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Abstract:

When proposing and processing uncertainty decision making algorithms of various kinds and purposes we meet more and more often probability distributions ascribing to random events non-numerical uncertainty degrees. The reason is that we have to process systems of uncertainties for which the classical conditions like σ-additivity or linear ordering of values are too restrictive to define sufficiently closely the nature of uncertainty we would like to specify and process. For the case of non-numerical uncertainty degrees at least the two criteria may be considered. First systems with rather complicated, but sophisticated and nontrivially formally analyzable uncertainty degrees. E.g., uncertainties supported by some algebras or partially ordered structures. Contrary, we may consider more easy non-numerical, but on the intuitive level interpretable relations. Well-known examples of such structures are set-valued possibilistic measures. Some perhaps interesting particular results in this direction will be introduced and analyzed in the contribution.

Keywords:
probability measures, possibility measures, non-numerical uncertainty degrees, set-valued uncertainty degrees, possibilistic uncertainty and set-valued entropy functions.

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1 Introduction

Since the very origins of modern mathematics, measure theory may be taken as almost the synonymum for mathematical theory of size quantification with the most general and abstract structures in which sizes take their values. On the other side, the structures over quantity degrees should be rich and flexible enough to enable to define and process non-trivial deductions with non-trivial results on quantity degrees and their relations.

In the measure theory and, consequently, in probability theory, the sizes of sets and uncertainty (in the sense of randomness as well as of fuzziness and possibility degrees) were quantified by numbers, going from finite natural numbers to rational and then real (or, perhaps, complex-valued numbers). The development of real-valued probability theory took their tops by Kolmogorov axiomatic theory of probability as systematically explained and applied in [4], [6] or elsewhere.

On the other side, the correctness and legality of application of the classical probability theory and its consequences (mathematical statistics, Shannon entropy and information theory, ...) to problems from real life is based on the assumption that certain non-trivial assumptions are satisfied and verified (the precise knowledge of apriori probabilities, statistical independence of some random variables and/or precisely known type and degrees of their dependencies together with the detailed conditional probabilities, ...). Even when enormous lot of work has been done till now within the framework of classical probability theory and statistics as well as reasonable processing of non-fully given and known input probabilistic data are processed, the demands for qualitatively different alternative tools for uncertainty (in the sense of randomness as well as fuzziness) processing are requested.

Qualitatively different models of uncertainty quantification and processing, even if still with numerical degrees, are real-valued fuzzy sets, defined by mappings taking the basic space Ω into the unit interval $[0,1]$, hence, extending the binary-valued characteristic functions of standard set, to functions with values in the closed interval $[0,1]$.

The pioneering Zadeh's idea of fuzzy sets emerged in 1965 in [12, 5] and, as soon as in 1967. J. A. Goguen entered on scene with the further step – fuzzy sets with non-numerical membership degrees. In particular, J. A. Goguen considered uncertainty, in the sense of fuzziness degrees, as elements of complete lattice, let us recall that complete lattice is defined as the p.o.set (partially ordered set) in which for each nonempty subset supremum and infimum are defined.

Till now, we have re-called models for uncertainty quantification and processing where the uncertainty degrees take their values in more and more general and less intuitive structures (natural numbers, rationals, real numbers, lattices, semilattices, ...), so that set-valued possibility degrees, occuring in the title of this text seem to be a rather strong back step which deserves a rather persuasive explanation. When quantifying sizes by numbers we have to keep in mind that this introduces into the model the complete ordering of numbers which need not correspond to sizes of pieces of uncertainty in question. Among the structures working with uncertainties and keeping in mind the idea to classify as incomparable also set-quantified degrees of uncertainty with the same values of real-valued measures, set-valued possibilistic measures seem to be sufficiently elastic and resilient to be taken as intuitively acceptable non-numerical size-quantifying mathematical model.

Let us survey, very briefly, the contents of particular sections. Our aim will be to minimize the quantity and complexity of preliminaries necessary for a non-fully oriented reader in order to understand the text. In Section 2 we introduce the structures for quantifying uncertainty (or uncertainties) by set values. It is perhaps worth being so-called just now that probability measure and probability theory is based on standard combination of set-valued uncertainty quantification (random events are sets) with also the standard real-valued quantification of the set-valued random events.

In Section 3 we introduce three alternative ways how to define mappings keeping at least some properties of conditional probabilities. This problem seems to be promising for some new and interesting results In Section 4 we define and analyze set-valued entropy function over set-valued possibilistic function with the aim to solve the problem arising when the possibilistic distribution takes the maximum value $\mathbf{1}_{\mathcal{T}}(=X)$ for at last two different arguments. Analogously to the case of real-valued probability measure the Shannon entropy function [10] takes the maximum value $\mathbf{1}_{\mathcal{T}}(=X)$ so that the qualities of this entropy function cannot be used as a tool for neither a partial ordering of different alternatives of possibilistic distrubution when choosing the best one for the application in question.

Very roughly speaking, the idea is to modify the space of values in which set-valued entropy function takes its values, in such a way that the supremum value of the set-valued entropy function is taken for just one value ω_0 from the basic space Ω of the possibilistic distribution in question. This goal will be achieved by introducing a sophisticated equivalence relation on the basic possibilistic space of our model, such that in the resulting factor space the supremum is reached in only one value, so that the case of unit entropy value for the entropy function does not menace our application of set-valued entropy functions.

Finally, in Section 5 we consider the compositions of set-valued possibilistic distributions.

2 Set-valued possibilistic distributions

Let Ω and X be nonempty sets, let $\mathcal{P}(X)$ be the set of all subsets of X (the power-set over X), let $\pi : \Omega \to \mathcal{P}(X)$ be a mapping ascribing to each $\omega \in \Omega$ a subset $\pi(\omega) \subset X$ (i.e., $\pi(\omega) \in \mathcal{P}(X)$). The mapping π is called *set-valued possibilistic distribution* on Ω, if $\bigcup_{\omega\in\Omega} \pi(\omega) = X$.

For each $A \subset \Omega$, set $\Pi(A) = \bigcup_{\omega\in A} \pi(\omega)$. The mapping $\Pi : \mathcal{P}(\omega) \to \mathcal{P}(X)$ is called the $\mathcal{P}(X)$-*valued possibilistic measure* induced on $\mathcal{P}(\Omega)$ by the set-valued possibilistic distribution π on Ω. The important characteristic of the $\mathcal{P}(X)$-valued possibilistic distribution π (and of the related $\mathcal{P}(X)$-valued possibilistic measure Π induced by π) is the so called *possibilistic* (or *Sugeno*) *entropy* defined by the *Sugeno integral* $I(\pi)$. For the particular case of the set-valued possibilistic distribution π on Ω defined as above the definition reads as follows:

$$I(\pi) = \bigcup_{\omega\in\Omega} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] \subset X. \tag{2.1}$$

E.g., in the most simple case when $\Omega = X$ and $\pi(\omega) = \{\omega\}$, we obtain that $\Pi(A) = \bigcup_{\omega\in A} \pi(\omega) = \bigcup_{\omega\in A}\{\omega\} = A$. For the entropy $I(\pi)$ we obtain that

$$I(\pi) = \bigcup_{\omega\in\Omega} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] = \bigcup_{\omega\in\Omega} ((\Omega - \{\omega\}) \cap \{\omega\}) = \emptyset \tag{2.2}$$

let us recall that the empty subset of X denotes the zero element of the complete lattice (complete Boolean algebra, as a matter of fact) $\langle \mathcal{P}(X), \subseteq \rangle$.

