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Abstract

Theory of special types of (fuzzy) filters on different algebras of non-classical logics has been intensively studied in the last decade. This contribution provides a generalization which covers many particular results and allows us to deal with special types of (fuzzy) filters in a uniform way. Our approach is based on simple definitions of a t-filter and a fuzzy t-filter. We are going to state and prove some basic properties of (fuzzy) t-filters and formulate generalizations of the most typical kinds of results occurring in the literature. We show that these results in this field can be generated via simple principles described in this paper.

1. Preliminaries and Basic Definitions

In this section we are going to recall the notion of a filter on a residuated lattice (more preciously a *bounded pointed commutative integral residuated lattice*).

In the whole text we are going to use often the following comfortable convention: the symbol $\overline{x}$ will be used as an abbreviation of $x, y, \dots$ i. e. for a listing of variables (neither a sequence, nor a vector) – therefore we can correctly write $\overline{x} \in L$ instead of $x, y, \dots \in L$.

At the very beginning we recall the definition of a residuated lattice.

Definition 1 *A* residuated lattice *is a structure*

$$\boldsymbol{L} = (L, \&, \to, \wedge, \vee, \overline{0}, \overline{1})$$

of type $(2, 2, 2, 2, 2, 0, 0)$ *satisfying the following conditions:*

1. $(L, \wedge, \vee, \overline{0}, \overline{1})$ *is a bounded lattice.*
2. $(L, \&, \overline{1})$ *is commutative semigroup with the unit element* $\overline{1}$.
3. $(\&, \to)$ *form an adjoint pair, i.e.* $x \,\&\, z \leq y$ *if and only if* $z \leq x \to y$ *for all* $x, y, z \in L$.

A comprehensive overview on residuated lattices and their subvarieties is provided by [1].

Since now we assume that $\boldsymbol{L}$ is a residuated lattice and L its domain.

Definition 2 *A non-empty subset* F *of* $\boldsymbol{L}$ *is called a* filter *on* $\boldsymbol{L}$ *if it satisfies these two conditions:*

1. *if* $x, y \in F$, *then* $x \,\&\, y \in F$,
2. *if* $x \in F$, $x \leq y$, *then* $y \in F$,

for all $x, y \in L$.

The equivalent definition of a filter on a residuated lattice is presented in the following theorem:

Theorem 3 ([2]) *A non-empty subset* F *of* $\boldsymbol{L}$ *is a filter on* $\boldsymbol{L}$ *if and only if it satisfies this following conditions:*

1. $\overline{1} \in F$,
2. *if* $x \in F$ *and* $x \to y \in F$, *then* $y \in F$.

for all $x, y \in L$.

Roughly said, this theorem shows that filters are just 'deductively closed' subsets of L. Therefore some authors uses the name 'deductive systems' – [3]. The connection between the notion of a filter in logic and the notion of a filter in algebra is described in [4], [5] or [6].

Filters can be also defined by many other equivalent ways, for example as a subsets of L containing $\overline{1}$ and satisfying one of these following conditions:

1. if $x \to y \in F$ and $y \to z \in F$, then $x \to z \in F$,
2. if $x \to y \in F$ and $x \,\&\, z \in F$, then $y \,\&\, z \in F$,
3. if $x, y \in F$ and $x \leq y \to z$, then $z \in F$.

for all $x, y, z \in L$ (see [2]).

2. Notion of a t-Filter

In the literature there is a great amount of papers concerning different types of filters on residuated lattices (or subvarieties of residuated lattices, such as BL-algebras, MTL-algebras, etc.). The notion of a t-filter on a residuated lattice was set up in order to generalize these particular results about special types of filters (implicative, boolean, etc.).

Definition 4 ([7]) *Let t be an arbitrary term in the language of residuated lattices. A filter F on $\boldsymbol{L}$ is a t-filter if $t(\overline{x}) \in F$ for all $\overline{x} \in L$.*

The definition in the submitted paper [7] uses slightly more general underlying structure which does not require $\overline{1}$ to be the greatest element, but for the purposes of this contribution we can conveniently restrict ourselves on residuated lattices.

