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Derviz, Alexis
2003

WORKING PAPER SERIES

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6/2003

THE WORKING PAPER SERIES OF THE CZECH NATIONAL BANK

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Project Coordinator:	Tibor Hlédik	

FOREX Microstructure, Invisible Price Determinants, and the Central Bank's Understanding of Exchange Rate Formation

Alexis Derviz[*]

Abstract

The paper investigates the transmission of macroeconomic factors into the price-setting behavior of a specific dealer in the FX market. This problem is viewed from the perspective of a central banker who observes the price evolution but does not make the market in the home currency. The central banker's task is to explain the forex behavior in terms of conventional economic logic. The analysis is based on a model of a multiple dealer market under two organizations: direct inter-dealer and brokered. The model is constructed in such a way as to reflect the most prominent features of the market for the Czech koruna and, accordingly, to address some issues of key relevance to the Czech National Bank's exchange rate policy.

We show that the totality of the exchange rate-relevant fundamental factors influence the market maker's behavior through a single sufficient statistic, his "marginal" valuation of foreign currency holdings. Under the two studied trading mechanisms, the marginal valuations across market participants determine the equilibrium exchange rate by means of different trade patterns. Specifically, the brokered market is inferior to the direct one in terms of welfare improvement through trade. It takes a higher inter-dealer trade volume in the brokered market to absorb a new price impulse. Therefore, the central banker would do best by monitoring the brokered segment (as the only partially transparent one available), but by conducting interventions in the direct segment, where the desired impact is easier to achieve.

JEL Codes: F31, G15, C72.
Keywords: forex microstructure, multiple dealership, order flow, pricing schedule.

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[*] The Czech National Bank, Monetary and Statistical Departmant, and Insitute for Information Theory and Automation, Prague, E-mail:Alexis.Derviz@cnb.cz.

Nontechnical Summary

Reconciling the monetary policymaker's understanding of observed exchange rate behavior with that of the market practitioner within the same central bank requires finding a common denominator for the fundamental characteristics of the economy and the logic of price formation in a specific FX market. More generally, it is often hard to capture the mechanism whereby the generally accepted macroeconomic reasoning is reflected in the price-setting behavior of FX market makers. This is why the methods of forex microstructure theory are being employed by policymakers to understand how macroeconomic factors and policy measures are channeled into the national currency valuation in the really existing market.

The present paper uses the techniques of microstructure finance to model a multi-dealer FX market. We analyze two trading organizations: a direct inter-dealer market and a brokered market. The information that the FX dealers learn from their received order flow determines how they set their quotes. The model is constructed in such a way as to reflect the most prominent features of the market for the Czech koruna and, accordingly, to address the issue of the Czech National Bank's ability to influence the koruna exchange rate by policy measures. The problem is viewed from the perspective of a central banker who observes the price evolution but does not make the market in the home currency.

The model demonstrates that the exchange rate-relevant fundamental factors influence the market maker's behavior through a single sufficient statistic, his "marginal" valuation of foreign currency holdings. From this theoretical result, we derive consequences for the observed patterns of currency trade, and show, in particular, that the same exchange rate movement can be caused by informational, institutional and inventory asymmetry across dealers and between dealers and non-dealer traders. Moreover, even fully identical dealers are shown to trade non-zero volumes between themselves, since this is the only way for them to learn the marginal currency valuation of other market participants.

The contribution to the exchange rate modeling literature consists in:

1. formulating a realistic exchange rate formation model in which dealers learn about the currency demand (which is well-defined) and not the fundamental currency value (which is not well-defined),
2. visualizing the role of "global" investors – trading with all FX dealers – in disseminating the fundamental currency supply and demand shocks,
3. comparing the direct and brokered FX trading mechanisms and consequences in terms of the trade-off between higher transparency in the brokered market against higher gains from trade in the direct market,
4. connecting the price impact of the forex order flow with the indirect foreign currency utilities of market participants,
5. providing a rationale for the policymaker observing a more transparent (brokered) segment of the forex but conducting exchange rate interventions in a more efficient (direct) one.

1. Introduction

It is often difficult to reconcile the international macroeconomist's understanding of observed exchange rate behavior with that of the market practitioner. The former seeks to explain the currency price by the generally accepted fundamental characteristics of the economy (such as growth, productivity, inflation or interest rate differentials against the reference economy, and trade and investment flows). The practitioner's view of the same price is determined by his or her knowledge of the current demand or supply overhang. The currency trader usually perceives only the immediate reason for this overhang, such as the liquidity need of a major investor or a position shift of a significant group of speculators. Even the key economic and policy announcements, which both categories follow and interpret, fail to create the desired link between these two exchange rate views. This is because, differently from common stock, it is usually impossible to find "hard" information concerning any investor's earnings in the messages about the economy as a whole as disseminated by news agencies. Accordingly, the impact of economic announcements, if any, often proves to be short-lived or even counter-intuitive, making these announcements nearly irrelevant in the longer perspective for economists and traders alike. More generally, it is hard to define the nature of the information that the FX dealers learn from their received order flow when they set their quotes. One way is to assume that there is a "fair" value of the currency known by a well-informed customer, which the dealer discovers in the process of executing the customer's order(s). Most practitioners suspect that there is no such thing in reality.

This absence of a common language has generated interest in a novel discipline, *forex microstructure theory*, which promises to explain the exchange rate by the economically justifiable behavior of market participants who utilize a specific trading mechanism. Thus, forex microstructure theory has taken on the extremely challenging task of applying abstract microstructure finance to an object, the FX market, which is substantially more complicated than the markets for most other financial assets. Beside the natural objective of justifying the actions of those who buy and sell the currency using comprehensive choice-theoretic motives, this new branch of financial economics has had another – quite thankless – task: to establish a link between its results and the prevailing orthodoxies of international economics. The present paper contributes to both objectives by proposing a model with a "common micro–macro denominator". Specifically, we show that the *marginal (indirect) utility of foreign currency* of an optimizing dealer reflects both the fundamental factors behind the exchange rate and the supply/demand curve that the dealer operates with when setting quotes or placing orders.

Principal Outputs of the Paper

This paper contains a formal toolkit that allows one to link liquidity, inventory and information in the Czech koruna market with fundamental shifts in the external position of the Czech economy. The model constructed to this end captures the preferences and behavior of international dealers, domestic banks, their clients and international investors. The dealers have the same preference structure as other investors; their market-maker role is exogenous. The model proposes an external shock propagation mechanism through the market participant's indirect utility into changes in the Czech koruna market value. We abstain from the artificial notion of the "fair currency value" discovery. Instead, we model strategic interaction in a quite basic setting of a normal form game, without a pre-defined statistical law to be discovered or learned.

To avoid the problem of separating customer–dealer from inter-dealer trades[1], the model handles all components of the received order flow as potentially informative. It is able to characterize the outcome of an active order of one party reaching the other party in a certain state. Consequently, an order can have a big or small price impact irrespective of whether it comes from a client or a dealer. Instead, the impact-determining factor is the marginal currency valuation by the order-receiving market maker. This marginal valuation can have a low level (prompting the dealer to reduce position), a high level (expand position) or a neutral level (not to change position). In the last-mentioned "no-trade" case the marginal valuation induces the agent to abstain from using the market for own buys or sales. If the received order is big enough to cause a move from this neutral level to a low marginal valuation level (dictating a position reduction as the optimal action) or to a high level (creating a preference for position expansion), then the order evidently has a price impact. This is because the market maker would change his quotes in order to invite the corresponding trades with other market users. Conversely, if the received order results in a shift in the originally non-neutral marginal valuation towards the no-trade value, the price impact is negligible, since the market maker has no need to "provoke" additional trades by shifting quotes. Altogether, we replace a one-dimensional picture of the order flow impact on prices with a two-dimensional one, where the price change is a function of both the incoming order and the current marginal valuation.

The model defines a "global" market user, i.e. an agent who has access to all quotes and trades with all existing market makers. That is, such an agent has the same possibility of using the inter-dealer market as the dealers themselves. This feature of the model guarantees straightforward demand-shock propagation. Indeed, the existence of global investors is the reason why the observed order flow can serve as a source of future price-relevant information for the dealer. Since the global investors' orders are split among all market makers, whilst the order signs and volumes are determined by their marginal valuations, privately received orders from global investors confer information about the overall direction in which the market is moving. This is much more than what can be read off the orders from "local" market users (exclusive clients of a given dealer); the latter only affect the dealer's endowment with which the subsequent inter-dealer trading round is entered.

We construct two variants of the model, one for the direct trading mechanism and the other for brokered trading mechanism. So, the institutional factors (in particular, the relative transparency of the brokered trading mechanism against the high liquidity of direct dealership) can be separated from the tastes, technologies and endowments of economic agents. In the brokered market, the dealer–client role difference only concerns the limit order submission (in the form of pricing schedules), which is reserved for the dealers. Both dealers and non-dealers submit market orders.

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[1] The main motive behind this is that, as has followed from informal discussions with a number of dealers in prominent CZK-market participant institutions, the dealers themselves rarely make this distinction. They also claimed it to be irrelevant for how they operate in the market themselves. That is, at least for the Czech koruna, the anecdotal evidence collected by the author suggests that the counter-party's status as a market maker or user plays no particular role in deciding the information value of the received order.