Fact 2.1 *Let Ω and X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω such that, for each $\omega_1, \omega_2, \omega_1 \neq \omega_2$, $\pi(\omega_1)\cap\pi(\omega_2) = \emptyset$ holds. Then for each $A, B \subset X, A\cap B = \emptyset$, we obtain that $\Pi(A) \cap \Pi(B) = \emptyset$ holds.*

Proof: An easy calculation yields that

$$\Pi(A) \cap \Pi(B) = \left(\bigcup_{\omega\in A} \pi(\omega)\right) \cap \left(\bigcup_{\omega\in B} \pi(\omega)\right) =$$

$$= \bigcup_{\omega_1\in B} \left[\left(\bigcup_{\omega\in A} \pi(\omega)\right) \cap \pi(\omega_1)\right] = \bigcup_{\omega_1\in B} \bigcup_{\omega\in A} (\pi(\omega_1) \cap \pi(\omega)) = \emptyset, \tag{2.3}$$

as the sets A and B are disjoint. The assertion is proved. □

Lemma 2.1 *Let Ω and X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω. Then $I(\pi) = \emptyset$ iff $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ for each $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$.*

Proof: If $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ for each $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$, then

$$I(\pi) = \bigcup_{\omega\in\Omega} [\pi(\Omega - \{\omega\}) \cap \pi(\omega)] = \bigcup_{\omega\in\Omega} [\pi(\Omega - \{\omega\}) \cap \Pi(\{\omega\})] = \emptyset \tag{2.4}$$

holds, due to Fact 2.1.

On the other side, let $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$, be such that $\pi(\omega_1) \cap \pi(\omega_2) \neq \emptyset$ Then $\omega_2 \in \Omega - \{\omega_1\}$ holds, so that

$$\Pi(\Omega - \{\omega_1\}) \cap \pi(\omega_1) \supset \pi(\omega_2) \cap \pi(\omega_1) \neq \emptyset, \tag{2.5}$$

consequently,

$$I(\pi) \supset \Pi(\Omega - \{\omega_1\}) \cap \pi(\omega_1) \supset \pi(\omega_2) \cap \pi(\omega_1) \neq \emptyset, \tag{2.6}$$

follows. The assertion is proved. □

Theorem 2.1 *Let Ω and X be nonempty sets, let π_1, π_2 be $\mathcal{P}(X)$-valued possibilistic distributions such that, for each $\omega \in \Omega, \pi_1(\omega) \subset \pi_2(\omega)$ holds. Then $I(\pi_1) \subset I(\pi_2)$ holds.*

Proof: By definition,

$$I(\pi_1) = \bigcup_{\omega \in \Omega} [\pi_1(\Omega - \{\omega\}) \cap \pi_2(\omega)]. \tag{2.7}$$

For each $\omega \in \Omega$, the inclusion

$$\begin{aligned} \Pi_1(\Omega - \{\omega\}) &= \bigcup_{\omega^* \in \Omega - \{\omega\}} \pi_1(\omega^*) \\ &\subset \bigcup_{\omega^* \in \Omega - \{\omega\}} \pi_2(\omega^*) = \Pi_2(\Omega - \{\omega\}) \end{aligned} \tag{2.8}$$

is valid, as $\pi_1(\omega^*) \subseteq \pi_2(\omega^*)$ holds for each $\omega^* \in \Omega$. Consequently, the inclusion

$$\Pi_1(\Omega - \{\omega\}) \cap \pi_1(\omega) \subset \Pi_2(\Omega - \{\omega\}) \cap \pi_2(\omega) \tag{2.9}$$

holds for each $\omega \in \Omega$, so that the inclusion $I(\pi_1)$ immediately follows. The assertion is proved. □

The following fact is almost trivial, but perhaps worth being explicitly recalled. In the space of set-valued possibilistic distributions it may easily happen that $\pi_1(\omega) \subset \pi_2(\omega)$ holds for each $\omega \in \Omega$, at least for some $\omega \in \Omega$ this inclusion is strict (i.e., $\pi_1(\omega) \neq \pi_2(\omega)$, but the identity $\bigcup_{\omega \in \Omega} \pi_1(\omega) = \bigcup_{\omega \in \Omega} \pi_2(\omega) = X$ is valid.

This property qualitatively differs from finite probability distributions, where the inequality $p_1(\omega_i) \leq p_2(\omega_i)$ for each $i = 1, 2, \ldots$, together with $\sum_{i=1}^{n} p_1(\omega_i) = \sum_{i=1}^{n} p_2(\omega_i) = 1$ implies that the probability distributions p_1 and p_2 are indentical on $\{\omega_1, \omega_2, \ldots, \omega_n\}$.

Lemma 2.2 *Let Ω, X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution. If there are $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$, such that $\pi(\omega_1) = \pi(\omega_2) = X$, then $I(\pi) = X = \mathbf{1}_{\mathcal{P}(X)}$.*

Proof: Let $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$, be such that $\pi(\omega_1) = \pi(\omega_2) = X$, consider the set $\Pi(\Omega - \{\omega_1\}) \cap \pi(\omega_1)$. Then $\omega_2 \in \Omega - \{\omega_1\}$ holds, hence,

$$\Pi(\Omega - \{\omega_1\}) = \bigvee^{\omega^* \in \Omega - \{\omega_1\}} \pi(\omega^*) \supset \pi(\omega_2) = X \tag{2.10}$$

holds and $\Pi(\Omega - \{\omega_1\}) = X$ follows. Replacing mutually ω_1 and ω_2 we obtain that $\Pi(\Omega - \{\omega_2\}) = X$ holds as well, hence,

$$X = \Pi(\Omega - \{\omega_j\}) \cap \pi(\omega_j) = \bigcup_{\omega \in \Omega} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] = I(\pi) \tag{2.11}$$

holds for any j and the assertion is proved. □

Denote by Q the set of all $\mathcal{P}(X)$-valued possibilistic distributions on Ω. If π_1, π_2 are $\mathcal{P}(X)$-possibilistic distributions on Ω such that $\pi_1(\omega) \subset \pi_2(\omega)$ holds for each $\omega \in \Omega$, we write $\pi_1 \leq \pi_2$ and say that π_1 is *majorized* by π_2 or that π_2 is an upper bound for π_1. As proved in Theorem 2.1 if $\pi_1 \leq \pi_2$ holds, then $I(\pi_1) \subseteq I(\pi_2)$ holds as well.

