq \frac{1}{\varepsilon}\sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta} \omega_\alpha\omega_\beta J_{\partial T_{hi}}^{\alpha\beta}\left(\left[v_{hn}^{\alpha\beta}\right]^{sm} - \left(d_n^{\alpha\beta}\right)^{sm}\right)_+ \sum_{i=1}^{3} w_{hi}^{\alpha\beta} n_i^{\alpha\beta},
\end{aligned} \tag{4.12}$$

and for the frictional functionals we have

$$j_h^{LM}(\mathbf{w}_h) \simeq$$

$$\simeq \int_{\Gamma_{ch}^{LM}} \mathcal{F}_{ch}^{LM}\left([\mathbf{v}_{ht}]^{LM}\right)\frac{1}{\varepsilon}\left([v_{hn}]^{LM} - d_{hn}^{LM}\right)_+ \nabla\Psi_\delta\left([\mathbf{v}_{ht}]^{LM}\right).[\mathbf{w}_{ht}]^{LM}\, ds,$$

$$j_h(\mathbf{w}_h) = \sum_{s,m} j_h^{sm}(\mathbf{w}_h),\, s,m \in [1,...,r],\ where$$

$$j_h^{sm}(\mathbf{w}_h) \simeq$$

$$\simeq \int_{\Gamma_{ch}^{sm}} \mathcal{F}_{ch}^{sm}([\mathbf{v}_{ht}]^{sm})\frac{1}{\varepsilon}([v_{hn}]^{sm} - d_{hn}^{sm})_+ \nabla\Psi_\delta([\mathbf{v}_{ht}]^{sm}).[\mathbf{w}_{ht}]^{sm}\, ds,$$

for the generalized Coulombian friction case and (4.13)

$$j_h^{LM}(\mathbf{w}_h) = \int_{\Gamma_{ch}^{LM}} g_{ch}^{LM}\left|[\mathbf{w}_{ht}]^{LM}\right| ds \simeq \int_{\Gamma_{ch}^{LM}} g_{ch}^{LM}\nabla\Psi_\delta\left([\mathbf{w}_{ht}]^{LM}\right) ds,$$

$$j(\mathbf{w}_h) = \sum_{s,m} j_h^{sm}(\mathbf{w}),\, s,m \in [1,...,r],\ where$$

$$j_h^{sm}(\mathbf{w}) = \int_{\Gamma_{ch}^{sm}} g_{ch}^{sm}\left|[\mathbf{w}_{ht}]^{sm}\right| ds \simeq \int_{\Gamma_{ch}^{sm}} g_{ch}^{sm}\nabla\Psi_\delta([\mathbf{w}_{ht}]^{sm})\, ds,$$

for the Tresca model of friction,

referring to the GLL points, where g_{ch}^{LM} and g_{ch}^{sm} are approximations of g_c^{LM} and g_c^{sm} that were defined above, and where in the terms $j_h^{LM}(\mathbf{w}_h)$ and $j_h(\mathbf{w}_h)$ we replaced the norms in the frictional terms by the differentiable, convex approximations $\Psi_\delta(.)$ satisfying $|\Psi_\delta(.) - |.|| \le \delta$ and $\nabla\Psi_\delta = 0$ as usual (Eck et al. (2005)). In our study we will use the Tresca model of friction.

Thus

$$j_h^{LM}(\mathbf{w}_h) \simeq \int_{\Gamma_{ch}^{LM}} g_{ch}^{LM}\nabla\Psi_\delta\left([\mathbf{w}_{ht}]^{LM}\right) ds \simeq \sum_{\partial T_{hi}} \int_{\partial T_{hi}} g_{ch}^{LM}\nabla\Psi_\delta\left([\mathbf{w}_{ht}]^{LM}\right) ds,$$

$$\int_{\partial T_{hi}} g_{ch}^{LM}\nabla\Psi_\delta\left([\mathbf{w}_{ht}]^{LM}\right) ds = \int_{-1}^{1}\int_{-1}^{1} g_{ch}^{LM}\nabla\Psi_\delta\left([\mathbf{w}_{ht}]^{LM}\right) J_{\partial T_{hi}} d\xi d\eta \simeq$$

$$\simeq \sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta} \omega_\alpha\omega_\beta J_{\partial T_{hi}}^{\alpha\beta}\left(g_{ch}^{LM}\right)^{\alpha\beta} . \sum_{i=1}^{3}\mathbf{e}_{xi}$$