First, dealer pricing schedules are automatically crossed by the broker and then the integral "standing pricing schedule" is announced. Therefore, the dealer cannot fully determine the market user order by means of his own pricing schedule, even if he takes the market user demand as given. The non-dealer investor in the brokered market is only given access to the standing pricing schedule. Her order is split proportionally between dealers, based on the totality of their quoting parameters. Therefore, the market user cannot split orders herself. Altogether, the presumed single price advantage of the brokered market has a cost: we show that the ability of investors to adjust their marginal foreign cash utility by trade is limited compared to the direct market.

In the direct market, a market user's optimal behavior consists in splitting orders between all market makers in proportions that equalize the marginal values of the purchased/sold quantities across contacted dealers (see later). This property of our model is similar to that of Bernhardt and Hughson, 1997, and Menkveld, 2001.

The model possesses a Nash equilibrium that equalizes the market users' indirect marginal utilities of foreign cash and the effective quotes of the market makers. The properties of this equilibrium show that the order flows (of both dealers in the inter-dealer market and their customers) are functions of these marginal indirect utilities, and so are the quoting parameters (i.e. the pricing schedule slope and intercept) of the dealers. Although we do not study imperfect or incomplete information game settings, the chosen benchmark concept of the Nash equilibrium in the normal form inter-dealer game still broadens our understanding of the learning by dealers from the order flow. This learning is compressed to one period and is, therefore, more implicit than is usual in the theoretical microstructure literature. Nevertheless, it allows us to model equilibrium outcomes in the inter-dealer market in situations where one cannot make a reasonable assumption about the "true" statistical law of uncertainty. That is, the dealers in our model do not learn about a stationary distribution of a fundamental factor behind the currency value, but go directly to learning the instantaneous marginal valuation of the currency by the counter-parties.

Formally, the set of model parameters that can cause dealer heterogeneity is formed by their exogenously given endowments of domestic and foreign cash. The same variable defined for the non-dealer investor is the source of the customer order flow. However, the present model does not belong to the inventory trade category, since the equilibrium outcome properties are driven by the investor marginal valuations, not the inventory levels directly. This marginal valuation discovery property manifests itself in many ways, among them the ability of the model to generate non-trivial hot potato trades in equilibrium, even when dealer inventories are identical. The dealer directs orders toward other dealers even though he knows that the very same order will be directed back to him. This is so because a non-zero inter-dealer trade in equilibrium is necessary to infer the marginal currency values of the counter-parties.

Since the closed form solution could have only be obtained at the cost of many counter-intuitive simplifications, we have given up the quest for explicitness in favor of numerical solutions of a few practically relevant cases. Specifically, we compute Nash equilibria of the normal form game between two dealers and an investor in the FX market. We then conduct comparative statics analysis of the two market structures for different levels of investor demand.

Contributions to Czech National Bank Policymaking

The general contribution of the conducted research can be found in an improvement of the monetary authority's ability to understand the motives behind the FX market maker quotes and market user orders. Although the observed "price", i.e. the exchange rate, always settles down after any major event in the forex with a relatively short delay, the price formation processes can vary substantially across dealers and the parts of the market in which they operate. We discuss two aspects of this difference in particular:

a) between dealers, depending on their marginal foreign currency valuations,

b) between market segments with different trade mechanisms, i.e. direct and brokered.

From the FX risk management perspective, our model points at the latent variable that is primarily responsible for exchange rate deviations from the "fundamental" levels. Namely, increased volatility can originate in the marginal valuation heterogeneity across market participants. The formal theoretical result could be employed in the design of procedures to extract the unobserved distributions of the marginal currency values of the selling and the buying group of dealers from time series of their quotes. This would provide a refinement of the standard VaR measure of FX exposure.

A popular question among regulators concerns the preferred trade institution. A recognized cost of using a direct inter-dealer market is price non-transparency and possible idiosyncratic price distortions due to dispersed information across dealers. Therefore, some authors produced models that claimed the inevitable advent of an electronic open limit order book as the sole trade platform (Glosten, 1994), given its presumed transparency and lack of price discrimination. This was not directly expressed with regard to the forex and, indeed, this market has proven to be incompatible with the said prophecy. Direct dealership and various types of brokers have coexisted in this market for years, and the Czech koruna is no exception to this rule (cf. Section 3). In our paper, we show that the cost of price transparency in the brokered market can be higher than the benefit. Specifically, since the brokered market's transparency only refers to the pricing schedule and not the counter-party identity (at least, not until after the transaction has been completed), the strategic interaction between the order sender and order recipient is hampered. With our analysis of the direct inter-dealer market, we demonstrate that this interaction not only leads to price discovery, but also has gains-from-trade effects superior to the brokered market. This means that policy measures should not be wasted in an attempt to support one or several brokers with the objective of "draining" direct dealership of its users.

On the other hand, the co-existence of the direct and brokered markets should be used by the monetary authority to:

a) *monitor the more transparent* of the two FX trading institutions (i.e. *brokers*) to extract information about the market as a whole,

b) overcome undesired asymmetry in the dispersed information in the forex by designing *interventions in the direct inter-dealer segment*.

Structure of the paper: Section 2 sums up the prominent features of the current FX microstructure research relevant to the work at hand. Section 3 gives a brief overview of the Czech koruna market institutions and participants. Section 4 defines the model, the agents' optimization problems and optimal active trade rules at given quotes (pricing schedules). This is done in both the direct inter-dealer and brokered inter-dealer market settings. In Section 5, we derive the optimal quoting conditions in the direct and brokered markets and discuss consequences concerning the role of marginal currency values for the price impact of the order flow. Section 6 outlines an imbedding of the one-shot inter-dealer game into a differential game between the same players. This embedding facilitates the Nash equilibrium calculation, described in the technical Appendix together with several proposition proofs. The second subsection of Section 6 discusses the outcomes of numerical NE calculations for the one-period game and the corresponding "reduced form" relations between order flows, prices and marginal FX values. Section 7 concludes.

2. State of the Art in the Forex Microstructure Research

Most existing FX microstructure models are anchored in abstract information microeconomics. So, these models remain too far from both mainstream macro and practitioners' thinking. To bridge the gap successfully, one needs a model combining the optimizing paradigm of standard macroeconomics with the key institutional features of the exchange rate setting taking place in the inter-dealer market with explicit rules of interaction. (As one of the founders of the FX microstructure economics, R. Lyons, 2001, puts it, "*It is a stubborn fact that there is no other exchange rate than that set by these people* [i.e. dealers]…")

Another existing conceptual gap is in the discrepancy between the formal FX microstructure models of an agent who *sets the fundamental asset (foreign currency) price* and an agent *who discovers that price* by participating in a specific trading institution.

Application of the theory is impaired by two artificial dividing lines:

a) between the customer/investor and dealer/market maker roles of a financial institution,

b) between the inventory handling and informational asymmetry/adverse selection factors behind price setting by an individual market maker.

The most popular models prefer to view the inter-dealer market as a location for the redistribution of customer order flow risks. To our knowledge, these models have not yet satisfactorily explained the co-existence of multiple brokerage systems in the forex, nor have they established the relative importance of dealers and other traders in generating persistent exchange rate movements. Technical difficulties lead to simplifying assumptions about dealer rationality and the precision of their knowledge of customer behavior. The outcomes of the models based on these assumptions convey marginally richer descriptions of inter-dealer trade than the classical auction market models, where agents condition on the market-clearing price (of which Kyle, 1985, is a common prototype). The complexities of decentralized quoting and trading have forced some authors to abandon the full rationality assumption. Instead, they model dealer interaction on a bounded rationality basis and simulate the outcomes of their adaptive learning. Finally, not enough work has been done so far on the distinctions between the brokered and direct segments of the forex (each covers about half of the inter-dealer market transactions). Ideally one should be able to model a mix of multiple direct dealerships and a brokered market with several institutional – as well as "cyberspace" – venues.

The pivotal notion of the modern FX microstructure theory is the twofold concept of the customer–dealer and dealer–dealer order flows. Both, although each in its own way, are responsible for dissemination of price impulses across the market. Put concisely, it has been convincingly demonstrated that order flow determines the exchange rate. *But what determines the order flow?* Accordingly, the principal problem in the practical application of FX microstructure theory is to identify the theoretical order flow variables known from the models with the publicly available information on the actions of a given financial intermediary.

Most contributions to the current FX microstructure literature identify the client order flow as the prime source of the new exchange rate information that is impounded in the prices set by the dealers (Evans and Lyons, 2002). At the same time, it is recognized that the basic Evans–Lyons paradigm of exchange rate formation implies a zero net client order flow, if aggregated across all market makers. This happens because the optimizing dealer price-setting behavior dictates the shift in the trade price to the level allowing the dealers to unwind undesired positions. Therefore, their equilibrium quotes induce the clients to generate "second stage" orders that would offset that part of their "first stage" orders made at the original quotes which is in excess of the dealer-acceptable value. So, if the net client order flow may erroneously evoke an impression of no-relation to the price change, what is the right order flow to look at?

One choice is the inter-dealer order flow, since it is generated by the dealers, who redistribute the initially taken position risk coming from client trades. However, the inter-dealer order flow would only reflect fundamental price information if the latter were dispersed asymmetrically among the clients of individual dealers. On the contrary, there would not be any information revelation if all dealers received the same client order flow signal. Therefore, one of the directions in which the basic Evans–Lyons paradigm can be extended is the introduction of investors who choose between, and trade with, several dealers almost simultaneously. Quite often, such traders are the most important carriers of fundamental price signals.