The universe implication does not hold in general, i.e., if $I(\pi_1) \subseteq I(\pi_2)$ is valid, then $\pi_1 \leq \pi_2$ need not hold. Indeed, let $\Omega = \{\omega_1, \omega_2\}$, let $\pi_1(\omega_1) = \pi_2(\omega_2) = X, \pi_1(\omega_2) = \pi_2(\omega_1) = \emptyset$, so that neither $\pi_1 \leq \pi_2$ nor $\pi_2 \leq \pi_1$ holds. For entropy $I(\pi_1)$ we obtain that

$$\begin{aligned} I(\pi_1) &= \bigcup_{\omega \in \Omega} [\Pi_1(\Omega - \{\omega\}) \cap \pi_2(\omega)] = \\ &= [\Pi_1(\Omega - \{\omega_1\}) \cap \pi_1(\omega_1)] \cup [\Pi_1(\Omega - \{\omega_2\}) \cap \pi_1(\omega_2)] = \\ &= (\pi_1(\omega_2) \cap \pi_1(\omega_1)) \cup (\pi_1(\omega_1) \cap \pi_1(\omega_2)) = \\ &= (\emptyset \cap X) \cup (X \cap \emptyset) = \emptyset. \end{aligned} \tag{2.12}$$

For $I(\pi_2)$ the calculations and the results are the same, so that $I(\pi_1) = I(\pi_2)$, but neither $\pi_1 \leq \pi_2$ nor $\pi_2 \leq \pi_1$ holds.

Lemma 2.3 *Let π be a $\mathcal{P}(X)$-valued distribution on Ω. Then for each $\mathcal{S} \subset \mathcal{P}(\Omega)$ the relation*

$$\begin{aligned} \Pi\left(\bigcup \mathcal{S}\right) &= \Pi\left(\bigcup \{A : A \in \mathcal{S}\}\right) = \bigvee^{\mathcal{T}} \left\{\pi(\omega) : \omega \in \bigcup \mathcal{S}\right\} = \\ &= \bigcup \{\{\pi(\omega) : \omega \in A\} : A \in \mathcal{S}\} : A \in \mathcal{S}\} = \\ &= \bigvee^{\mathcal{T}} \{\Pi(A) : A \in \mathcal{S}\} = \bigcup \{\Pi(A) : A \in \mathcal{S}\} \end{aligned} \tag{2.13}$$

holds.

Proof: Obvious. □

Let us denote by $Q(\Omega, X)$ the space of all $\mathcal{P}(A)$-valued possibilistic distributions over the space Ω, in symbols,

$$Q(\Omega, X) = \{\pi : \pi : \Omega \to \mathcal{P}(X), \bigcup \{\pi(\omega) : \omega \in \Omega\} = \mathbf{1}_{\mathcal{T}} = X\}. \tag{2.14}$$

Let $\leq^*$ be the binary relation, on $Q(\Omega, X)$, i.e., the subset of the Cartesian product $Q(\Omega, X) \times Q(\Omega, X)$ defined in this way: for each $\pi_1, \pi_2 \in Q(\Omega, X), \pi_1 <^* \pi_2$ holds iff $\pi_1(\omega) \subseteq \pi_2(\omega)$ holds for each $\omega \in \Omega$. It is possible that $\pi_1 <^* \pi_2$ holds for two $\mathcal{P}(X)$-distribution π_1, π_2 such that $\pi_1(\omega) \subset \pi_2(\omega)$ is the case for some $\omega \in \Omega$ and, of course, $\pi_1(\omega^*) \subseteq \pi_2(\omega^*)$ holds for two $\mathcal{P}(X)$-distributions π_1, π_2 such that $\pi_1(\omega) \subset \pi_2(\omega)$ is the case for some $\omega \in \Omega$ and, of course, $\pi_1(\omega^*) \subseteq \pi_2(\omega^*)$ holds for each $\omega^* \in \Omega$.

Lemma 2.4 *The ordered pair $\mathcal{D} = \langle Q(\Omega, X), \leq^* \rangle$ is a p.o.set which defines a complete upper semi-lattice, so that for each nonempty subset $E \subset D$ the supremum $\pi(E) = \bigvee^{\mathcal{D}} \{\pi : \pi \in E\}$ is defined. Given explicitly, π^E is the mapping which takes Ω into $\mathcal{P}(X)$ in such a way that for each $\omega \in \Omega$*

$$\pi^E(\omega) = \bigcup \{\pi \in E : \pi(\omega)\} \tag{2.15}$$

This mapping obviously defines a π-valued possibilistic distribution on Ω.

Proof: Obvious. □

However, the situation with the infimum of a set E of $\mathcal{P}(X)$-distributions is not dual to $\bigvee^D E$. We may define the mapping $M(E) : \Omega \to \mathcal{P}(X)$ in such a way that, for each $\omega \in \Omega$, $M(E)(\omega) = \bigcap \{\pi(\omega) : \pi \in E\}$, but this mapping does not meet the condition $\bigvee^{\mathcal{D}} \{M(E)(\omega) : \omega \in \Omega\} = \mathbf{1}_{\mathcal{P}(X)} = X$. Indeed, let $E = \{\pi_1, \pi_2\}$ be such that $\pi_1(\omega) = \mathbf{1}_{\mathcal{P}(X)}$ for $\omega \in \Omega_0$, $\emptyset \neq \Omega_0 \neq \Omega$, $\pi_1(\omega) = \emptyset_{\mathcal{P}(X)}$ otherwise, and $\pi_2(\omega) = \mathbf{1}_{\mathcal{P}(X)}$ for $\omega \in \Omega - \Omega_0$, $\pi_2(\omega) = \emptyset_{\mathcal{P}(X)}$ for $\omega \in \Omega$. Obviously, $M(E)(\omega) = \emptyset_{\mathcal{P}(X)}$ for each $\omega \in \Omega$, so that $M(E)$ is not a $\mathcal{P}(X)$-distribution. Neither the operation of $\mathcal{P}(X)$-valued complements, defined by $\pi^C(\omega) = X - \pi(\omega)$ yields the results meeting the conditions imposed on $\mathcal{P}(X)$-distributions.

Lemma 2.5 *Let $E \subset Q$ be a nonempty set of $\mathcal{P}(X)$-distributions, for each $\pi \in Q$ let $\Pi_\pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ denote the corresponding induced $\mathcal{P}(X)$-possibilistic measure on $\mathcal{P}(\Omega)$. Then, for each $A \subset \Omega$, the relation $\Pi_\pi E(A) = \bigvee^{\mathcal{T}} \{\Pi_\pi(A) : \pi \in E\}$ holds.*

Proof: For each $A \subset \Omega$ we obtain that

$$\begin{aligned} & \bigvee^{\mathcal{T}} \{\pi_\pi(A) : \pi \in E\} = \bigvee^{\mathcal{T}} \{\{\bigvee^{\mathcal{T}} \pi(\omega) : \omega \in A\} : \pi \in E\} = \\ = & \bigvee^{\mathcal{T}} \{\pi(\omega) : \omega \in \Omega, \pi \in E\} = \bigvee^{\mathcal{T}} \{\{\bigvee^{\mathcal{T}} \pi(\omega) : \pi \in E\} : \omega \in A\} \\ = & \bigvee^{\mathcal{T}} \{\pi^E(\omega) : \omega \in A\} = \Pi_{\pi^E}(A). \end{aligned} \tag{2.16}$$