$$\left[(\partial_i\xi)^{\alpha'\beta'\gamma'}\sum_{\alpha=0}^{\eta_\alpha}\Psi_\delta^{\alpha\beta'\gamma'}\theta'_\alpha(\xi_{\alpha'}) + (\partial_i\eta)^{\alpha'\beta'\gamma'}\sum_{\beta=0}^{\eta_\beta}\Psi_\delta^{\alpha'\beta\gamma'}\theta'_\beta(\xi_{\beta'}) + (\partial_i\zeta)^{\alpha'\beta'\gamma'}\sum_{\gamma=0}^{\eta_\gamma}\Psi_\delta^{\alpha'\beta'\gamma}\theta'_\gamma(\xi_{\gamma'})\right], \tag{4.14}$$

referring to the GLL points, and where $\Psi_\delta^{\alpha\beta'\gamma'} \equiv \Psi_\delta\left([\mathbf{w}_{ht}(\xi_\alpha, \eta_{\beta'}, \zeta_{\gamma'})]^{LM}\right)$, etc. Similar expressions we will obtain for the frictional term $j_h^{sm}(\mathbf{w}_h)$.

The gravity terms lead to the form

$$\int_{\Omega_h^L \cup \Omega_h^M} \rho_h \mathbf{v}_h.\overline{\mathbf{H}}_h.\mathbf{w}_h d\mathbf{x} \simeq \sum_{T_{hi}} \int_{T_{hi}} \rho_h \mathbf{v}_h.\overline{\mathbf{H}}_h.\mathbf{w}_h d\mathbf{x},$$

$$\int_{T_{hi}} \rho_h \mathbf{v}_h.\overline{\mathbf{H}}_h.\mathbf{w}_h d\mathbf{x} \simeq \sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma} \omega_\alpha\omega_\beta\omega_\gamma J_{T_{hi}}^{\alpha\beta\gamma}\rho_h^{\alpha\beta\gamma}\sum_{i,j=1}^{3} w_{hi}^{\alpha\beta\gamma}\overline{H}_{hij}^{\alpha\beta\gamma}v_{hj}^{\alpha\beta\gamma} \tag{4.15}$$

and the terms coupling between fluid and solid regions lead to

$$\int p_h \mathbf{n}.\mathbf{w}_h ds \simeq \sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta} \omega_\alpha\omega_\beta J_{\partial T_{hi}}^{\alpha\beta} p_h^{\alpha\beta}(t)\sum_{i=1}^{3} w_{hi}^{\alpha\beta} n_i^{\alpha\beta}. \tag{4.16}$$

To approximate the weak formulation of the problem in the outer fluid core, we introduce the approximation of the potential φ_h in terms of Lagrange polynomials as follows

$$\varphi_h\left(\mathbf{x}\left(\xi,\eta,\zeta\right),t\right)\simeq\sum_{\sigma,\tau,\nu=0}^{\eta_\sigma,\eta_\tau,\eta_\nu}\varphi_h^{\sigma\tau\nu}\left(t\right)\theta_\sigma\left(\xi\right)\theta_\tau\left(\eta\right)\theta_\nu\left(\zeta\right), \tag{4.17}$$

and the scalar test function w_h by

$$w_h\left(\mathbf{x}\left(\xi,\eta,\zeta\right)\right)\simeq\sum_{\alpha,\beta,\gamma=0}^{\eta_\sigma,\eta_\tau,\eta_\nu}w_h^{\alpha\beta\gamma}\theta_\alpha\left(\xi\right)\theta_\beta\left(\eta\right)\theta_\gamma\left(\zeta\right). \tag{4.18}$$

Then the first term on the left-hand side in (3.7) corresponding to the weak formulation of the wave equation for the outer core can be written as

$$\begin{gathered}\int_{\Omega_h^{OC}}\rho_h k_h^{-1}w_h\frac{\partial^2\varphi_h\left(t\right)}{\partial t^2}d\mathbf{x}\simeq\sum_{T_{hi}}\int_{T_{hi}}\rho_h k_h^{-1}w_h\frac{\partial^2\varphi_h\left(t\right)}{\partial t^2}d\mathbf{x},\\ \int_{T_{hi}}\rho_h k_h^{-1}w_h\frac{\partial^2\varphi_h\left(t\right)}{\partial t^2}d\mathbf{x}\simeq\sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma}\omega_\alpha\omega_\beta\omega_\gamma J_{T_{hi}}^{\alpha\beta\gamma}\rho_h^{\alpha\beta\gamma}\left(k_h^{\alpha\beta\gamma}\right)^{-1}w_h^{\alpha\beta\gamma}\frac{\partial^2\varphi_h^{\alpha\beta\gamma}\left(t\right)}{\partial t^2}\end{gathered} \tag{4.19}$$

and the corresponding "mass" matrix for the element T_{hi} is diagonal from the same reason as in the elemental "mass" matrix derived above.