On the empirical side, the inter-dealer order flow is usually difficult to separate from the one coming from clients.[2] Orders of both categories arrive in random sequence and, moreover, a dealer may be a carrier of even stronger exchange rate information than any of the non-dealer investors, so his active trades are difficult to classify in accordance with the scheme outlined above. In other words, the inter-dealer order flow includes, but is not limited to, "hot potato" trades of unwanted currency quantities. Therefore, the primary and secondary reasons for the inter-dealer order flow are impossible to distinguish in the data.

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[2] It is totally impossible in the brokered trade data. A number of researchers had access to the complete trade books of a given dealer, which included information on the counter-party. Even then, the decision to classify the same party as a dealer or a customer sometimes had to be taken on an *ad hoc* basis. It would be particularly difficult in the case of the Czech koruna (or any other minor currency), where certain banks perform a market maker function for a while and then stop doing so for a prolonged period. The present model does not have to solve the problem, since it classifies any such bank as a "global market user" when modeling its active trades.

Regardless of the crucial role of order flow in the price formation process, established theoretically, the latter is publicly unobservable and has to be proxied. Equally unobservable are the actual transaction prices. Therefore, most of the ground-breaking empirical research in FX microstructure derives from studies of transaction data obtained privately on an *ad hoc* basis (see, for example, the monograph by Frankel, Galli and Giovannini, 1996, for a collection of now classic FX microstructure papers). That is, they cannot be reproduced on a regular basis for policymaking purposes.

Several major international FX-dealer banks collect their own data on received client and inter-dealer order flows. The best known example is Citibank with its G*lobal Flow* database. Such activity can have a contributive value to the bank's FX-position management, provided the order flow data it collects are sufficiently representative, i.e. it begins to make sense only starting from a certain market share level. Just a few institutions in the world qualify, and there is not a single central bank among them. A central bank of a small open economy must look for alternative sources of interpretable data. The present paper can serve as a guide as to what kind of information can be useful or appropriate.

3. Participants and Institutions in the CZK Market

Spot transactions in the Czech koruna exist predominantly in the koruna–euro pair (important investors wishing to exchange for CZK a position in another currency – USD, JPY, GBP, etc. – usually create a EUR-position first). Derivative instruments in both the CZK/EUR and CZK/USD pairs are more or less evenly distributed in terms of volume; historically, however, sufficient liquidity in forwards and swaps could only be found in the CZK/USD pair. Currently, spot, outright forward and FX swap quotes and trades on these two and a number of other currency pairs are available from major dealers. On the other hand, the inter-bank market only operates with spot and swap transactions (outright forwards for clients are constructed synthetically by entering an FX swap and offsetting its spot leg with the inverted spot trade).

There is a domestic inter-bank FX market for koruna, where the formal number of participants is eighteen, although at most eight execute more than 5 per cent of the trade volume in a given segment (identified by currency pair and instrument, i.e. spot, swap or option). Many of the trades they effectuate are initiated by parent companies outside the country. Each of the inter-bank participants utilizes the direct dealership form of inter-bank trade interchangeably with voice- and electronic FX-broker services (of the latter, Reuters Dealing-3000 is the only one actually used; there is no evidence of EBS being in any way involved in CZK trading).

There is a considerable amount of offshore koruna trading. Some of it is difficult to separate from the onshore operations described above, since the parent company of a domestically licensed bank may decide to execute the same type of CZK sell or buy order either through this bank domestically or itself, abroad. Specifically, direct inter-dealer and client–dealer koruna trades without the participation of a domestically licensed entity are undertaken by a small number of major internationally active dealer banks, such as Citibank, ABN–Amro, ING–Barings and Deutsche Bank, who all have branches licensed in the Czech Republic. Still, there also seems to be a permanent offshore koruna market segment in the proper sense, which exists, in part, due to a number of minor internet brokers (e.g. Bear Sterns, Rada Forex, Tullett&Tokyo, Olsen and ass., etc.).

Our information about the operation of direct dealerships is limited to posted indicative dealer quotes (on Reuters or Bloomberg screens). However, it has been documented in the empirical FX microstructure literature that the deviations of the indicative from the effective quotes are rarely significant and, under all circumstances, are very short-lived (about 10 minutes). Therefore, indicative dealer quotes offer a relatively accurate measure of individual dealer pricing behavior. On-line information on brokered forex is restricted to its users, so the exact picture of standing quotes, the depth of the limit order book, and the direction and intensity of market orders is rarely at the disposal of either regulators or analysts. (Our research is a step towards overcoming this difficulty by analyzing the direct and brokered segments with the help of the same model.)

Given this variety of market structures and participants, modeling the operation of the Czech koruna market may seem analytically insurmountable. Heavy stylizing offers the only chance of discerning tangible objects of interest and clear-cut results. Therefore, the toolkit of microstructure finance theory will be adopted in order to identify a modeling approach that will, on the one hand, accommodate the key phenomena of interest and, on the other hand, be amenable to formal treatment. In the model to be introduced next, we employ some of the most recent achievements of market microstructure theory.

4. The Model

This section proposes a model of multiple dealer forex trading in two variants: for direct and brokered market organization. In both variants, we shall consider a dealer, i.e. an agent who makes the market by posting quotes, and an investor, who trades at dealers' quotes without placing her own. However, both types of agents will act as *market users* when they place their own orders. The state variables and preferences of all agents will have the same structure. It is assumed for definiteness that all agents are domestic residents, but this assumption is not pivotal for the results.

The equilibrium order flow pattern is derived as a function of the shadow prices (marginal valuations) of FX holdings across market participants. The shadow valuation heterogeneity (due to differences in preferences, endowments or asset payoff information) gives rise to currency purchases/sales initiated by those market users whose marginal currency value is high/low in relation to the marginal valuation by the market maker. As a consequence, one would observe a strong price impact of the order flow when it moves the marginal value of the recipient dealer away from the no-trade level, and a weak or zero order flow impact when the received position reverts the marginal value back to it. The Nash equilibrium of the studied one-period inter-dealer game is obtained as a steady state Nash equilibrium of a differential game between the same players. We also find that differences in equilibrium quoting behavior result in higher volumes of inter-dealer trade in the brokered market compared to the direct market. Under both trading mechanisms, however, hot potato trades are non-zero even if the dealers are perfectly symmetric, since inter-dealer transactions are a necessary condition for price discovery. The brokered market gains from trade are lower for a given level of investor demand for foreign currency.

Let x^m be the domestic cash holdings and x^i the foreign cash holdings of a given investor. There is an exogenous endowment y^m of domestic cash and y^i of foreign cash. The liquid wealth of the market participant is valued by means of the utility function $(x^m, x^i) \mapsto v(x^m, x^i)$. This function is strictly increasing and strictly concave in each argument and converges to $-\infty$ when either argument goes to $-\infty$. An example is

$$v(x^m, x^i) = \gamma - \beta_m e^{-\alpha_m x^m} - \beta_i e^{-\alpha_i x^i}, \tag{1}$$

with positive constants α_m, α_i, β_m, β_i and γ. This definition of preferences implies that the agent's domestic and foreign assets (including, but not limited to, cash) are imperfect substitutes. The agent may have short selling constraints, limits on open positions or other reasons to avoid imbalances in the currency composition of his/her portfolio. Thus, negative cash holdings in either currency (debt, short position) are penalized whereas positive holdings have decreasing marginal value.

Each dealer quotes a pricing schedule that is a smooth convex transform of a linear schedule of the form

$$p=\rho+\sigma q,$$

where p is the transformed price, q is the quantity bid/offered ($q<0$ corresponds to purchases from the client and $q>0$ to sales to the client) and ρ, σ are constants, $\sigma>0$. The price is obtained by the rule $P=f(p)$, where f is a strictly positive, increasing and convex function on the real line. Our standard example will be $f(p)=e^{cp}$, for some positive constant c. The fact that we do not use linear pricing schedules directly, as, for example, in Kyle, 1985, and many papers following Kyle's, is explained by our effort to obtain internal solutions for individual investor problems and later – a well defined Nash equilibrium - without the need to check whether the resulting prices are positive. This is particularly important in the settings where a closed form solution cannot be derived and numerical methods are used to calculate equilibrium transaction patterns.

In order to economize on notation and highlight the most important qualitative results, we shall concentrate on the case where there are two dealers in the FX market (indexed by 1 and 2). There is also a single representative *global* non-dealer investor, indexed by U. The word "global" means that this market user approaches both dealers for quotes and trades.

4.1 Decentralized Dealership

In the market considered in this subsection, only bilateral quoting and trade between dealers and other investors exists. That is, the market is fully non-transparent: although every agent obtains both dealers' quotes, the *volumes* of effectuated trades other than one's own are unobservable. Therefore, the effective transaction prices are unobservable as well. We do not assign informational significance to any price signal, since the model does not consider stationary distributions of uncertainty factors. (Were such distributions a part of the considered equilibrium, they would be inferred by the dealers jointly from the orders and prices in the course of Bayesian learning. However, in most real-life FX markets, it is doubtful whether the assumption of stationary risk factors driving market user demands is justified.)

4.1.1 Global Non-Dealer Market User

We denote by ρ^j, σ^j the parameters of the pricing schedule quoted by dealer j, j=1,2. The orders placed by the investor at these quotes with the two dealers are denoted by Q^j, j=1,2 ($Q^j>0$ if the order is for a foreign cash purchase). Then the end-of-period domestic and foreign cash holdings of the investor are:

$$x^m = y^m - f(\rho^1 + \sigma^1 Q^1)Q^1 - f(\rho^2 + \sigma^2 Q^2)Q^2\,, \tag{2a}$$

$$x^i = y^i + Q^1 + Q^2\,. \tag{2b}$$

The investor maximizes (1) subject to (2). The parameters (ρ^j,σ^j),j=1,2, of the market-maker quoting schedules are taken as given by the market user.