The assertion is proved. □

According to the way in which $\mathcal{P}(X)$-valued possibilistic measure Π on $\mathcal{P}(\Omega)$ induced by a $\mathcal{P}(X)$-valued possibilistic distribution π on Ω is defined, the set function Π is extensional with respect to the supremum operation $\bigvee^{\mathcal{T}}$ on $\mathcal{T} = \mathcal{P}(X)$ in the sense that for each nonempty system $\mathcal{A}$ of subsets of Ω the identity

$$\Pi\left(\bigcup \mathcal{A}\right) = \bigvee^{\mathcal{T}} \{\Pi(A) : A \in \mathcal{A}\} \tag{2.17}$$

holds. In particular, for $\mathcal{A} = \{A_1, A_2\}$, $\Pi(A_1) \cup \Pi(A_2) = \Pi(A_1 \cup A_2)$. For the operation of infimum the relation dual to (2.17) is not the case, in general, only the inclusion $\Pi(A \cap B) \subseteq \Pi(A) \cap \Pi(B)$ is valid, as $\Pi(A \cap B) \subset \Pi(A)$ and $\Pi(A \cap B) \subset \Pi(B)$ holds trivially. The difference between the values $\Pi(A_1 \cap A_2)$ and $\Pi(A_1) \cap \Pi(A_2)$ may range over all the Boolean interval $\langle \emptyset, X \rangle$ of $\mathcal{T} = \langle \mathcal{P}(X), \subseteq \rangle$. Indeed, let $X = \{0, 1\}$, let $\pi(\omega_1) = \pi(\omega_2) = 1$, let $A_1 = \{\omega_1\}$, $A_2 = \{\omega_2\}$. Then $\Pi(A_1) = \Pi(A_2) = 1$, so that $\Pi(A_1) \wedge \Pi(A_2) = 1$, but $\Pi(A_1 \cap A_2) = \Pi(\emptyset) = 0$.

As the most simple $\mathcal{P}(X)$-valued possibilstic distribution π for which the induced $\mathcal{P}(X)$-measure Π on $\mathcal{P}(\Omega)$ is extensional also w.r.to the operation of infimum $\bigwedge$ let us consider the identity mapping on $\mathcal{P}(\Omega)$. Take $\Omega = X$, take $\pi(\omega) = \{\omega\}$ for every $\omega \in \Omega$, so that, for each $A \subset \Omega$, $\Pi(A) = \bigcap_{A \in \mathcal{A}} \Pi(A)$ follows, in particular, $\Pi(A \cap B) = \Pi(A) \cap \Pi(B)$ holds.

Definition 2.1 *$\mathcal{P}(X)$-valued possibilistic distribution π taking a nonempty set Ω into the power-set $\mathcal{P}(X)$ over a nonempty set X is called completely extensional, if for each nonempty system $\mathcal{A}$ of subsets of Ω the relation*

$$\Pi\left(\bigcap \mathcal{A}\right) = \Pi\left(\bigcap_{A \in \mathcal{A}} A\right) = \bigcap_{A \in \mathcal{A}} \Pi(A) \tag{2.18}$$

holds. The $\mathcal{P}(X)$-distribution π is called extensional, if

$$\Pi(A \cap B) = \Pi(A) \cap \Pi(B) \tag{2.19}$$

holds for each $A, B \subset \Omega$.

Lemma 2.6 *Let π be a $\mathcal{P}(X)$-valued possibilistic distribution defined on a nonempty space Ω, taking its values in the power-set $\mathcal{P}(X)$ over a nonempty space X and such that $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ holds for each $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$. Then the induced $\mathcal{P}(X)$-possibilistic measure on $\mathcal{P}(\Omega)$ is extensional in the sense that $\Pi(A) \cap \Pi(B) = \Pi(A \cap B)$ is valid for each $A, B \subset \Omega$.*

Proof: First of all, let us consider the case when the sets A, B are disjoint. Then

$$\begin{aligned}
\Pi(A) \cap \Pi(B) &= \left(\bigvee_{\omega_1 \in A} \pi(\omega_1)\right) \cap \left(\bigvee_{\omega_2 \in B} \pi(\omega_2)\right) = \\
&= \bigcup_{\langle \omega_1, \omega_2 \rangle, \omega_1 \in A, \omega_2 \in B} (\pi(\omega_1) \cap \pi(\omega_2)) = \emptyset = \\
&= \Pi(\emptyset) = A \cap B = \Pi(A \cap B), \qquad (2.20)
\end{aligned}$$

as for each $\omega_1 \in A, \omega_2 \in B, \omega_1 \neq \omega_2$ and $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ holds.

For each $A, B \subset \Omega, A = (A - B) \cup (A \cap B), B = (B - A) \cup (A \cap B)$ holds, so that

$$\begin{aligned}
& \Pi(A) \cap \Pi(B) = [\Pi((A - B) \cup (A \cap B))] \cap [\Pi((B - A) \cup (A \cap B))] = \\
= \; & [\Pi(A - B) \cup \Pi(A \cap B)] \cap [\Pi(B - A) \cup \Pi(A \cap B)] = \\
= \; & [\Pi(A - B) \cap \Pi(B - A)] \cup [\Pi(A \cap B) \cap \Pi(B - A)] \cup \\
\cup \; & [\Pi(A \cap B) \cap \Pi(A - B)] \cup \Pi(A \cap B) = \Pi(A \cap B), \qquad (2.21)
\end{aligned}$$

as

$$(A - B) \cap (B - A) = (A \cap B) \cap (B - A) = (A \cap B) \cap (A - B) = \emptyset, \qquad (2.22)$$

so that, due to (2.20)

$$\Pi(A - B) \cap \Pi(B - A) = (A \cap B) \cap (B - A) = (A \cap B) \cap (A - B) = \emptyset, \qquad (2.23)$$

The assertion is proved. □

3 Conditioned set-valued possibilistic distributions and measures

Conditioned (or conditional) probability distributions are very important tools in probability theory. Roughly speaking, conditioned probabilities enable, on the ground of a newly obtained evidence, to transform the probability values in such a way that the random events incompatible with the new pieces of evidence are eliminated from evidence – they obtain the zero-valued conditioned probability. Within the framework of the standard Kolmogorov axiomatic probability theory the mathematical formalization of this transformation is very simple and well-known. Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, hence, Ω is a nonempty space (no relations to the support set of the possibilistic distribution introduced above being supposed at this moment), $\mathcal{A}$ is a nonempty σ-field of subsets of Ω, and $P : \mathcal{A} \to [0, 1]$ is a σ-additive real-valued set function. Subsets of Ω belonging to $\mathcal{A}$ are called *random events*, hence, for each $A \in \mathcal{A}$ the real number $P(A) \in [0, 1]$ is ascribed and called the *probability of (the random event) A*. Given another random event $B \in \mathcal{A}$ such that $P(B) > 0$ holds, the conditioned probability of (the random event) A under the condition that (the random event) B holds is denoted by $P(A/B)$ and defined by the well-known formula

$$P(A/B) = P(A \cap B)/P(B). \qquad (3.1)$$

This definition cannot the immediately translated into the model and language of $\mathcal{T}$-valued possibilistic distributions because of the fact that operation of division between the values $P(A \cap B)$ and $P(B)$ cannot be defined in $\mathcal{T}$. Let us proceed in this way: we introduce three alternative approaches and for each of them we will examine its role when taken as conditioned probability and measure.