The first term on the right-hand side of (3.7) (i.e., the term $\int_{\Omega_h^{OC}}\nabla w_h.\nabla\varphi_h d\mathbf{x}\simeq\sum_{T_{hi}}\int_{T_{hi}}\nabla w_h.\nabla\varphi_h d\mathbf{x}$) after approximation in terms of Lagrange polynomials can be written as

$$\begin{gathered}\int_{T_{hi}}\nabla w_h.\nabla\varphi_h d\mathbf{x}\simeq\sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma}w_h^{\alpha\beta\gamma}[\omega_\beta\omega_\gamma\sum_{\alpha'=0}^{\eta_{\alpha'}}\omega_{\alpha'}J_{T_{hi}}^{\alpha'\beta\gamma}\left(\frac{\partial\varphi_h}{\partial x_1}\right)^{\alpha'\beta\gamma}\theta'_\alpha\left(\xi_{\alpha'}\right)+\\ +\omega_\alpha\omega_\gamma\sum_{\beta'=0}^{\eta_{\beta'}}\omega_{\beta'}J_{T_{hi}}^{\alpha\beta'\gamma}\left(\frac{\partial\varphi_h}{\partial x_2}\right)^{\alpha\beta'\gamma}\theta'_\beta\left(\xi_{\beta'}\right)+\omega_\alpha\omega_\beta\sum_{\gamma'=0}^{\eta_{\gamma'}}\omega_{\gamma'}J_{T_{hi}}^{\alpha\beta\gamma'}\left(\frac{\partial\varphi_h}{\partial x_3}\right)^{\alpha\beta\gamma'}\theta'_\gamma\left(\xi_{\gamma'}\right)],\end{gathered} \tag{4.20}$$

where

$$\left(\frac{\partial\varphi_h}{\partial x_i}\right)^{\alpha\beta\gamma}=\sum_{\alpha=0}^{\eta_\alpha}\varphi_h^{\alpha\beta'\gamma'}\theta'_\alpha\left(\xi_{\alpha'}\right)\frac{\partial\xi}{\partial x_i}+\sum_{\beta=0}^{\eta_\beta}\varphi_h^{\alpha'\beta\gamma'}\theta'_\beta\left(\eta_{\beta'}\right)\frac{\partial\eta}{\partial x_i}+\sum_{\gamma=0}^{\eta_\gamma}\varphi_h^{\alpha'\beta'\gamma}\theta'_\gamma\left(\zeta_{\gamma'}\right)\frac{\partial\zeta}{\partial x_i}.$$

The second term in (3.7) will be approximated in terms of Lagrange polynomials as

$$\begin{gathered}\int_{T_{hi}}\rho_h k_h^{-1}w_h\left[\mathbf{g}_h.\left(\nabla\varphi_h+\boldsymbol{\psi}_h\right)-\Phi_h\right]d\mathbf{x}\simeq\\ \simeq\sum_{\alpha,\beta,\gamma=0}^{\eta_\alpha,\eta_\beta,\eta_\gamma}\omega_\alpha\omega_\beta\omega_\gamma J_{T_{hi}}^{\alpha\beta\gamma}\rho_h^{\alpha\beta\gamma}\left(k_h^{\alpha\beta\gamma}\right)^{-1}w_h^{\alpha\beta\gamma}.\sum_{i=1}^{3}g_{ih}^{\alpha\beta\gamma}\left[\left(\frac{\partial\varphi_h}{\partial x_i}\right)^{\alpha\beta\gamma}+\boldsymbol{\psi}_{hi}^{\alpha\beta\gamma}-\Phi_h^{\alpha\beta\gamma}\right],\end{gathered} \tag{4.21}$$

where the Cowling approximation was applied.