Let ξ_m, ξ_i be the investor's *marginal utilities* of the domestic and foreign cash, i.e.

$$\xi_m = \frac{\partial v}{\partial x^m}(x^m, x^i) = v_m(x)\,,\ \xi_i = \frac{\partial v}{\partial x^i}(x^m, x^i) = v_i(x)\,.$$

We shall also call these partial derivatives the *investor's shadow prices* of the domestic and foreign currency respectively.

For us, the most important part of optimal policies is the one which describes the currency purchases/sales $Q^{1,2}$ from/to both dealers. Let us denote by

$$\theta = \frac{\xi_i}{\xi_m} = \frac{v_i(x)}{v_m(x)} = \frac{\alpha_i \beta_i}{\alpha_m \beta_m} e^{\alpha_m x^m - \alpha_i x^i}$$

the relative *marginal valuation* of foreign currency by the market user. Then the optimal trades can be written in the form

$$Q^j = \frac{g(\rho^j, \theta)}{\sigma^j}\,, j=1,2, \tag{3}$$

where g is the implicit function solving the equation

$$f'(\rho^j + g(\rho^j,\theta))g(\rho^j,\theta) + f(\rho^j + g(\rho^j,\theta)) = \theta \tag{4}$$

identically for all ρ^j and θ. Such a solution exists if $0 < 2f' + f''g$ in the range of considered parameters ρ^j and θ. Then g, which is the optimal traded quantity per unit of pricing schedule slope σ, is a smooth function with partial derivatives (in future we denote them with subscripts) given by

$$g_\rho = -\frac{f' + f''g}{2f' + f''g},\ g_\theta = \frac{1}{2f' + f''g}. \tag{5}$$

Expressions (5) show that the optimal currency demand/supply "per unit of spread", denoted by g, is decreasing in the mid-quote and increasing in the marginal foreign currency valuation. These are intuitively correct properties of optimal orders: the market user buys more (sells less) of the foreign currency when the pricing schedule intercept ρ that she faces goes down and when her marginal valuation of the foreign currency goes up. The marginal valuation parameter θ itself has an interpretation of the market user demand intercept.

For our specific example of quoting rule $f(p)=e^{cp}$, the condition for the existence of internal solution for optimal orders is given by $g>-2/c$, equivalently, $Q>-2/(c\sigma)$, and the partial derivative of g w.r.t. ρ is equal to

$$g_\rho = -\frac{1+cg}{2+cg} = -\frac{1+c\sigma Q}{2+c\sigma Q}.$$

4.1.2 Dealer's Active Trades

Formally, the decision problem faced by the dealer is an extension of the one solved by the non-dealer investor. For definiteness, the exposition in this subsection concerns dealer 1, with the formulations for dealer 2 being obtained by substitution of indices.

At the start of the trading period, dealer 1 quotes a pricing schedule (ρ^1,σ^1), which is good for both the other dealer and the non-dealer global market user. Symmetrically, dealer 2 does the same, so that his pricing schedule (ρ^2,σ^2) is good for dealer 1 to trade at. The commonality of the dealer's quotes given to all market users, even in the decentralized market, allows us to simplify the price impulse propagation modeling. We justify this assumption by referring to the reputational considerations on the part of the dealer.

As is common in strategic trade models, some sort of noise- (i.e. not fully rational) traders are needed to generate non-trivial transactions in the market. Here, noise traders will be introduced to ensure non-zero exogenous quoting costs of the dealers. Specifically, dealers will always face a situation in which the optimal choice of the pricing schedule slope σ will be strictly positive and finite. Any attempt to increase utility by reducing σ to zero or infinitely increasing it will result in an increasingly costly position received from a noise trader.

Formally, we shall assume that, after the active trades of dealers 1 and 2 and market user U have been decided upon, the pricing schedule of each market maker can be randomly matched directly with a similar schedule of an outside trader (i.e. someone whom we do not model explicitly). In this way, a specific additional trade is generated at the dealer's quote. This external matching could be a consequence of the dealer's bargaining with an important local customer (i.e. someone trading only with this dealer, as opposed to the global investor, whom we model explicitly), automatic crossing by a voice broker or another unspecified reason. (In the part describing the brokered FX market, a similar automatic schedule crossing between contributors to the order book will be an essential part of the brokerage process leading to a common pricing schedule, or state of the book.) It is convenient to assume that the outside trader's pricing schedule slope is equal to that of the dealer. Technically, this element of the model allows one to obtain an internal solution for the price schedule slope without affecting the main qualitative properties of equilibrium.

Matching happens by adding up the pricing schedules horizontally and effectuating the transaction at the price that equalizes the dealer's demanded quantity with the noise trader's supplied quantity. Specifically, let $q=\frac{p-\rho^j}{\sigma^j}$ be the pricing schedule of dealer *j* and $q=\frac{p-\gamma^j}{\sigma^j}$ be the schedule of the noise trader (we will work in the space of transformed prices *p*). Then their joint schedule is $q=\frac{p-\rho^{cj}}{\sigma^{cj}}$, where $\rho^{cj}=\frac{\rho^j+\gamma^j}{2}$, $\sigma^{cj}=\frac{\sigma^j}{2}$. By substituting the transformed price ρ^{cj} into the two dealers' pricing schedules, we see that at this price, dealer *j* is willing to transact quantity $q^{cj}=\frac{\gamma^j-\rho^j}{2\sigma^j}$ (sell if $\gamma^j>\rho^j$, buy in the opposite case), whereas the noise trader is willing to transact minus this quantity, i.e. $\frac{\rho^j-\gamma^j}{2\sigma^j}$.

Matching of the price schedules happens at a cost, which we define in domestic cash units. These costs can be associated with the required provisions that the dealer must make for the eventuality of noise trader matching. Specifically, let *h* be a strictly increasing and strictly concave function on the real line with $h(0)=0$, $h'(0)=1$. When, as a result of price schedule matching with a noise trader, the dealer sells (buys) q^c units of foreign currency, he receives (pays the negative of) $f(q^c)h(q^c)$ units of domestic cash, which is less (more in absolute value) than $f(q^c)q^c$. Our principal example of the *transaction function* will be the linear-quadratic function $h(q)=q-aq^2/2$, $a>0$ a constant. The origin of the non-linear cost of matching is in the precautions the dealer must take in order to cope with situations where the transacted volume resulting from this trade gets out of his control. That is the reason for the convex growth of costs with volume. Because of the non-linearly growing matching costs, the dealer prefers to set a strictly positive pricing schedule slope σ. Otherwise, he might be exposed to suboptimally high transaction volumes with the noise traders.

Further, define by q^{12} and q^{21} the volumes of regular active trades directed by dealer 1 towards dealer 2 and vice versa. We shall assume that dealer 1 has a certain degree of market power over both market users who trade with him at his quotes. Namely, in his optimization problem, dealer 1 takes into account the demand/supply schedule (3) of the market user U and the similar schedule of dealer 2 (we shall see shortly that each dealer's active trades indeed satisfy an analogue of (3)). We shall denote by θ^2 the marginal currency valuation of dealer 2 and use superscript U for the market user variables. Dealer 1 has end-of-period cash positions

$$x^m = y^m + f\left(\rho^1 + g(\rho^1,\theta^2)\right)\frac{g(\rho^1,\theta^2)}{\sigma^1} + f\left(\rho^1 + g(\rho^1,\theta^U)\right)\frac{g(\rho^1,\theta^U)}{\sigma^1}$$
$$- f\left(\rho^2 + \sigma^2 q^{12}\right)q^{12} + f(\rho^{c1})h(q^{c1}) - c\,, \qquad (6a)$$

$$x^i = y^i - \frac{g(\rho^1,\theta^2) + g(\rho^1,\theta^U)}{\sigma^1} + q^{12} - q^{c1}. \qquad (6b)$$

The objective function of dealer 1 is defined by (1) as for any other market participant. It is now obvious that the equations for optimal active trades are the same as (3), but with Q^2 replaced by q^{12} and θ^U by θ^1. Therefore, by symmetry, the assumptions made in (6) about the dependence of q^{21} on ρ^1 and σ^1 are validated.

The discussion of optimal quoting by the dealers is postponed until Section 5.

4.2 Brokered Market

The brokered inter-dealer FX market (real-life prototypes being either the EBS or Reuters Dealing 3000 electronic brokerage systems) will be modeled as an institution to which both dealers submit their pricing schedules $q = \frac{p-\rho^j}{\sigma^j}$, $j=1,2$, as defined earlier. These schedules are added up horizontally to generate the *standing market schedule* $q = \frac{p-\rho^b}{\sigma^b}$ with $\rho^b = \frac{\sigma^2\rho^1 + \sigma^1\rho^2}{\sigma^1 + \sigma^2}$, $\sigma^b = \frac{\sigma^1\sigma^2}{\sigma^1 + \sigma^2}$. At this automatic pricing schedule crossing occasion, dealer 1 sells to (if $\rho^2 > \rho^1$, otherwise buys the negative of this from) dealer 2 the amount $q^{b1} = \frac{\rho^2 - \rho^1}{\sigma^1 + \sigma^2}$ at price $f(\rho^b)$.[3]

In the same way as in the direct inter-dealer market case, we need to define the external costs of quoting, caused by the existence of noise traders. We assume that each dealer can be randomly matched by the same noise trader as defined in 4.1.2 for the direct market, instead of being matched with the other dealer(s) by the broker. The prudential risk management rules of the dealer bank then require the dealer to create a "corrective item" equal to $f(\rho^{c1})h(q^{c1})$ in x^m and another corrective item $-q^{c1}$ in x^i to cover for this contingency. These terms then appear in the dealer's state-transition equations. Their presence restricts the domain of the admissible pricing schedule slopes σ^1 and guarantees the existence of an internal solution to the dealer optimization problem.