So, let $\mathcal{T} = \langle X, \subseteq \rangle, \Omega, \pi : \Omega \to \mathcal{P}(X)$ such that $\bigcup_{\omega \in \Omega} \pi(\omega) = X = \mathbf{1}_{\mathcal{T}}$ and $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ defined by $\Pi(A) = \bigcup_{\omega \in A} \pi(\omega)$ for each $A \subset X$ be as above. Given $B \subset \Omega$, let us define three mappings $\pi^i(\omega/B) : \Omega \to \mathcal{P}(X)$ in this way.

$$(i) \quad \pi^1(\omega/B) = \pi(\omega) \cap \Pi(B), \tag{3.2a}$$

$$(ii) \quad \pi^2(\omega/B) = \pi(\omega), \text{ if } \omega \in B, \pi^2(\omega/B) = \emptyset (= \emptyset_{\mathcal{T}}), \\ \text{if } \omega \in \Omega - B, \tag{3.2b}$$

$$(iii) \quad \pi^3(\omega/B) = \Pi(\Omega - B) \cup \pi(\omega) = \Pi((\Omega - B) \cup \{\omega\}). \tag{3.2c}$$

Let us investigate the most elementary properties of these three mappings. Define, for each $i = 1, 2, 3$ and each $B \subset \Omega$ the mapping $\Pi^i(\cdot/B) : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ in this way: for each $A \subset \Omega$,

$$\Pi^i(A/B) = \bigvee_{\omega \in A}^{\mathcal{T}} \pi^i(\omega/B) = \bigcup_{\omega \in A} \pi^i(\omega/B). \tag{3.3}$$

Hence, for each $i = 1, 2, 3$ we obtain explicitly that

$$\begin{aligned} \Pi^1(A/B) &= \bigcup_{\omega \in A} \pi^1(\omega/B) = \bigcup_{\omega \in A} (\pi(\omega) \cap \Pi(B)) = \left(\bigcup_{\omega \in A} \pi(\omega) \right) \cap \Pi(B) = \\ &= \Pi(A) \cap \Pi(B), \end{aligned} \tag{3.4}$$

$$\Pi^2(A/B) = \bigcup_{\omega \in A} \pi^2(\omega/B) = \bigcup_{\omega \in A \cap B} \pi(\omega) = \Pi(A \cap B), \tag{3.5}$$

$$\begin{aligned} \Pi^3(A/B) &= \bigcup_{\omega \in A} \pi^3(\omega/B) = \bigcup_{\omega \in A} (\Pi(\Omega - B) \cup \pi(\omega)) = \\ &= \Pi(\Omega - B) \cup \bigcup_{\omega \in A} \pi(\omega) = \\ &= \Pi(\Omega - B) \cup \Pi(A) = \Pi((\Omega - B) \cup A) \end{aligned} \tag{3.6}$$

For the extremum values $A = \Omega$ or $B = \Omega$ we obtain that

$$\begin{aligned} \Pi^1(\Omega/B) &= \Pi(\Omega) \cap \Pi(B) = \Pi(B), \\ \Pi^2(\Omega/B) &= \Pi(\Omega \cap B) = \Pi(B), \\ \Pi^3(\Omega/B) &= \Pi((\Omega - B) \cup \Omega) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}, \\ \Pi^1(A/\Omega) &= \Pi(A) \cap \Pi(\Omega) = \Pi(A), \\ \Pi^2(A/\Omega) &= \Pi(A \cap \Omega) = \Pi(A), \\ \Pi^3(A/\Omega) &= \Pi((\Omega - \Omega) \cup A) = \Pi(A), \end{aligned} \tag{3.7}$$

So, $\pi^1(\cdot/B)$ and $\pi^2(\cdot/B)$ define $\mathcal{T}$-possibilistic distribution on B (supposing that $B \neq \emptyset$), $\pi^3(\cdot/B)$ defines a $\mathcal{T}$-possibilistic distribution on Ω. Moreover, if $B = \Omega$, then $\Pi^i(\cdot/B)$ is identical with the apriori possibilistic distribution π on Ω for each $i = 1, 2, 3$. Let us recall that in standard probability theory, if $B \subset \Omega$ is such that $P(B) = 1$, then for each $A \subset \Omega$ the identity $P(A/B) = P(A \cap B)/P(B) = P(A)$ holds. The intuition behind is quite simple – the occurence of certain (i.e., which the probability 1 valid) random event does not bring any new information, so that no modification of the apriori probability measure results. All the three set functions $\Pi^i(\cdot/B), i = 1, 2, 3$, also possess this important property.

More generally, not only for $A = \Omega$, but for each $A \supseteq B$ the result $\Pi^i(A/B) = \Pi(B)$ (for $i = 1, 2$) or $\Pi^3(A/B) = \mathbf{1}_{\mathcal{T}}$ holds, as may be easily checked by inspection of the formulas (3.4), (3.5), and (3.6).

When approaching to a more detailed analysis of the three $\mathcal{P}(X)$-valued mappings $\pi^i(\omega/B), i = 1, 2, 3$, let us begin with the mapping $\pi^3(\omega/B)$ defined by (3.2c), so that

$$\pi^3(\omega|B) = \Pi(\Omega - B) \cup \pi(\omega) = \Pi((\Omega - B) \cup \{\omega\}). \tag{3.8}$$

Hence, for each $A, B \subset \Omega$,

$$\begin{aligned}\pi^3(A/B) &= \bigcup_{\omega\in A} \pi^3(\omega/B) = \bigcup_{\omega\in A} (\Pi(\Omega - B) \cup (\omega)) = \\ &= \Pi(\Omega - B) \cup \bigcup_{\omega\in A} \pi(\omega) = \Pi(\Omega - B) \cup \Pi(A) = \\ &= \Pi((\Omega - B) \cup A). \end{aligned} \tag{3.9}$$

The reason for this preference given to $\pi^3(\cdot/B)$ consists in the fact that $\pi^3(\omega/B)$ is, for each B, the only of the three mappings in question which meets the condition of normalization, i.e., for which

$$\begin{aligned}&\bigcup_{\omega\in\Omega} \pi^3(\omega/B) = \bigcup_{\omega\in\Omega} (\pi(\Omega - B) \cup \pi(\omega)) = \\ &= \Pi(\Omega - B) \cup \bigcup_{\omega\in\Omega} \pi(\omega) = \Pi(\Omega - B) \cup X = X = \mathbf{1}_{\mathcal{T}}. \end{aligned} \tag{3.10}$$

So, the $\mathcal{P}(X)$-valued entropy $I(\pi^3(\cdot/B))$ is defined and, writing $\hat{\pi}(\omega)$ for $\pi^3(\omega/B)$ in order to simplify the rotation, may be written by

$$I(\pi^3(\cdot/B)) = I(\hat{\pi}) = \bigcup_{\omega\in\Omega} (\Pi^3(\Omega - \{\omega\}) \cap \hat{\pi}(\omega)). \tag{3.11}$$

Let $\omega_0 \in \Omega$ be such that $\hat{\pi}(\omega_0) = X$. Then

$$\begin{aligned}I(\pi^3(\cdot/B)) &= I(\hat{\pi}) = \bigcup_{\omega\in\Omega,\omega\neq\omega_0} \hat{\Pi}((\Omega - \{\omega\}) \cap \hat{\pi}(\omega)) \cup \hat{\Pi}(\Omega - \{\omega_0\}) \cap \hat{\pi}(\omega_0) = \\ &= \bigcup_{\omega\in\Omega,\omega\neq\omega_0} (X \cap \hat{\pi}(\omega)) \cup (\hat{\Pi}(\Omega - \{\omega_0\}) \cap X = \\ &= \bigcup_{\omega\in\Omega,\omega\neq\omega_0} \hat{\pi}(\omega) \cup \hat{\Pi}(\Omega - \{\omega_0\}) = \hat{\Pi}(\Omega - \{\omega_0\}) = \\ &= \Pi^3((\Omega - \{\omega_0\})/B). \end{aligned} \tag{3.12}$$