The further two coupling terms in (3.7) will be approximated as follows

$$\begin{gathered}\int p_h\mathbf{n}.\mathbf{w}_h ds\simeq\sum_{\partial T_{hi}}\int_{\partial T_{hi}}p_h\mathbf{n}.\mathbf{w}_h ds,\\ \int_{\partial T_{hi}}p_h\mathbf{n}.\mathbf{w}_h ds\simeq\sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta}\omega_\alpha\omega_\beta J_{\partial T_{hi}}^{\alpha\beta}p_h^{\alpha\beta}\left(t\right)\sum_{i=1}^{3}w_{hi}^{\alpha\beta}n_i^{\alpha\beta},\end{gathered} \tag{4.22}$$

$$
\begin{aligned}
\int w_h.\mathbf{n}.\mathbf{v}_h ds &\simeq \sum_{\partial T_{hi}} \int_{\partial T_{hi}} w_h.\mathbf{n}.\mathbf{v}_h ds, \\
\int_{\partial T_{hi}} w_h.\mathbf{n}.\mathbf{v}_h ds &\simeq \sum_{\alpha,\beta=0}^{\eta_\alpha,\eta_\beta} \omega_\alpha \omega_\beta J_{\partial T_{hi}}^{\alpha\beta} w_h^{\alpha\beta} \sum_{i=1}^{3} v_{hi}^{\alpha\beta} n_i^{\alpha\beta},
\end{aligned} \tag{4.23}
$$

referring to the GLL points, where $J_{\partial T_{hi}}^{\alpha\beta}$ is the surface Jacobian $J_{\partial T_{hi}}$.

In the variational formulation (3.8) the separate terms can be approximated by a similar way as in the cases of similar terms introduced above.

Let $\tilde{\varphi}_h$ be an admissible potential test function which is continuous throughout the approximated interfaces $\partial\Omega_h \cup \Gamma_{ch}$, where by Γ_{ch} we denote the approximations of all interface boundaries inside the Earth, then

$$
\begin{aligned}
&\int_{\Sigma_h} \nabla\Phi_h.\nabla\tilde{\varphi}_h d\mathbf{x} - \int_{\partial\Sigma_h} \nabla\Phi_h^{ext}.\mathbf{n}.\tilde{\varphi}_h ds = \\
&= -4\pi G \left[\int_{\Omega_h^L \cup \Omega_h^M \cup \Omega_h^{IC}} \rho_h \mathbf{v}_h.\nabla\tilde{\varphi}_h d\mathbf{x} + \int_{\Omega_h^{OC}} \rho_h \left(\nabla\varphi_h + \boldsymbol{\psi}_h\right).\nabla\tilde{\varphi}_h d\mathbf{x} \right],
\end{aligned} \tag{4.24}
$$

where

$$
\int_{\partial\Sigma} \nabla\Phi_h^{ext}.\mathbf{n}\tilde{\varphi}_h ds = -\frac{1}{a} \sum_{l=0}^{\infty} (l+1) \sum_{m=-l}^{l} \Phi_{hlm}^{int}(a)\, \tilde{\varphi}_{hl}^{m}(a).
$$

In realization the infinite sum can be limited by some $l = l_{\max}$ (see Chaljub and Valette (2004)).

Then, after the approximations in terms of Lagrange polynomials, the problem leads to the second-order ordinary equation at each GLL point with diagonal "mass" matrix (see e.g. Komatitsch et al. (2005), Chaljub and Valette (2004)), which can be solved by numerical methods for systems of ordinary differential equations (see e.g. Hartman (1973), Butcher (1987), Wood (1990), Belytschko et al. (2000)). We saw that the choice of the Gauss-Lobatto-Legendre integration is very useful for numerical computation because the matrix representations of the L^2-scalar products lead to diagonal matrices, that are advantageous of the used algorithms. The equation (4.24) will be approximated by the Galerkin-SEM method.

Let $\mathbf{U}_{sh}$ denote the displacement vector of the global solid parts of the Earth, $\mathbf{U}_{\varphi h}$ and $\mathbf{U}_{\psi h}$ denote the nodal values of the displacement potential and of vector field in the fluid of the outer core and $\boldsymbol{\Phi}_h$ stand for the nodal values of the Eulerian perturbation of the gravitational potential (the PGP) and let $\mathbf{U}_h = (\mathbf{U}_{sh}, \mathbf{U}_{\varphi h}, \mathbf{U}_{\varphi h}, \boldsymbol{\Phi}_h)^T$ denote the global vector of the nodal values in the Earth. Then after the spatial discretization we have to solve a system of ordinary differential equations in time with coupling terms