Upon completion of the automatic crossing procedure, the broker announces the standing pricing schedule to the two dealers and the non-dealer investor. This schedule is viewed as a horizontal sum of the *reduced schedules of the individual dealers*. This means that the original dealer schedule is reduced horizontally by the volume sold (or bought if $\rho^2 < \rho^1$) in the course of

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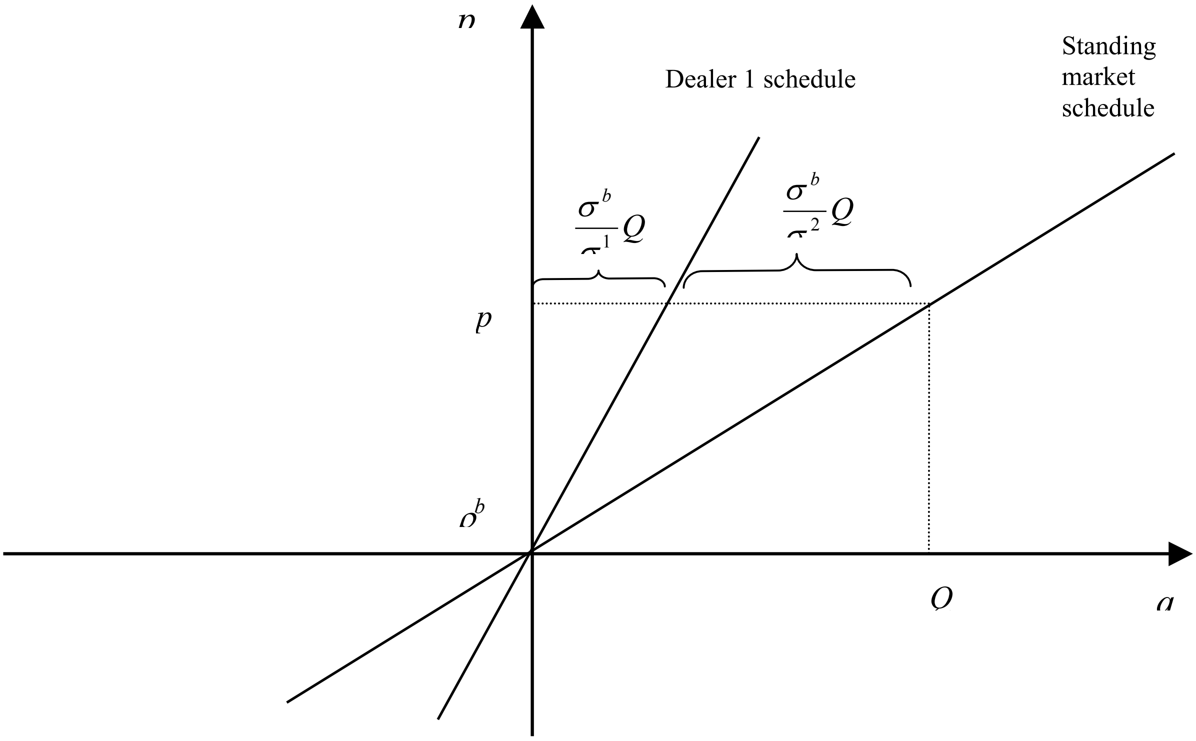
[3] The order in which the automatic pairwise crossing is conducted by the broker affects the involved transaction prices, but not the automatically determined overall transaction volume of a given dealer with the rest. So, if one intends to extend the current set-up to the case of three or more dealers, it is easiest to think of a broker who adds up the individual pricing schedules horizontally in one batch, effectuating the transactions at a singe clearing price.

automatic crossing. The resulting schedule is then $q=\frac{p-\rho^1}{\sigma^1}-q^{b1}=\frac{p-\rho^b}{\sigma^1}$ for dealer 1 and $q=\frac{p-\rho^2}{\sigma^2}-q^{b2}=\frac{p-\rho^b}{\sigma^2}$ for dealer 2, giving the standing schedule in sum.

Each of the three market users (two dealers and the non-dealer investor) then submit their orders to the broker, who executes them at the standing schedule, by splitting each order Q in proportions $\left(\frac{\sigma^b}{\sigma^1},\frac{\sigma^b}{\sigma^2}\right)$ among the two constituent schedules. Geometrically, this is equivalent to setting the order proportion of a dealer by letting his reduced schedule intersect the horizontal line which goes through the standing schedule point with 1st coordinate Q (see Fig. 1).

Figure 1: State of the Limit Order Book (the Standing Pricing Schedule Structure) and Market Order Matching by the Broker



Note: Q denotes the incoming market order (originating from either a dealer or non-dealer investor) at the standing market schedule with parameters (ρ^b, σ^b). This schedule is obtained by crossing the two dealer pricing schedules and adjusting to a common intercept ρ^b. The order is cleared at price $f(p)=f(\rho^b+\sigma^b Q)$, whereas the order proportions are given by the relative slopes of the individual dealer pricing schedules, equal to $\left(\frac{\sigma^b}{\sigma^1},\frac{\sigma^b}{\sigma^2}\right)$, adding up to unity.

Note that the formulation of the brokered market set-up requires a resolution of the following dilemma. Should the market order submitted by the dealer be partially returned to himself in accordance with the general rule defined above, or canceled out in some way by the dealer, who is supposed to decide on market orders and quotes (limit orders) simultaneously? Fortunately, the formal outcome of the dealer optimization problem resolution does not depend on which variant is chosen. Here, we are taking the former view (a representative dealer partially "trading with himself"), for reasons of expositional transparency.[4]

The individual market participants' optimization problems in the brokered market are defined next.

4.2.1 Non-Dealer Global Market User

At the beginning of the trading period, the non-dealer investor who uses the brokered market faces only the standing pricing schedule $P = f\left(\rho^b + \sigma^b Q\right)$, where Q is the market-user's order size. She maximizes (1) with respect to Q, given the current domestic and foreign cash holdings. Her end-of-period cash holdings are

$$x^m = y^m - f(\rho^b + \sigma^b Q)Q - c\,, \tag{7a}$$

$$x^i = y^i + Q\,. \tag{7b}$$

Therefore, the first-order condition of optimality is

$$f'\left(\rho^b + \sigma^b Q\right)\sigma^b Q + f\left(\rho^b + \sigma^b Q\right) = \theta^U\,,$$

similarly to what we have established in 2.1.1 for the direct market case, and the optimal order size is equal to

$$Q = \frac{g\left(\rho^b, \theta^U\right)}{\sigma^b}\,. \tag{8}$$

4.2.2 Dealer's Problem

Dealer 1, who uses the brokered market, decides upon the same variable as the non-dealer investor (his order size, denoted by q^1), but also sets the parameters (ρ^1, σ^1) of his pricing schedule that would be incorporated into the standing pricing schedule by the broker. He knows and takes into account the functional form of the non-dealer order (8) and a similar order $q^2 = \dfrac{g\left(\rho^b, \theta^2\right)}{\sigma^b}$ which is placed with the broker by dealer 2 (we shall see in a moment that his own optimal order is consistent with this assumption). From the broker, dealer 1 receives an order of size q^{b1} at price

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[4] One can justify this by assuming that active trades and quotes are determined by separate units within the firm.

$f(\rho^b)$ resulting from automatic price schedule crossing and proportion $\frac{\sigma^b}{\sigma^1}$ of orders Q and q^2 (see above). Altogether, his end-of-period cash holdings are

$$
\begin{aligned}
x^m &= y^m + \frac{f\left(\rho^b + g(\rho^b,\theta^1)\right)g(\rho^b,\theta^1)}{\sigma^1} \\
&+ \frac{\sigma^b}{\sigma^1}\left\{ f\left(\rho^b + g(\rho^b,\theta^2)\right)\frac{g(\rho^b,\theta^2)}{\sigma^b} + f\left(\rho^b + g(\rho^b,\theta^U)\right)\frac{g(\rho^b,\theta^U)}{\sigma^b} \right\} \\
&- f\left(\rho^b + \sigma^b q^1\right)q^1 + f(\rho^b)q^{b1} + f(\rho^{c1})h(q^{c1}) - c, \qquad (9a)
\end{aligned}
$$

$$
x^i = y^i - \frac{\sigma^b}{\sigma^1}\frac{g(\rho^b,\theta^1) + g(\rho^b,\theta^2) + g(\rho^b,\theta^2)}{\sigma^b} + q^1 - q^{b1} - q^{c1}. \qquad (9b)
$$

Observe the double appearance of the dealer 1 order $q^1 = \frac{g\left(\rho^b,\theta^1\right)}{\sigma^b}$ in the above equations. As mentioned earlier, this order returns to the dealer as the proportion $\frac{\sigma^b}{\sigma^1}$ of the current market order and at the same time is processed as his own market order. We shall see immediately that this understanding is internally consistent.

The first-order condition of optimality for the dealer’s market order is

$$
f'\left(\rho^b + \sigma^b q^1\right)\sigma^b q^1 + f\left(\rho^b + \sigma^b q^1\right) = \theta^1.
$$

Therefore, $q^1 = \frac{g\left(\rho^b,\theta^1\right)}{\sigma^b}$, $q^2 = \frac{g\left(\rho^b,\theta^2\right)}{\sigma^b}$, which is consistent with the assumption

(regarding dealer 2 order size) used in (9).