It can be shown that many special types of filters are just t-filters for suitably chosen term t: in BL-algebras implicative filters are just t-filters for

$$t = x \to x \,\&\, x,$$

positive implicative filters are t-filters for

$$t = (\neg x \to x) \to x,$$

and fantastic filters are t-filters for

$$t = \neg\neg x \to x.$$

Recall that in BL-algebras $\neg x$ is defined as $x \to \overline{0}$.

This follows from the corresponding (technical) results in [8]. These special types of filters are defined by condition '$\overline{1}$ is in the filter' and some additional specific condition. After stating this definition, authors prove that defined special type of filter is a filter. This approach is in some sense a bit unnatural: we usually expect that a special type of filter is a filter having some additional properties. This intuition is rendered in the definition of a t-filter.

Remark: the question whether we can find the corresponding term t for a given special type of filter is closely related to the question of axiomatizability of the logics involved.

Note that the answer of the question whether the class of t_1-filters and the class of t_2-filters (for different terms t_1 and t_2) are equal, depends on the algebra we are working on.

3. Fuzzy Case

Definition 5 *Let X be an arbitrary non-empty set. A function $\mu : X \longrightarrow [0,1]$ is called a* fuzzy set *on X. If μ is a fuzzy set on the set X, then for any $\alpha \in [0,1]$ we denote the set $\{x \in X \mid \mu(x) \geq \alpha\}$ by the symbol μ_α.*

Note that a characteristic function of an arbitrary set A can be viewed as a fuzzy set on A.

Definition 6 *A fuzzy set μ of L is a* fuzzy filter *on $\boldsymbol{L}$ if and only if it satisfies the following two conditions:*

1. $\mu(x \,\&\, y) \geq \min\{\mu(x), \mu(y)\}$,
2. *if $x \leq y$, then $\mu(x) \leq \mu(y)$,*

for all $x, y \in L$.

As presented in [2], this definition can be alternatively formulated in the following way:

Theorem 7 *A fuzzy set μ of L is a fuzzy filter on $\boldsymbol{L}$ if and only if it satisfies the following two conditions:*

1. $\mu(y) \geq \min\{\mu(x), \mu(x \to y)\}$,
2. $\mu(x) \leq \mu(\overline{1})$,

for all $x, y \in L$.

The first condition in this theorem is in fact a 'fuzzy version of modus ponens'.

The relationship between fuzzy filters and filters on $\boldsymbol{L}$ is illustrated by the next theorem:

Theorem 8 ([2]) *A fuzzy set μ on $\boldsymbol{L}$ is a fuzzy filter if and only if for each $\alpha \in [0,1]$ the (crisp) set μ_α is either empty or a filter on $\boldsymbol{L}$.*

One of the key notions of this contribution is a notion of a fuzzy t-filter, which is a natural fuzzification of the concept of a t-filter.

Definition 9 *Let t be an arbitrary term in the language of residuated lattices. A fuzzy filter μ on $\boldsymbol{L}$ is called a* fuzzy t-filter *on $\boldsymbol{L}$, if it satisfies $\mu(t(\overline{x})) = \mu(\overline{1})$ for all $\overline{x} \in L$.*

According to the previous definition, we can see that fuzzy boolean filters ([2] and [9]) are just fuzzy t-filters for

$$t = x \vee \neg x.$$

Analogously, regular fuzzy filters ([2]) are fuzzy t-filters for

$$t = \neg\neg x \to x.$$

These filters on MTL-algebras (in the crisp case) are known as IMTL-filters (see [10]).

Similarly as in the crisp case, special types of fuzzy filters are usually presented in a slightly different way – for example fuzzy fantastic filters (fuzzy MV-filters) are defined as a fuzzy filters satisfying

$$\mu(((x \to y) \to y) \to x) \geq \mu(y \to x).$$

However, as shown in [2] again, fuzzy fantastic filters are just fuzzy t-filters where condition

$$\mu(((x \to y) \to y) \to ((y \to x) \to x)) = \mu(\overline{1})$$

holds for all $x, y \in L$.

There is a very close relationship between t-filters and fuzzy t-filters. It can be described in the terms of 'cut-consistency', which is the content of the next theorem.

