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## Institute of Computer Science Academy of Sciences of the Czech Republic

# A Bendixson-Type Theorem for Eigenvalues of Interval Matrices 

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Technical report No. V-1184
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## Abstract:

We describe a rectangle in complex plane which encloses all eigenvalues of an interval matrix. Special cases of symmetric and skew-symmetric interval matrices are also considered. ${ }^{11}$


Keywords:
Interval matrix, eigenvalues, enclosure, symmetric interval matrix, skew-symmetric interval matrix.

[^0]
## 1 Introduction

In this report we describe a rectangle in complex plane enclosing all eigenvalues of an interval matrix (Theorem 1 and Corollary 2). Special cases of a symmetric or skew-symmetric interval matrix are handled in Corollaries 3 and 4. These results are obtained as simplifications of Theorem 2 in [2].

## 2 The results

Our main result is formulated as follows.
Theorem 1. Let $\boldsymbol{A}=\left[A_{c}-\Delta, A_{c}+\Delta\right]$ be a square interval matrix. Then for each eigenvalue $\lambda$ of each $A \in \boldsymbol{A}$ we have

$$
\begin{align*}
\lambda_{\min }\left(A_{c}^{\prime}\right)-\varrho\left(\Delta^{\prime}\right) & \leq \operatorname{Re} \lambda \leq \lambda_{\max }\left(A_{c}^{\prime}\right)+\varrho\left(\Delta^{\prime}\right)  \tag{2.1}\\
-\sigma_{\max }\left(A_{c}^{\prime \prime}\right)-\varrho\left(\Delta^{\prime}\right) & \leq \operatorname{Im} \lambda \leq \sigma_{\max }\left(A_{c}^{\prime \prime}\right)+\varrho\left(\Delta^{\prime}\right) \tag{2.2}
\end{align*}
$$

where

$$
\begin{aligned}
A_{c}^{\prime} & =\frac{1}{2}\left(A_{c}+A_{c}^{T}\right) \\
A_{c}^{\prime \prime} & =\frac{1}{2}\left(A_{c}-A_{c}^{T}\right) \\
\Delta^{\prime} & =\frac{1}{2}\left(\Delta+\Delta^{T}\right)
\end{aligned}
$$

Proof. In [2, Thm. 2] it is proved that under the current assumptions and notation there holds

$$
\begin{aligned}
\lambda_{\min }\left(A_{c}^{\prime}\right)-\varrho\left(\Delta^{\prime}\right) & \leq \operatorname{Re} \lambda \leq \lambda_{\max }\left(A_{c}^{\prime}\right)+\varrho\left(\Delta^{\prime}\right) \\
\lambda_{\min }\left(A_{c}^{\prime \prime \prime}\right)-\varrho\left(\Delta^{\prime \prime}\right) & \leq \operatorname{Im} \lambda \leq \lambda_{\max }\left(A_{c}^{\prime \prime \prime}\right)+\varrho\left(\Delta^{\prime \prime}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{c}^{\prime} & =\frac{1}{2}\left(A_{c}+A_{c}^{T}\right) \\
\Delta^{\prime} & =\frac{1}{2}\left(\Delta+\Delta^{T}\right) \\
A_{c}^{\prime \prime \prime} & =\left(\begin{array}{cc}
0 & A_{c}^{\prime \prime} \\
A_{c}^{\prime \prime T} & 0
\end{array}\right) \\
\Delta^{\prime \prime} & =\left(\begin{array}{cc}
0 & \Delta^{\prime} \\
\Delta^{\prime T} & 0
\end{array}\right)
\end{aligned}
$$

This proves (2.1). To prove (2.2), we shall use the Jordan-Wielandt theorem [3, Thm. 4.2] according to which a matrix

$$
\left(\begin{array}{cc}
0 & B \\
B^{T} & 0
\end{array}\right)
$$

(with $B \in \mathbb{R}^{n \times n}$ ) has eigenvalues $\sigma_{1}(B) \geq \cdots \geq \sigma_{n}(B) \geq-\sigma_{n}(B) \geq \ldots \geq-\sigma_{1}(B)$. Thus $\lambda_{\max }\left(A_{c}^{\prime \prime \prime}\right)=\sigma_{\max }\left(A_{c}^{\prime \prime}\right), \lambda_{\min }\left(A_{c}^{\prime \prime \prime}\right)=-\sigma_{\max }\left(A_{c}^{\prime \prime}\right)$, and $\varrho\left(\Delta^{\prime \prime}\right)=\sigma_{\max }\left(\Delta^{\prime}\right)=\varrho\left(\Delta^{\prime}\right)$ (because $\Delta^{\prime}$ is symmetric and nonnegative), whereby we are done.