5. Dealer’s Optimal Quoting Policy

We return to the dealer 1 optimization problem from Section 4 to characterize the optimal quoting strategies (i.e. the choice of parameters ρ and σ). As before, we discuss the decisions of dealer 1 so that the other’s optimal moves are derived by symmetry.

Let us introduce the following notations:

$$\alpha(\rho,\theta)=1+g_\rho(\rho,\theta)=\frac{f'(\rho+g(\rho,\theta))}{2f'(\rho+g(\rho,\theta))+g(\rho,\theta)f''(\rho+g(\rho,\theta))},$$

$$P(\rho,\theta)=f(\rho+g(\rho,\theta)),\ P^b=f(\rho^b),\ P^c=f(\rho^c).$$

$P(\rho^1,\theta^2)$ ($P(\rho^1,\theta^U)$, P^b, P^{c1}) is the effective transaction price at which the order of dealer 2 (non-dealer market user, dealer 2 in the course of the price schedule crossing by the broker, noise trader by price schedule crossing) is executed by dealer 1 in the direct market (or brokered market in the case of P^b). We consider prices $P(\rho^1,\theta^2)$ and $P(\rho^1,\theta^U)$, as well as auxiliary parameters α as functions of the quoting parameter (transformed mid-quote price) ρ^1 and the marginal valuation parameters θ^2 and θ^U respectively. A similar understanding should be applied in the brokered market context.

Further, let us define functions

$$C^1(\rho,\sigma)=\left[1+\frac{a(\rho-\gamma^1)}{2\sigma}\right]f\left(\frac{\rho+\gamma^1}{2}\right)-\theta^1,\ D^1(\rho,\sigma)=\frac{\rho-\gamma^1}{2}\left[1+\frac{a(\rho-\gamma^1)}{4\sigma}\right]f'\left(\frac{\rho+\gamma^1}{2}\right),$$

$$L^{p1}(\rho,\theta)=\alpha(\rho,\theta^2)\theta^2+\alpha(\rho,\theta^U)\theta^U+(2-\alpha(\rho,\theta^2)-\alpha(\rho,\theta^U))\theta^1)-P(\rho,\theta^2)-P(\rho,\theta^U),$$

$$L^{s1}(\rho,\theta)=g(\rho,\theta^2)P(\rho,\theta^2)+g(\rho,\theta^U)P(\rho,\theta^U)-[g(\rho,\theta^2)+g(\rho,\theta^U)]\theta^1.$$

Similar variables (index 2 everywhere replacing 1, and vice versa) are defined for dealer 2.

Direct Inter-Dealer Market

Proposition 1 *The first-order conditions for the objective function optimization w.r.t. quoting parameters ρ^1 and σ^1 of dealer 1 are given by equations*

$$L^{p1}(\rho^1;\theta^1,\theta^2,\theta^U)=\frac{1}{2}[C^1(\rho^1,\sigma^1)+D^1(\rho^1,\sigma^1)],\tag{10a}$$

$$L^{s1}(\rho^1;\theta^1,\theta^2,\theta^U)=\frac{\rho^1-\gamma^1}{2}C^1(\rho^1,\sigma^1).\tag{10b}$$

The calculations leading to (10) are given in Section A1 of the Appendix.

An analogous pair of equations is valid for the optimization of ρ^2 and σ^2 by dealer 2.

Brokered Inter-Dealer Market

We will need additional notation, which we introduce by putting

$$B^1(\rho^1,\rho^2,\sigma^1,\sigma^2)=\frac{\sigma^1}{\sigma^2}[f(\rho^b)-P(\rho^b,\theta^1)]+\frac{\sigma^1(\rho^1-\rho^2)}{\sigma^1+\sigma^2}f'(\rho^b).$$

Proposition 2 *The first-order conditions for the current value Hamiltonian optimization w.r.t. quoting parameters* ρ^1 *and* σ^1 *of dealer 1 are given by equations*

$$B^1(\rho^1,\rho^2,\sigma^1,\sigma^2)+\frac{\sigma^1+\sigma^2}{2\sigma^2}\left[C^1(\rho^1,\sigma^1)+D^1(\rho^1,\sigma^1)\right]=L^{p1}(\rho^b;\theta^1,\theta^2,\theta^U)\,, \tag{11a}$$

$$\frac{\rho^1-\gamma^1}{2}C^1(\rho^1,\sigma^1)-\frac{\sigma^1\sigma^2(\rho^1-\rho^2)}{\left(\sigma^1+\sigma^2\right)^2}\left[L^{p1}(\rho^b;\theta^1,\theta^2,\theta^U)-B^1(\rho^1,\rho^2,\sigma^1,\sigma^2)\right]$$
$$=L^{s1}(\rho^b;\theta^1,\theta^2,\theta^U)\,. \tag{11b}$$

See Section A1 of the Appendix for the proof.

We now define a normal form game between dealer 1 and dealer 2, with the action space of each player consisting of his quoting parameters ρ and σ, $\sigma>0$, and his payoff given by (1) subject to (6) (in the direct market) or (9) (in the brokered market). The optimal active trade rules of the dealers, the same as the non-dealer investor, stated in (3) or (8), are taken as given. The investors' endowments y in both currencies are exogenous. For given marginal currency valuation parameters θ^1, θ^2 and θ^{U}, the four equations (10) for the direct inter-dealer market and (11) for the brokered market would give a pair of quoting rules. (The latter themselves determine the end-of-period cash positions of the players and, through them and the marginal utilities, also the marginal valuations θ^1, θ^2, θ^{U}.) By fixing the parameters of the non-dealer investor and the other dealer, one can regard the optimal quoting rule (ρ,σ) of each dealer that solves (10) (or (11)) as a *reaction function*. Their intersection gives the Nash equilibrium of the one-period trading game between dealers 1 and 2. Accordingly, the Nash equilibrium is the 10-dimensional vector (x, ρ,σ) of the investor cash holdings and the dealer quoting parameters that satisfy the ten equations (2), (6), (10) for the direct market, and (7), (9), (11) for the brokered market.

Observe that (10b) and (11b) imply the following expression for the marginal currency valuation of the dealer in the case where the noise trades are negligible:

$$\theta^1=\frac{q^{21}}{q^{21}+Q^1}P(\rho^1,\theta^2)+\frac{Q^1}{q^{21}+Q^1}P(\rho^1,\theta^U)\,,\ \theta^1=\frac{q^2}{q^2+Q}P(\rho^b,\theta^2)+\frac{Q}{q^2+Q}P(\rho^b,\theta^U)\,. \tag{12}$$

Qualitatively, (12) and its generalizations[5] mean that the dealer's marginal currency value in any one-shot equilibrium is equal to *the sum of the effective transaction prices paid by the remaining market participants weighted by their normalized orders*. This opens the way to empirical estimates of the dealer marginal values.

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[5] A straightforward analogue of this equation happens to be valid for generalizations of the model with more than two dealers.

6. Equilibrium of the Static Inter-Dealer Game and the Steady State Equilibrium of a Dynamic Game

This section first explains the method of equilibrium derivation in the inter-dealer game and then discusses the properties of equilibrium prices and trades obtained numerically. The possibility of numerical solution is a direct consequence of the relation between the original static game and the more general differential game.

6.1 Embedding of the One-Period Game

Calculation of the inter-dealer game Nash equilibrium as defined in the previous section requires solving a system of non-linear equations for the choice variables of market participants. Such a solution cannot be obtained in closed form, and even its numerical calculation by standard methods proves to be computationally difficult. There exists a way around this difficulty, based on constructing an alternative procedure for equilibrium derivation. It consists in defining a dynamic (differential) game for the same set of players and, basically, the same rules as the one-shot inter-dealer game discussed before. The exogenous parameters of this differential game can then be chosen in such a way that any steady state equilibrium of the dynamic game is equivalent to the Nash equilibrium of the original static one. However, the dynamic perspective will help us prove a number of properties of the steady state equilibrium that will simplify its calculation. Thanks to these simplifications, we are able to obtain the numerical solution to the Nash equilibrium problem. The dynamic game is also of interest in its own right, since it broadens our understanding of possible forex behavior patterns in continuous time. Unfortunately, the complete characterization of Nash equilibria is complicated by the highly involved stability structure of the corresponding dynamic system. Nevertheless, by finding out this complex structure in the dynamic model, we should be less surprised by the excess volatility observed in the real-life forex. We also see that there should be a link between the institutional arrangements prevailing in the market and the volatility magnitude.

Below, we give a brief outline of the differential game in which the one-period game of Section 4 can be embedded as a steady state.

Now we will deal with the flow variables (trade orders q, Q) in the form of rates per infinitesimal time period dt. Also, we define an exogenous *endowment rate* y^m (endowment per period dt) of domestic cash and a similar endowment rate y^i of foreign cash. For simplicity, we only consider constant cash endowment rates in this paper.

A consumption/dividend payment rate c is subtracted from the current domestic cash holdings. The dividends are evaluated by the period utility function $c \mapsto u(c)$ with standard properties (increasing, strictly concave). With a certain infinitesimal probability, the investor may need to stop operations in the current period and submit his/her book to an audit, which evaluates the cash holdings by means of a "liquidation" value function $(x^m, x^i) \mapsto v(x^m, x^i)$, i.e. the same as the wealth utility function of the static problem of Sections 4 and 5.