Theorem 10 *A fuzzy filter μ on $\boldsymbol{L}$ is a fuzzy t-filter if and only if for each $\alpha \in [0, 1]$ the (crisp) set μ_α is either empty or a t-filter on $\boldsymbol{L}$.*

Proof: Let μ be a fuzzy t-filter on $\boldsymbol{L}$, $\alpha \in [0, 1]$. If $\alpha > \mu(x)$ for all $x \in L$, then μ_α is obviously empty. Otherwise let $z \in \mu_\alpha$. Thus $\mu(z) \geq \alpha$. From Theorem 8 we already know that μ_α is a filter on $\boldsymbol{L}$.

Since μ is a fuzzy t-filter, then $\mu(t(\overline{x})) = \mu(\overline{1})$ for all $\overline{x} \in L$, so

$$\mu(t(\overline{x})) = \mu(\overline{1}) \geq \mu(z) \geq \alpha$$

for all $\overline{x} \in L$, hence $\mu(t(\overline{x})) \geq \alpha$, so $t(\overline{x}) \in \mu_\alpha$ for all $\overline{x} \in L$. Thus μ_α is a t-filter.

Conversely, we assume that μ_α is a t-filter or an empty set for each $\alpha \in [0, 1]$. Let us choose $\mu(\overline{1})$ as α. Since $\overline{1} \in \mu_{\mu(\overline{1})}$, then $\mu_{\mu(\overline{1})}$ is non-empty. So $\mu_{\mu(\overline{1})}$ is – due to the assumption – a t-filter, thus $t(\overline{x}) \in \mu_{\mu(\overline{1})}$ for all $\overline{x} \in L$. Hence $\mu(t(\overline{x})) \geq \mu(\overline{1})$ and therefore $\mu(t(\overline{x})) = \mu(\overline{1})$ for all $\overline{x} \in L$. ■

As the corollary we obtain a relationship between characteristic functions of t-filters and fuzzy t-filters.

Theorem 11 *Let F be a filter of $\boldsymbol{L}$. Then F is a t-filter if and only if χ_F is a fuzzy t-filter of $\boldsymbol{L}$.*

Proof: Straightforward consequence of the previous theorem. ■

There is one more simple theorem showing the relationship between t-filters and fuzzy t-filters generalizing Theorem 5.20 in [2].

Theorem 12 *Let F be a t-filter on $\boldsymbol{L}$. Then there exists a fuzzy t-filter μ on $\boldsymbol{L}$ such that $\mu_\alpha = F$ for some $\alpha \in (0, 1)$.*

Proof: Let us define μ on L by cases:

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in F \\ 0 & \text{if } x \notin F, \end{cases}$$

where α is an arbitrary number ($0 < \alpha < 1$). Clearly, $\mu_\alpha = F$. Now we simply apply Theorem 10. ■

4. Core Results about t-Filters and Fuzzy t-Filters

In this section we are going to present generalizations of many statemements about special types of filters via our notion of a (fuzzy) t-filter.

Let us start with a crisp case.

Theorem 13 ([7]) *Let F and G be filters on a residuated lattice $\boldsymbol{L}$ such that $G \supseteq F$. If F is a t-filter, then so is G.*

Proof: Thanks to the definition of a t-filter the proof is obvious. ■

This theorem is often referred as an 'extension theorem'. Example of a particular result is provided in the next theorem.

Theorem 14 ([11]) *If F is a positive implicative filter, then every filter G containing F is also a positive implicative filter in any* BL*-algebra.*

Theorem 15 ([7]) *Let $\mathbb{B}$ be a subvariety of residuated lattices and $\boldsymbol{L} \in \mathbb{B}$. Moreover let $\mathbb{C}$ be a subvariety of $\mathbb{B}$ such that in all $\boldsymbol{C} \in \mathbb{C}$ the equation $t = \overline{1}$ holds. Then the following statements are equivalent:*

1. *Every filter on $\boldsymbol{L}$ is a t-filter.*
2. *$\{\overline{1}\}$ is a t-filter.*
3. *$\boldsymbol{L} \in \mathbb{C}$.*