4 Refined set-valued entropy functions

Let us re-consider and analyse, in more detail, Lemma 2.2. According to this result, if there are $\omega_1, \omega_1 \in \Omega, \omega_1 \neq \omega_2$, such that $\pi(\omega_1) = \pi(\omega_2) = X$, then $I(\pi) = X = \mathbf{1}_{\mathcal{P}(X)}$. This fact may be understood in the sense that set-valued entropy function I as defined above is a very weak, poor and rough quantitative tool when seeking for some $\omega_0 \in \Omega$ which could be preferred for the most expectable state of the universe Ω on the ground of criteria which may be formalized within the framework of possibilistic distributions and measures taking their values in the power-set $\mathcal{P}(X)$. Hence, each decision rule picking up just one $\omega_0 \in \Omega$ must be based on more input parameters than those expressible by the values of the entropy function $I(\pi)$. However, the same is the situation in the most simple probability space $\langle \Omega, \mathcal{A}, P \rangle$, where $\Omega = \{\omega_1, \omega_2\}$ and $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$. When we have to pick up just one of the states ω_1, ω_2 as the better solution of a problem in question, we have to do so on the ground of some more data and criteria than the some two values $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$. The following lemma may be taken as a complementary formulation of the conditions when $I(\pi) \neq 1_{\mathcal{P}(X)} = X$ is the case.

Lemma 4.1 *Let Ω, X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-possibilistic distribution on Ω, let $\omega_0 \in \Omega$ be such that $\pi(\omega_0) = X$. Then*

$$I(\pi) = \pi(\Omega - \{\omega_0\}) \tag{4.1}$$

holds. Consequently, if $\Pi(\Omega - \{\omega_0\}) \subsetneq X$ holds, then $I(\pi) \subsetneq X$ follows.

Proof: For $I(\pi)$ we have

$$\begin{aligned} I(\pi) &= \bigcup_{\omega\in\Omega} (\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] = \\ &= \bigcup_{\omega\in\Omega,\omega\neq\omega_0} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] \cup \Pi(\Omega - \{\omega_0\}) \cap \pi(\omega_0). \end{aligned} \tag{4.2}$$

If $\omega \neq \omega_0$, then $\omega_0 \in (\Omega - \{\omega\})$ and $\Pi(\Omega - \{\omega\}) = X = \pi(\omega_0)$ holds, so that

$$\begin{aligned} I(\pi) &= \left(\bigcup_{\omega\in\Omega,\omega\neq\omega_0} \pi(\omega) \right) \cup \Pi(\Omega - \{\omega_0\}) = \\ &= \Pi(\Omega - \{\omega_0\}). \end{aligned} \tag{4.3}$$

is valid and the assertion is proved. □

An easy corollary of Lemma 4.1 reads as follows. Let Ω, X and π be as in Lemma 4.1, let there exist $x_0 \in X$ such that there is only one $\omega_0 \in \Omega$ with the property $x_0 \in \pi(\omega_0)$ and $\Pi(\Omega - \{\omega_0\}) \subsetneq X$. Then $I(\pi) = \Pi(\Omega - \{\omega_0\}) \subsetneq X$ follows.

Inspired by Lemma 2.2 and Lemma 4.1 we propose in [7, 8, 9] some modifications of the space of values in which the mapping $\pi : \Omega \to T$ take its values in such a way that $\pi(\omega_0) = \mathbf{1}_{\mathcal{T}}$ is valid only for one $\omega_0 \in \Omega$. In [7], the mapping π, defined on Ω, takes its values in a *complete chained lattice*, let us recall, for the reader's convenience, the way leading to this notion.

A poset (partially ordered set) $\mathcal{T} = \langle T, \leq \rangle$ is called a *lattice*, if for each $t_1, t_2 \in T$ the elements $t_1 \vee t_2$ and $t_1 \wedge t_2$ are defined, and $\mathcal{T}$ is called a *complete lattice*, if $\bigvee S$ and $\bigwedge S$ are defined for each $S \subset T$ applying the convention according to which $\bigwedge \emptyset = \bigvee T$ and $\bigvee \emptyset = \bigwedge T$ for the empty subset of T ($\vee$ and $\wedge$ and $\bigvee$ and $\bigwedge$ are supremum and infimum operations related to the partial ordering relation $\leq$ on T). The element $\bigvee T$ ($\bigwedge T$, resp.) is called the *unit (element)* of $\mathcal{T}$ (the *zero (element)* of $\mathcal{T}$, resp) and is denoted by $\mathbf{1}_{\mathcal{T}}$ ($\emptyset_{\mathcal{T}}$, resp.).

A complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *distributive*, if for each $s \in T$ and $S \subset T$ the relations

$$s \wedge \left(\bigvee S \right) = \bigvee (s \wedge t),\ s \vee \left(\bigwedge S \right) = \bigwedge (s \vee t) \tag{4.4}$$

are valid. Complete lattice $\mathcal{T}$ is called *chained*, if the partial ordering $\leq$ on T is linear, so that either $t_1 \leq t_2$ or $t_2 \leq t_1$ holds for each $t_1, t_2 \in T$. Consequently, for each different $t_1, t_2 \in T$ either $t_1 < t_2$ or $t_2 < t_1$ holds.

Obviously, each complete chained lattice $\mathcal{T} = \langle T, \leq \rangle$ is distributive.

For more detail on binary relations, partial orderings and chains (linear orderings), semi-lattices and lattices, Boolean algebras, and related structures and notions cf. [1, 3, 11] or a more recent textbook and monograph.

So, in [7], the values of possibilistic distributions were taken from complete lattices, but bound by the condition of chained structure, so that each two possibility degrees are comparable by the partial ordering relation $\leq$ defined on $\mathcal{T} = \langle T, \leq \rangle$. In what follows, we use more intuitive space of values, namely, that of the power-set over the space X. However, the conditions imposed on chained lattices need not be valid in general, so that neither the structure from [7] nor that introduced in this text can be classified as one being a particular case or, in contrary, as a generalization of the other one.

Theorem 4.1 *Let Ω, X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω. Let $\omega_0 \in \Omega$ be such that $\pi(\omega_0) = X_0$ let $\Omega_0 = \{\omega \in \Omega : \omega \neq \omega_0, \pi(\omega) = X\}$. Define $\pi_0 : \Omega \to \mathcal{P}(X)$ in this way: if $\omega \in \Omega - \Omega_0$, then $\pi_0(\omega) = \pi(\omega)$, if $\omega \in \Omega_0$, then $\pi^0(\omega) = \emptyset$ $(= 0_{\mathcal{P}(X)})$. Then*

$$I(\pi^0) = \Pi(\Omega - \{\omega_0\}). \tag{4.5}$$

Proof: By definition,

$$\begin{aligned} I(\pi^0) &= \bigcup_{\omega \in \Omega} [\Pi^0(\Omega - \{\omega\}) \cap \pi^0(\omega)] = \\ &= \bigcup_{\omega \in \Omega - \Omega_0} [\Pi^0(\Omega - \{\omega\}) \cap \pi^0(\omega) \cup \\ &\cup \bigcup_{\omega \in \Omega_0} [\Pi^0(\Omega - \{\omega\}) \cap \pi^0(\omega)]. \end{aligned} \tag{4.6}$$