For other formulations, let us introduce the following notation for the set of all eigenvalues:

$$
\Lambda(\boldsymbol{A})=\left\{\lambda \in \mathbb{C} \mid A x=\lambda x, x \in \mathbb{C}^{n}, x \neq 0, A \in \boldsymbol{A}\right\}
$$

Notice that $\Lambda(\boldsymbol{A})$ is symmetric with respect to the real axis because, as well known, each $A \in \boldsymbol{A}$ together with an eigenvalue $\lambda=a+b i$ also possesses the eigenvalue $\bar{\lambda}=a-b i$.

Corollary 2. For each square interval matrix $\boldsymbol{A}=\left[A_{c}-\Delta, A_{c}+\Delta\right]$ there holds

$$
\Lambda(\boldsymbol{A}) \subseteq\left[\lambda_{\min }\left(A_{c}^{\prime}\right)-\varrho\left(\Delta^{\prime}\right), \lambda_{\max }\left(A_{c}^{\prime}\right)+\varrho\left(\Delta^{\prime}\right)\right] \times\left[-\sigma_{\max }\left(A_{c}^{\prime \prime}\right)-\varrho\left(\Delta^{\prime}\right), \sigma_{\max }\left(A_{c}^{\prime \prime}\right)+\varrho\left(\Delta^{\prime}\right)\right]
$$

where $A_{c}^{\prime}, A_{c}^{\prime \prime}$ and $\Delta^{\prime}$ are as in Theorem 1.
This is merely a reformulation of Theorem 1 expressing the set of all eigenvalues of $\boldsymbol{A}$ as a subset of a rectangle in complex plane. This rectangle is also symmetric with respect to real axis.

For an interval matrix $\boldsymbol{A}$, its transpose is defined by

$$
\boldsymbol{A}^{T}=\left\{A^{T} \mid A \in \boldsymbol{A}\right\}
$$

A square interval matrix $\boldsymbol{A}$ is called symmetric if $\boldsymbol{A}^{T}=\boldsymbol{A}$, and it is said to be skewsymmetric if $\boldsymbol{A}^{T}=-\boldsymbol{A}$. It is not difficult to prove that $\boldsymbol{A}=\left[A_{c}-\Delta, A_{c}+\Delta\right]$ is symmetric if and only if both $A_{c}$ and $\Delta$ are symmetric, and that it is skew-symmetric if and only if $A_{c}$ is skew-symmetric and $\Delta$ is symmetric.

The next two corollaries describe enclosures of sets of eigenvalues of symmetric and skewsymmetric interval matrices, respectively.

Corollary 3. For a symmetric interval matrix $\boldsymbol{A}=\left[A_{c}-\Delta, A_{c}+\Delta\right]$ we have

$$
\Lambda(\boldsymbol{A}) \subseteq\left[\lambda_{\min }\left(A_{c}\right)-\varrho(\Delta), \lambda_{\max }\left(A_{c}\right)+\varrho(\Delta)\right] \times[-\varrho(\Delta), \varrho(\Delta)]
$$

Proof. Obviously, in this case $A_{c}^{\prime}=A_{c}, A_{c}^{\prime \prime}=0$ and $\Delta^{\prime}=\Delta$, from which the result follows.

Corollary 4. For a skew-symmetric interval matrix $\boldsymbol{A}=\left[A_{c}-\Delta, A_{c}+\Delta\right]$ we have

$$
\Lambda(\boldsymbol{A}) \subseteq[-\varrho(\Delta), \varrho(\Delta)] \times\left[-\sigma_{\max }\left(A_{c}\right)-\varrho(\Delta), \sigma_{\max }\left(A_{c}\right)+\varrho(\Delta)\right]
$$

Proof. This is a consequence of the facts that $A_{c}^{\prime}=0, A_{c}^{\prime \prime}=A_{c}$, and $\Delta^{\prime}=\Delta$.
Notice that in the latter case the enclosure of the eigenvalue set is symmetric with respect to the origin.

## 3 Acknowledgement

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## Bibliography

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[2] J. Rohn, Bounds on eigenvalues of interval matrices, Zeitschrift für Angewandte Mathematik und Mechanik, Supplement 3, 78 (1998), pp. S1049-S1050. 1
[3] G. W. Stewart and J. Sun, Matrix Perturbation Theory, Academic Press, San Diego, 1990. [1


[^0]:    ${ }^{1}$ Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4] x_{1}+[-2,1] x_{2}=[-2,2],[-1,2] x_{1}+[2,4] x_{2}=[-2,2]$ (Barth and Nuding [1])).