This theorem generalizes many particular theorems like the following one:

Theorem 16 ([11]) *In any* BL*-algebra $\boldsymbol{A}$, the following conditions are equivalent:*

1. *Every filter on $\boldsymbol{A}$ is an implicative filter.*
2. *$\{\overline{1}\}$ is an implicative filter.*
3. *$\boldsymbol{A}$ is a Gödel algebra.*

It can be proved easily by the fact that implicative filters on BL-algebras are just t-filters for $t = x \to x \mathbin{\&} x$, the fact that Gödel algebras are just BL-algebras satisfying $x \to x \mathbin{\&} x = \overline{1}$ and our theorem about t-filters.

Now we are going to state and prove fuzzy counterparts of mentioned theorems.

Theorem 17 *Let μ, γ be fuzzy filters on $\boldsymbol{L}$, $\mu(x) \leq \gamma(x)$ for all $x \in L$, and moreover, $\mu(\overline{1}) = 1$. If μ is a fuzzy t-filter, then γ is also a fuzzy t-filter.*

Proof: μ is a fuzzy t-filter, hence $\mu(t(\overline{x}) = \mu(\overline{1})$ for all $\overline{x} \in L$. Since $\mu \leq \gamma$ (pointwise) and also $\mu(\overline{1}) = \gamma(\overline{1}) = 1$, we directly obtain

$$\gamma(t(\overline{x})) = 1 = \gamma(\overline{1}),$$

for all $\overline{x} \in L$, thus γ is a fuzzy t-filter. ■

Application of this theorem is straightforward again. If we we choose $t = \neg\neg x \to x$ for example, we get the following particular result taken from [2]:

Corollary 18 ([2]) *Let μ and ν be fuzzy filters of $\boldsymbol{L}$ with $\mu \leq \nu$ and $\mu(\overline{1}) = \nu(\overline{1})$. If μ is a fuzzy regular filter of $\boldsymbol{L}$, then so is ν.*

The set of all fuzzy filters on $\boldsymbol{L}$ is denoted by the symbol $FF(\boldsymbol{L})$. The next theorem shows how can we describe a subvariety of residuated lattices via the properties of (all/certain important) filters on $\boldsymbol{L}$.

Theorem 19 *Let $\mathbb{B}$ be a subvariety of residuated lattices and $\boldsymbol{B} \in \mathbb{B}$. Moreover let $\mathbb{C}$ be a subvariety of $\mathbb{B}$ such that in all $\boldsymbol{C} \in \mathbb{C}$ the equation $t = \overline{1}$ holds. Then the following statements are equivalent:*

1. *Every fuzzy filter on $\boldsymbol{B}$ is a fuzzy t-filter.*
2. *$\chi_{\{\overline{1}\}}$ is a fuzzy t-filter.*
3. *$\mu_{\mu(\overline{1})}$ is a t-filter for any $\mu \in FF(\boldsymbol{L})$.*
4. *$\boldsymbol{B} \in \mathbb{C}$.*

Proof: At first we are going to prove the 'circle' 1., 2., 4. and then we are going to connect the third statement.

1.⇒2.: $\chi_{\{\overline{1}\}}$ is a fuzzy filter (consequence of Theorem 8) and thanks to the assumption $\chi_{\{\overline{1}\}}$ is also a fuzzy t-filter.

2.⇒4.: If $\chi_{\{\overline{1}\}}$ is a fuzzy t-filter, then $\{\overline{1}\}$ is a t-filter (using Theorem 10). Therefore $t(\overline{x}) \in \{\overline{1}\}$ for all $\overline{x} \in L$, hence $t = \overline{1}$, so $\boldsymbol{B} \in \mathbb{C}$.

4.⇒1.: If $\boldsymbol{B} \in \mathbb{C}$, then $t(\overline{x}) = \overline{1}$ for all $\overline{x} \in L$, and hence also $\mu(t(\overline{x})) = \mu(\overline{1})$. Thus fuzzy filter μ is a fuzzy t-filter.

1.⇒3.: If $F \in FF(\boldsymbol{L})$, then by the assumption μ is a fuzzy t-filter. We apply Theorem 10 with $\alpha = \mu(\overline{1})$.