The last line in (4.6) is identical with $\emptyset$, as $\pi^0(\omega) = \emptyset$ for each $\omega \in \Omega_0$, so that, as $\pi^0(\omega)$ and $\pi(\omega)$ are identical for each $\omega \in \Omega - \Omega_0$, we obtain that

$$I(\pi^0) = \Pi(\Omega - \{\omega_0\}) \cap \pi(\omega_0) \cup \bigcup_{\omega \in (\Omega - \Omega_0) - \{\omega_0\}} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] \tag{4.7}$$

If $\omega \in (\Omega - \Omega_0) - \{\omega_0\}$ is the case, then $\omega_0 \in \Omega - \{\omega\}$ and $\Pi(\Omega - \{\omega\}) = X$ holds, moreover, $\pi(\omega_0) = X$ holds as well. Consequently,

$$\begin{aligned} I(\pi^0) &= \left(\bigcup_{\omega \in (\Omega - \Omega_0) - \{\omega_0\}} (\pi(\omega) \cap X) \cup (\Pi(\Omega - \{\omega_0\}) \cap X) \right) = \\ &= \Pi(\Omega - \{\omega_0\}). \end{aligned} \tag{4.8}$$

The assertion is proved. □

Let us consider another example of restriction of set-valued possibilistic distributions which may be taken as a more severe application of the reduction principle leading from π to π^0 in Theorem 4.1.

Theorem 4.2 *Let Ω, X be nonempty spaces, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω. Let $\pi(\omega_0) = X$ for some $\omega_0 \subset \Omega$, let $\pi(\omega) \subset X_0 \subsetneq X$ hold for some proper subset X_0 of X and for each $\omega \in \Omega, \omega \neq \omega_0$. Then $I(\pi) \subseteq X_0$ holds with equality being the case when there is $\omega_1 \in \Omega$ such that $\pi(\omega_1) = X_0$.*

Proof: By definition,

$$\begin{aligned} I(\pi) &= \bigcup_{\omega \in \Omega} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] = \\ &= \bigcup_{\omega \in \Omega, \omega \neq \omega_0} [\Pi(\Omega - \{\omega\}) \cap \pi(\omega)] \cup \\ &\cup \ [\Pi(\Omega - \{\omega_0\}) \cap \pi(\omega_0)] = \\ &= \bigcup_{\omega \in \Omega, \omega \neq \omega_0} (X \cap \pi(\omega)) \cup [\Pi(\Omega - \{\omega_0\}) \cap X] = \\ &= \bigcup_{\omega \in \Omega, \omega \neq \omega} \pi(\omega) \cup \Pi(\Omega - \{\omega_0\}) = \Pi(\Omega - \{\omega_0\}) \cup \Pi(\Omega - \{\omega_0\}) = \\ &= \Pi(\Omega - \{\omega_0\}) \subset X_0. \end{aligned} \tag{4.9}$$

as for each $\omega \neq \omega_0$ the relations $\omega_0 \in \Pi(\Omega - \{\omega\})$ and $\Pi(\Omega - \{\omega\}) = X$ hold. The inclusion $I(\pi) \subset X_0$ with the equality in the particular case when $\pi(\omega_1) = X_0$ for some $\omega_1 \in \Omega$ easily follows.

It is perhaps worth being quoted explicitly, that for each $\mathcal{P}(X)$-valued possibilistic distribution $\pi : \Omega \to= \mathcal{P}(X)$ we may obtain reduced possibilistic distribution π^0, setting $\pi^0(\omega_0) = \pi(\omega_0) = X$, and setting $\pi^0(\omega) = \pi(\omega) \cap X_0$ for a fixed proper subset $X_0 \subset X$ and for each $\omega \in \Omega, \omega \neq \omega_0$. □

5 Compositions of set-valued possibilistic distributions

Let Ω and X nonempty sets, let H be a nonempty set of parameters. For each $i \in H$, let $\pi_i : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω, hence, for each $\omega \in \Omega$, $\pi_i(\omega) \subset X$ and $\bigcup_{\omega\in\Omega} \pi_i(\omega) = X$ holds. Let π^H be the $\mathcal{P}(X)$-valued mapping defined on Ω in this way: for each $\omega \in \Omega$,

$$\pi^H(\omega) = \bigcup_{i\in H} \pi_i(\omega). \tag{5.1}$$

This mapping is called the supremum of the $\mathcal{P}(X)$-valued possibilistic distribution over the set H of parameters. Instead of π^H we write also $\bigvee^H \pi_i$ or $\bigvee_{i\in H} \pi_i$ (the symbol for supremum being used in order to save the symbol $\bigcup$ of set union just for subsets of the spaces Ω and X). In order to apply (2.1) we obtain for the $\mathcal{P}(X)$-valued entropy of π_i, $i \in H$, the value

$$I(\pi_i) = \bigcup_{\omega\in\Omega} [\Pi^i(\Omega - \{\omega\}) \cap \pi_i(\omega)]. \tag{5.2}$$

The mapping π^H obviously meets the conditions imposed on $\mathcal{P}(X)$-valued possibilistic distribution, so that the related entropy value $I(\pi^H)$ is defined by

$$I(\pi^H) = \bigcup_{\omega\in\Omega} [\Pi^H(\Omega - \{\omega\}) \cap \pi^H(\omega)], \tag{5.3}$$

here Π^i is the $\mathcal{P}(X)$-valued possiblistic measure on $\mathcal{P}(\Omega)$ defined by the distribution π_i on Ω and Π^H is the $\mathcal{P}(X)$-valued possibilistic measure defined by the distribution π^H on Ω. As proved in Section 2 for each $i \in H$ the inclusion $I(\pi_i) \subseteq I(\pi^H)$ holds, so that also the inclusion $\bigvee_{i\in H} I(\pi_i) \subseteq I(\pi^H)$ is valid. The equality need not hold, as the following very simple examples demonstrates.

Let $\Omega = \{\omega_1, \omega_2\}$, let $X \neq \emptyset$, let $\pi_1 : \Omega \to \mathcal{P}(X)$ be defined by $\pi_1(\omega_1) = X, \pi_1(\omega_2) = \emptyset$, let $\pi_2 : \Omega \to \mathcal{P}(X)$ be defined by $\pi_2(\omega_1) = \emptyset$, $\pi_2(\omega_2) = X$. For both $i = 1, 2$, the identity $\bigcup_{\omega\in\Omega} \pi_i(\omega) = X$ obviously holds. Moreover

$$\begin{aligned} I(\pi_1) = {} & \bigcup_{\omega\in\Omega} (\Pi_1(\Omega - \{\omega\}) \cap \pi_1(\omega)) = \\ = {} & (\Pi_1(\Omega - \{\omega_1\}) \cap \pi_1(\omega)) \cup (\Pi_1(\Omega - \{\omega_2\}) \cap \pi_1(\omega_2)) = \\ = {} & (\pi_1(\omega_2) \cap \pi_1(\omega_1)) \cup (\pi_1(\omega_1) \cap \pi_1(\omega_2)) = \\ = {} & (\emptyset \cap X) \cup (X \cap \emptyset) = \emptyset. \end{aligned} \tag{5.4}$$

Analogously, we obtain that $I(\pi_2) = \emptyset$, hence, $I(\pi_1) \vee I(\pi_2) = \emptyset$. For $\pi_1 \vee \pi_2$ we obtain that

$$(\pi_1 \vee \pi_2)(\omega_1) = \pi_1(\omega_1) \cup \pi_2(\omega_1) = X \cup \emptyset = X, \tag{5.5}$$

$$(\pi_1 \vee \pi_2)(\omega_2) = \pi_1(\omega_2) \cup \pi_2(\omega_2) = \emptyset \cup X = X, \tag{5.6}$$

so that $I(\pi_1 \vee \pi_2) = X \neq \emptyset = I(\pi_1) \vee I(\pi_2)$.