We shall assume that the arrival of the said liquidation event is a Poissonian random event with intensity β. With probability $e^{-\beta dt}$, the operation will be continued in the immediate infinitesimal time interval *dt* after the present moment, and with complementary probability 1- $e^{-\beta dt} \approx \beta dt$ the investor will have to liquidate within *dt*. A similar random termination feature, although in discrete time setting, is used in Foucault, 1999. It allows one to analyze the stationary equilibria of a dynamic order placement and execution model with a potentially infinite number of trading rounds.

The investor objective at every moment *t* is to maximize the performance index

$$J(x_t^m, x_t^i) = \int_t^\infty e^{-\beta\tau} \{u(c_\tau) + \beta v(x_\tau^m, x_\tau^i)\} d\tau \, , \tag{13}$$

subject to the appropriate state-transition equation and the initial cash holdings (x_t^m, x_t^i). Maximization is achieved by choosing the trajectory of dividends, active trades (purchases or sales of domestic against foreign currency through a market maker) and, if the agent is a dealer, the trajectory of quotes.

Note that, thanks to the presence of liquidation function v in the current utility in (13), the problem does not require the imposition of transversality conditions: explosive x-paths are excluded by the defined properties of function v.

The state transition equations will be different depending on the investor category (dealer or not) and the market structure (decentralized or brokered). They are similar to the cash holding variable definitions of Section 2, but describe cash change rates instead of levels. Specifically, the non-dealer market user in the direct market has the state-transition equations

$$\dot{x}^m = y^m - f(\rho^1 + \sigma^1 Q^1) Q^1 - f(\rho^2 + \sigma^2 Q^2) Q^2 - c \, , \tag{14a}$$

$$\dot{x}^i = y^i + Q^1 + Q^2 \, . \tag{14b}$$

Dealer 1 in the direct market has state-transition equations (cf. (6))

$$\begin{aligned} \dot{x}^m &= y^m + f\left(\rho^1 + g(\rho^1, \theta^2)\right) \frac{g(\rho^1, \theta^2)}{\sigma^1} + f\left(\rho^1 + g(\rho^1, \theta^U)\right) \frac{g(\rho^1, \theta^U)}{\sigma^1} \\ &- f\left(\rho^2 + \sigma^2 q^{12}\right) q^{12} + f(\rho^{c1}) h(q^{c1}) - c \, , \end{aligned} \tag{15a}$$

$$\dot{x}^i = y^i - \frac{g(\rho^1, \theta^2) + g(\rho^1, \theta^U)}{\sigma^1} + q^{12} - q^{c1} \, . \tag{15b}$$

The state-transition equations for the same dealer in the brokered market are (cf. (9))

$$\begin{aligned} \dot{x}^m &= y^m + \frac{f\left(\rho^b + g(\rho^b, \theta^2)\right) g(\rho^b, \theta^1)}{\sigma^1} \\ &+ \frac{\sigma^b}{\sigma^1} \left\{ f\left(\rho^b + g(\rho^b, \theta^2)\right) \frac{g(\rho^b, \theta^2)}{\sigma^b} + f\left(\rho^b + g(\rho^b, \theta^U)\right) \frac{g(\rho^b, \theta^U)}{\sigma^b} \right\} \\ &- f\left(\rho^b + \sigma^b q^1\right) q^1 + f(\rho^b) q^{b1} + f(\rho^{c1}) h(q^{c1}) - c \, , \end{aligned} \tag{16a}$$

$$\dot{x}^i = y^i - \frac{\sigma^b}{\sigma^1}\frac{g(\rho^b,\theta^1)+g(\rho^b,\theta^2)+g(\rho^b,\theta^2)}{\sigma^b} + q^1 - q^{b1} - q^{c1}. \qquad (16b)$$

Note that (14a)–(16a) contain the dividend rate c, which was not defined in the one-period case. Another substantial difference compared to (2), (6) and (9) lies in the definition of marginal values θ (with superscripts corresponding to the individual market participants). The latter are defined as $\theta = \frac{\xi_i}{\xi_m}$, ξ being the adjoint variables of the corresponding optimization problem (see Section A2 of the Appendix for details). With this caveat in mind, the remaining state-transition equations can be defined by building an analogy with (2), (7) and (9).

We proceed by using (13)–(16) and the analogous transition equations for the remaining market participants to define a differential game between the non-dealer investor, dealer 1 and dealer 2 (actually, two games: one for the direct and the other for the brokered market). The payoff is given by (13) for both market organizations, every participant and every time moment. We shall consider the open-loop Nash equilibria of this game. This choice seems intuitively more appropriate for modeling FX dealer interaction than closed-loop equilibria. The latter would imply that each dealer is able to evaluate the impact of his quoting and trading behavior on the actions of others. Such ability would not be plausible in a multi-dealer environment with a limited degree of transparency. Moreover, we will concentrate specifically on steady state Nash equilibria in this class.

Proposition 3 *Let the differential inter-dealer game defined by payoffs (13) and state transition equations (14)–(16) (and their appropriate analogues) possess an open-loop steady state equilibrium for, at least, a collection of finite intervals of constant cash endowment rates* y^{km}, y^{ki}, *k=1, 2, U. For each one-period inter-dealer game with initial cash endowments* $\tilde{y}^{km}$, $\tilde{y}^{ki}$, *k=1, 2, U, from a non-empty interval, there exists a set of parameters of the differential game such that the steady state Nash equilibrium of the latter corresponds to a Nash equilibrium of the former.*

This proposition and other properties of the steady state are proven in Section A3 of the Appendix. The immediate technical value of the result consists in the possibility of simplifying the Nash equilibrium search in the one-period game of Section 5. A direct calculation of the latter would involve a solution of a rather complex system of ten non-linear equations for ten unknowns (six cash holding variables and four quoting parameters). In the dynamic game, a part of the complexity is removed since one characterizes the Nash equilibria by means of three interrelated maximum principles (one for each market participant). Transition to the steady state Nash equilibrium means further simplification and dimensionality reduction.

There is one immediate application of Proposition 3. It exploits the said steady state Nash equilibrium feature of the one-period game in characterizing what are known as “hot potato” inter-dealer trades. In this setting, we define a hot potato trading pattern as an equilibrium outcome in which the gross inter-dealer order flow (the sum of net dealer orders) is bigger in absolute value than the net inter-dealer order flow. Formally, the gross inter-dealer flows in the direct and brokered market are equal to q^{12}+q^{21} and q^1+q^2 respectively, whereas the net inter-dealer flows are q^{12}-q^{21} and q^1-q^2. The absence of hot potato trades would mean that at most one of the orders q^{12}, q^{21} or q^1, q^2 is different from zero, i.e. a dealer would not place an order which is due to be offset by an opposite order in equilibrium. In the present model, however, the Nash equilibria generically involve hot potato trades. This fact can be established in any equilibrium with given exogenous parameter values, when it is calculated numerically. The easiest, but also most spectacular, outcome is obtained analytically, when one considers the equilibrium with perfectly symmetric dealers. It turns out that hot potato trades are present even then.

Proposition 4 *Assume that the external matching (cf. 4.1.2 and 4.2.2) happens according to the same rule with the same parameters for both dealers, and that the quoting rule is exponential, as defined in 4.1. If the structural parameters and cash endowments of both dealers are identical, then the Nash equilibrium of the one-period game is characterized by non-zero inter-dealer order flow* $q^{12}=q^{21}$ *in the direct market and* $q^1=q^2$ *in the brokered market every time the customer order flow* $Q^1=Q^2$ *or Q is itself non-zero.*

The proof, conducted directly for the steady state equilibrium of the dynamic game, is given in Section A4 of the Appendix.

The result of Proposition 4 follows from the fact that, for a given dealer, the strategy of making use of the market-making function of the other dealer always dominates the strategy of abstaining from inter-dealer trade. Since the hot potato trades between symmetric dealers are Pareto-inferior to the no inter-dealer trade outcome, in a repeated one-period game there would exist a possibility of coordinating on a collusive no-trade outcome. However, this is hardly an intuitive outcome in a market with many competing dealers. In such a market, either the counter-party transparency is incomplete (brokered organization) or the chance of collusion is undermined by competition from other dealers (direct organization). Although in this paper we model just two dealers, this is done mainly for reasons of computational tractability (a discussion of Nash equilibrium properties obtained numerically follows in the next subsection). The overall objective is to study FX markets where one part of the participants may develop an intrinsic need to buy and the other to sell, which is expressed by different marginal utilities of foreign cash in equilibrium. So, our dealer 1 and dealer 2 just perform the roles of representatives for bigger groups with homogenous marginal currency values. In such an environment, coordination on a no-hot-potato trade equilibrium in a repeated game is as good as irrelevant.

The possible equilibrium trajectories of the differential game defined above are not limited to saddle paths converging to a steady state. One can identify three Lyapunov functions (corresponding to Hamiltonians of the three market participants) that must be constant along any open-loop equilibrium trajectory. There are probably multidimensional attractors for the trajectories lying on the level surfaces of these three functions. The exact picture is so far unclear. In any case, almost all equilibrium trajectories of the dynamic game are periodic or quasi-periodic and do not have a single point of convergence. Therefore, the dynamic inter-dealer interaction in our model exhibits enough complexity to match the observed volatility of the real forex.