3.⇒2.: $\chi_{\{\overline{1}\}}$ is a fuzzy filter, so we choose $\chi_{\{\overline{1}\}}$ as μ in 3. We obtain that $\mu_{\mu(\overline{1})} = \{\overline{1}\}$ is a t-filter an also — by Theorem 11 — that $\chi_{\{\overline{1}\}}$ is a fuzzy t-filter. ■

Again we are going to show an application of this theorem on concrete particular example concerning fuzzy Boolean filters (Theorem 4.15 from [2]).

Theorem 20 *In any residuated lattice $\boldsymbol{L}$, the following assertions are equivalent, for all $x, y \in L$:*

1. *$\boldsymbol{L}$ is a Boolean algebra.*
2. *Every fuzzy filter of $\boldsymbol{L}$ is a fuzzy Boolean filter of $\boldsymbol{L}$.*
3. *$\chi_{\{\overline{1}\}}$ is a fuzzy Boolean filter of $\boldsymbol{L}$.*

Proof: Let t be the term $\neg x \vee x$. Recall that 'residuated lattices where $t = \overline{1}$ holds are just Boolean algebras'. The rest is provided by Theorem 19. ■

5. Application of the Theory and Obtaining 'New' Results

One of the main aims of this paper is to illuminate that many published particular results can be generated (using the theory presented in this paper) in an 'automatic' way.

We are going to demonstrate this fact on the examples arising in residuated lattices, resp. MTL-algebras. We are going to define a new special (and artificial!) kind of filter, called *prelinear filter*, as a t-filter for certain t, analogously also *fuzzy prelinear filter*.

Recall that MTL-algebras are just residuated lattices satisfying $(x \to y) \vee (y \to x) = \overline{1}$ (axiom of prelinearity).

Definition 21 *A filter F on $\boldsymbol{L}$ is called a* prelinear filter *on $\boldsymbol{L}$, if it satisfies $((x \to y) \vee (y \to x)) = \overline{1}$ for all $x, y \in L$.*

Definition 22 *A fuzzy filter μ on $\boldsymbol{L}$ is called a* fuzzy prelinear filter *on $\boldsymbol{L}$, if it satisfies*

$$\mu((x \to y) \vee (y \to x)) = \mu(\overline{1})$$

for all $x, y \in L$.

Theorem 23 *The following statements are equivalent for each residuated lattice $\boldsymbol{L}$:*

1. *Every filter on $\boldsymbol{L}$ is a prelinear filter.*
2. *$\{\overline{1}\}$ is a prelinear filter.*
3. *$\boldsymbol{L}$ is an MTL-algebra.*

Theorem 24 *Let μ and γ be fuzzy filters of $\boldsymbol{L}$ satisfying $\mu(x) \leq \gamma(x)$ for all $x \in L$ and $\mu(\overline{1}) = \gamma(\overline{1})$. If μ is a fuzzy prelinear filter on $\boldsymbol{L}$, then so is γ.*

Theorem 25 *The following assertions are equivalent, for all $x, y \in L$:*

1. *Every fuzzy filter of $\boldsymbol{L}$ is a fuzzy prelinear filter of $\boldsymbol{L}$.*
2. *$\chi_{\{\overline{1}\}}$ is a fuzzy prelinear filter of $\boldsymbol{L}$.*
3. *$\boldsymbol{L}$ is an MTL-algebra.*

Proof: All these theorems are simple consequences of theorems in the previous sections for

$$t = (x \to y) \vee (y \to x).$$ ■

6. Conclusion and Final Remarks

This work summarizes some results about generalizations of special types of filters. As we can see, provided proofs are simple — the significance of this work comes not from the complexity of the theory, but rather from the amount of particular results which it covers.

The submitted paper [7] contains also generalization of results about the relationship of special types of filters and quotient algebras. In one of the prepared papers we are going to state and prove analogous result about fuzzy quotients.

There is one more way of generalizations: the most widely used definition of a fuzzy filter is closely bind to the Gödel min-conjunction. In this context it is possible to investigate such generalizations where this min-conjunction is replaced by any t-norm.

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