Let Ω, X be nonempty spaces, let $\pi_1, \pi_2 : \Omega \to \mathcal{P}(X)$ be $\mathcal{P}(X)$-valued possibilistic distributions on Ω such that $\Pi_1 \subseteq \Pi_2$ holds, hence, the inclusion $\pi_1(\omega) \subseteq \pi_2(\omega)$ is valid for each $\omega \in \Omega$. As proved in

Section 2, in this case $I(\pi_1) \subseteq I(\pi_2)$ follows and $\pi_1(\omega) \cup \pi_2(\omega)$ is valid for each $\omega \in \Omega$. Consequently, $I(\pi_1 \vee \pi_2) = I(\pi_2) = I(\pi_1) \cup I(\pi_2)$ follows.

Two almost immediate consequences of the relations proved above are as follows.

Lemma 5.1 *Let Ω, X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω such that there exist different elements $\omega_1, \omega_2 \in \Omega$ for which the intersection $\pi(\omega_1) \cap \pi(\omega_2)$ defines a nonempty subset of X. Then the $\mathcal{P}(X)$-valued entropy function I ascribes to π the nonempty (i.e., nonzero in the sense of the structure $\mathcal{T}$ on $\mathcal{P}(\Omega)$) value*

$$I(\pi) = \bigcup_{\omega \in \Omega} (\pi(\Omega - \{\omega\}) \cap \pi(\omega)). \tag{5.7}$$

Proof: As $\omega_1 \neq \omega_2$, the membership relations $\omega_2 \in \Omega - \{\omega_1\}$ and $\omega_1 \in \Omega - \{\omega_2\}$ are valid. Hence, the relation

$$I(\pi) \supseteq \pi(\omega_1) \cap \pi(\omega_2) \neq \emptyset \tag{5.8}$$

holds and the assertion is proved. □

Lemma 5.2 *Let Ω, X and π be as in Lemma 4.1, let $\pi_1, \pi_2 : \Omega \to \mathcal{P}(X)$ be such that $\pi_1(\omega_1) = X, \pi_1(\omega) = \emptyset$, if $\omega \neq \omega_1, \pi_2(\omega_2) = X, \pi_2(\omega) = \emptyset$, if $\omega \neq \omega_2$. Then*

$$\emptyset = I(\pi_1) = I(\pi_2) = I(\pi_1) \cup I(\pi_2) \neq I(\pi_1 \cup \pi_2) \supseteq \pi(\omega_1) \cap \pi(\omega_2) = X. \tag{5.9}$$

Proof: $I(\pi_1) = \emptyset$, as $\pi(\omega_2)$ are disjoint subsets of X for different $\omega_1, \omega_2 \in \Omega$ (cf. Lemma 2.1). The right-hand side of (5.9) follows from (5.7). The assertion is proved. □

Let Ω and X be nonempty sets, let G be a nonempty set of parameters. For each $i \in G$, let $\pi_i : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-possibilistic distribution on Ω, hence, for each $\omega \in \Omega, \pi_i(\omega) \subset X$ and $\bigcup_{\omega \in \Omega} \pi_i(\omega) = X$ holds. Let $\pi^G : \Omega \to \mathcal{P}(X)$ be the $\mathcal{P}(X)$-valued mapping defined on Ω in this way: for each $\omega \in \Omega$,

$$\pi^G(\omega) = \bigcap_{i \in G} \pi_i(\omega). \tag{5.10}$$

The following lemma is obvious, but perhaps worth being introduced explicitly.

Lemma 5.3 *Let there exist $\omega_0 \in \Omega$ such that for each $i \in G$, $\bigvee_{\omega \in \Omega} \pi_i(\omega) = X$. Then the mapping $\pi^G : \Omega \to \mathcal{P}(X)$, defined by (5.10), meets the conditions imposed on $\mathcal{P}(X)$-valued possibilistic distributions.*

6 Conclusions

According to what we told in the introductory section, our aim was to introduce and analyze some possibilistic distributions and related possibilistic measures with non-numerical, but intuitive enough uncertainty (in the sense of fuzziness and vagueness) degrees – as the most simple structure for these purposes we have taken the classical Boolean algebra over the power-set of all subsets of a basic set Ω with sizes of elements of Ω and their collections quantified by subsets of another space X. The contents of particular sections as scheduled in the introductory one have been more or less tightly kept and it is why we do not take as necessary to repeat them now, rather focusing our attention to some inspirations for further developments.

First, worth of interest are set-valued distributions taking values in power-sets of particular sets X, interesting and important. E.g., take the map of a region with different subregions colored by different colors yielding some information on different regions due to the system according to which the system of colors is known to the user.

More theoretical but interesting enough are the problems of incomplete set-valued possibilistic distributions and measures. In [7], we proposed possibilistic distributions $\pi : \Omega \to \mathcal{P}(X)$ and possibilistic measures $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ as complete mappings, so that for each $\omega \in \Omega$ and each $A \subset \Omega$ the values $\pi(\omega) \in \mathcal{P}(X)$ and $\Pi(A) = \bigcup_{\omega \in A} \pi(\omega)$ are defined. However, in probability spaces $\langle \Omega, \mathcal{A}, P \rangle$ with finitely additive probability measure P on finite field $\mathcal{A}$ only the value $P(\bigcup_{A \in \mathcal{S}} A)$ for finite system $\mathcal{S}$ of disjoint subsystem of $\mathcal{A}$ may be defined and computed from values of P on $\mathcal{A}$. Hence, only probability spaces which may be fully described by relative frequences of their results may be fully defined by probability spaces and if this is the case, finitely additive probability measures suffices. For infinite spaces $\langle \Omega, \mathcal{A}, P \rangle$ and for the Borel or Lebesque subsets of real line the Borel measure defined for semi-open interval by their length may be in a consistent and conservative way extended to Borel or Lebesque sets, but there are subsets of the real line which are measurable neither in the Borel nor in the Lebesque sense, so that the system of all Borel and Lebesque subsets of the real line measurable in the Borel or Lebesque sense remains incomplete.

As it is well-known, in the competition of set-functions in general and measures, including the probabilistic ones, in particular much more applications in various practical computational and technical problems have been based on set-functions based on Borel and Lebesque real-valued measures. There are measures not defined on all subsets of the basic space, but keeping intuitive and easy to compute and process values on sets where their values are defined. Consequently, even when set-valued distributions and measures introduced and analyzed above lead to complete measures, it should be useful and interesting to admit the incompleteness of the resulting set-valued possibilistic distributions and measures from the very primary and axiomatic approach to set-valued possibilistic distributions and measures. Let us hope to have an opportunity to analyze this problem in more detail in some future work.

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