$$
\begin{aligned}
\mathbb{M}_{sh}\mathbf{U}_{sh}''(t) + \mathbb{K}_{sh}\mathbf{U}_{sh}(t) + \mathbb{C}_{sfh}\mathbf{U}_{\psi h}''(t) + \mathbb{C}_{Gh}\boldsymbol{\Phi}_h(t) &= \hat{\mathbf{F}}_h(t), \\
\mathbb{M}_{fh}\mathbf{U}_{\psi h}''(t) + \mathbb{K}_{fh}\left(\mathbf{U}_{\psi h}, \mathbf{U}_{\varphi h}\right)(t) + \mathbb{C}_{fsh}\mathbf{U}_{sh}(t) &= \mathbf{0}, \\
\mathbb{M}_{fh}\mathbf{U}_{\varphi h}''(t) + \mathbb{C}_{\psi\varphi\Phi h}\left(\mathbf{U}_{\psi h}'', \mathbf{U}_{\psi h}, \mathbf{U}_{\varphi h}, \boldsymbol{\Phi}_h\right)(t) &= \mathbf{0}, \\
\mathbb{K}_{\Phi h}\boldsymbol{\Phi}_h(t) &= \mathbb{C}_{s\psi\varphi h}\left(\mathbf{U}_{sh}, \mathbf{U}_{\psi h}, \mathbf{U}_{\varphi h}\right)(t),
\end{aligned} \tag{4.25}
$$

where $\mathbb{M}_{sh}$ is the matrix representation of the L^2-scalar product weighted by the density (the mass matrix in the solid parts of the Earth); $\mathbb{M}_{fh}$ is the matrix representation of the scalar product in the fluid part of the Earth with the weight $\rho_h^{-1}k_h$ representing the corresponding speed of sound; $\mathbb{K}_{sh}$, $\mathbb{K}_{fh}$ are the stiffness matrices arising from the space approximations of the volume integrals in the variational formulations; $\hat{\mathbf{F}}_h(t) = \mathbf{F}_h(t) + \mathbb{J}_h(\mathbf{U}_{sh}(t))$ is the approximation of the acting forces, that is, the approximations of earthquake source term $\mathbf{F}_h(t)$ and of the frictional and penalty terms summed in $\mathbb{J}_h(.)$. Discretizating the surface integrals in the variational formulations then that yield to the solid-fluid coupling matrices $\mathbb{C}_{sfh}$ and $\mathbb{C}_{fsh}$; $\mathbb{C}_{Gh}$ is the matrix representation of the gradient term, and moreover, $\mathbb{K}_{\Phi h}$ and $\mathbb{C}_{s\psi\varphi h}$ correspond to the matrix representation of the Poisson-Laplace operator and of the divergence operator, where $\mathbb{C}_{s\psi\varphi h}$ contains the factor $4\pi\rho_h G$. Moreover, $\mathbb{C}_{\psi\varphi\Phi h}$ corresponds to the discretization of the volume integral in terms of fluid outer core and involves the operation on $\mathbf{U}''_{\psi h}$, $\mathbf{U}_{\psi h}$, $\nabla\mathbf{U}_{\varphi h}$ and $\mathbf{\Phi}_h$.

Completing the system of ordinary differential equations (4.25) we then obtain the global second-order system of the form

$$\mathbb{M}_h \mathbf{U}''_h(t) + \mathbb{K}_h \mathbf{U}_h(t) + \mathbb{C}_h \mathbf{U}_h(t) = \mathbb{F}_h(t) \tag{4.26}$$

Since the global mass matrix is diagonal, then the time discretization of the second-order ordinary differential equation (4.26) can be based on a classical explicit second-order finite-difference scheme of the Newmark β-method ($\beta \geq \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2, \gamma \geq \frac{1}{2}$) with acceptable values $\gamma = 1$, $\beta \geq \frac{9}{16}$ (Hughes (1987), Ward (1990), Belytschko et al. (2000)). These schemes are conditionally stable, and that the Courant criterion is valid, that is, the Courant stability condition gives the relation between the size of mesh steps "h" and the longitudinal wave velocity "c_L" crossing the element,

$$\triangle t \leq \triangle t_{crit} \sim \delta \frac{h}{c_L}, \tag{4.27}$$

where $\delta \in (0.2, 0.9)$ is a reduction factor which was determined from the numerical experiments, which is necessary because of the destabilizing effect of round-off and the possibility of rapidly varying material properties.

Note that in the global algorithm the last equation of (4.25) due to the coupling terms, that are unknown at time t_i of the Newmark algorithm, will be solved firstly in every time step of the Newmark algorithm by using some of iterative methods (e.g. conjugate gradient method (CGM), preconditioned CGM, etc.). Subsequently the wave equation corresponding to the outercore will be solved, because the coupling term $\mathbb{C}_{sfh}\mathbf{U}''_{\psi h}$ is unknown at time t_i of the Newmark algorithm (for more details see Belytschko et al. (2000), Deville et al. (2002)).

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