To draw more specific qualitative conclusions, we now return to the one-period set-up of Sections 4 and 5 and comment upon the findings obtained by solving for its Nash equilibrium numerically. Given the result of Proposition 3, the qualitative discussion to follow in the next subsection will mainly concentrate on the outcomes of steady state calculations for the dynamic game, instead of the Nash equilibrium in the original one-period game. We will discuss numerical results for dealer-symmetric NE in particular, since they can be relatively easily illustrated graphically.

6.2 Equilibria of the One-Period Game: the Role of Customer Order Flow Under Different Market Organizations

As was already mentioned, closed-form solutions for Nash equilibria in the one-period inter-dealer game do not exist. Numerically, one can derive equilibrium trading patterns by solving for the steady state Nash equilibrium in the "enclosing" differential inter-dealer game. We have done this for the interval of non-dealer foreign currency endowments that corresponds to her order flow (OF) values in the interval [-1,1].

Let us recall the pivotal role of the indirect marginal utility of foreign cash for the quoting and order placement behavior. To illustrate that role, we show in Figs. 2–5 how the NE trading pattern depends on this marginal value of the non-dealer investor and, equivalently, on her order flow (in the dealer-symmetric equilibrium we are discussing now, there is a one-to-one correspondence between the two). These figures contain marginal FX values, order flows and transaction prices/exchange rates (ER) obtained by calculating Nash equilibria numerically for different exogenous characteristics of the non-dealer investors. The NE calculation results demonstrate the following similarities and differences of the two market structures, "in reduced form":

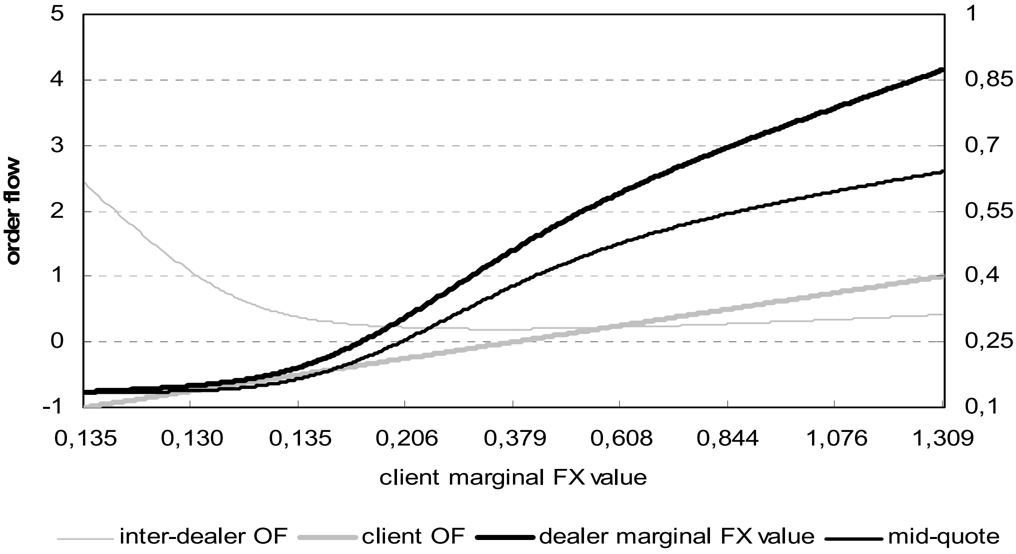
a) The effective transaction prices, the mid-quote and the marginal FX values of dealers and clients are all increasing functions of the client order flow, regardless of the trading mechanism. Accordingly, both the direct and brokered market show an intuitively correct dependence on the market user "demand intercept", represented by the marginal foreign cash value.
b) Since the dealer parameters in the equilibrium we are discussing are identical, the inter-dealer trading is reduced to hot potato transactions. In the discussed equilibrium, they are represented by purchase orders that have the objective of compensating for the foreign currency holding reduction caused by customer purchases. This is true for both trading mechanisms.
c) Surprisingly, the hot potato orders in this equilibrium are still buying ones even when the client OF becomes negative (i.e. the non-dealer investors sell). This is so because client sales depress the price to levels that encourage dealers to buy.
d) Under any client order flow, the inter-dealer OF in the brokered market is higher than in the direct market. That is, the same volume of client orders induces higher inter-dealer activity in the brokered market. This means that this market is more "effort-consuming".
e) Quoted dealer spreads (parameters σ of the model) are lower in the direct market when clients sell and in the brokered market when clients buy, per unit of client order flow.
f) Effective spreads: Clients in the direct market pay a higher price margin over the mid-quote than dealers when they place a big buying order, i.e. *effective* client ask half-spreads in the direct market are higher than effective inter-dealer ask half-spreads, except for small orders. In the brokered market, the buy order volume for which the effective client ask half-spreads become higher than the inter-dealer half-spreads is more elevated. (Effective bid half-spreads are hard to compare since, in this equilibrium, dealers only place buy orders.)
g) Altogether, the same level of client demand or supply corresponds to a smaller adjustment of the client marginal utility of foreign cash, which indicates that the brokered market has a

weaker ability of investor welfare improvement by means of market order placement compared to the direct market.

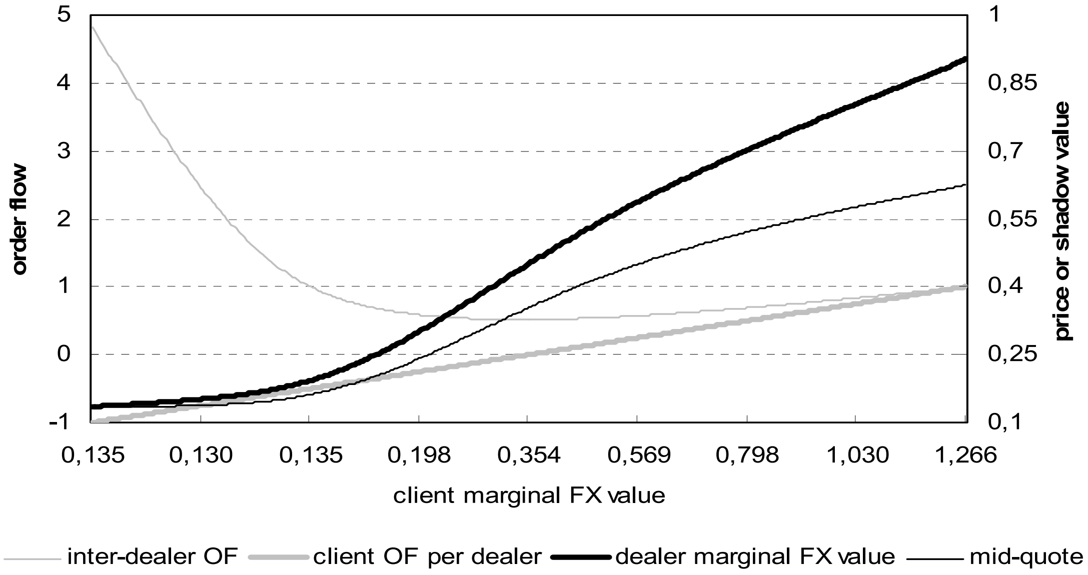
We see that the two markets, at least in the equilibrium considered above, behave similarly in the qualitative sense but are characterized by different effective prices, spreads and inter-dealer trade volumes.

Figure 2: Trade patterns as a function of the investor indirect marginal utility of foreign currency holdings

(a) Direct inter-dealer market



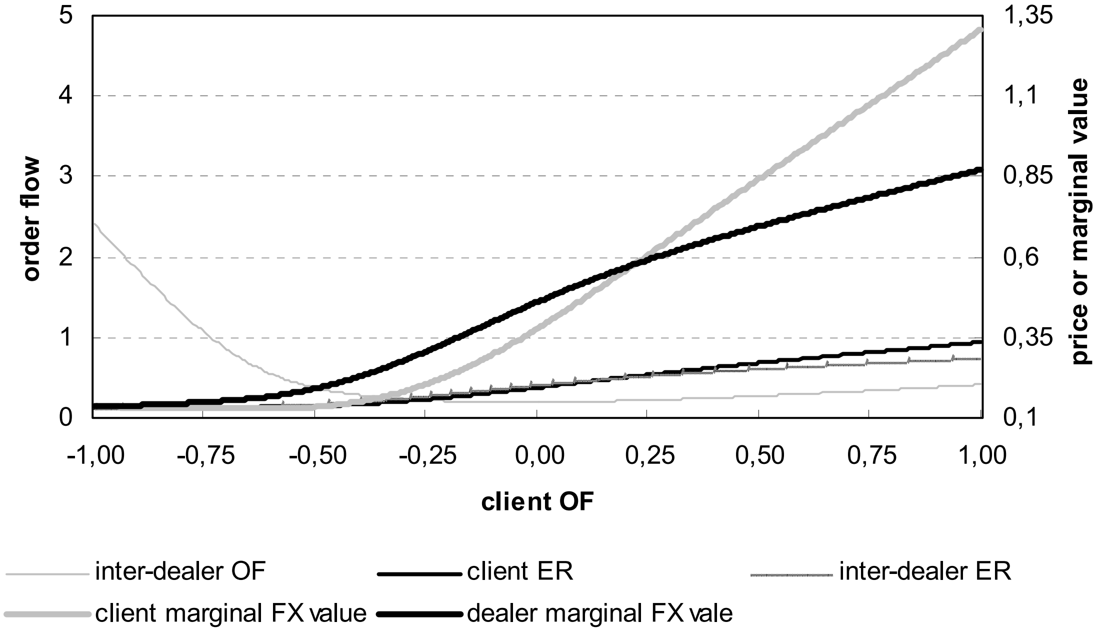
(b) Brokered inter-dealer market



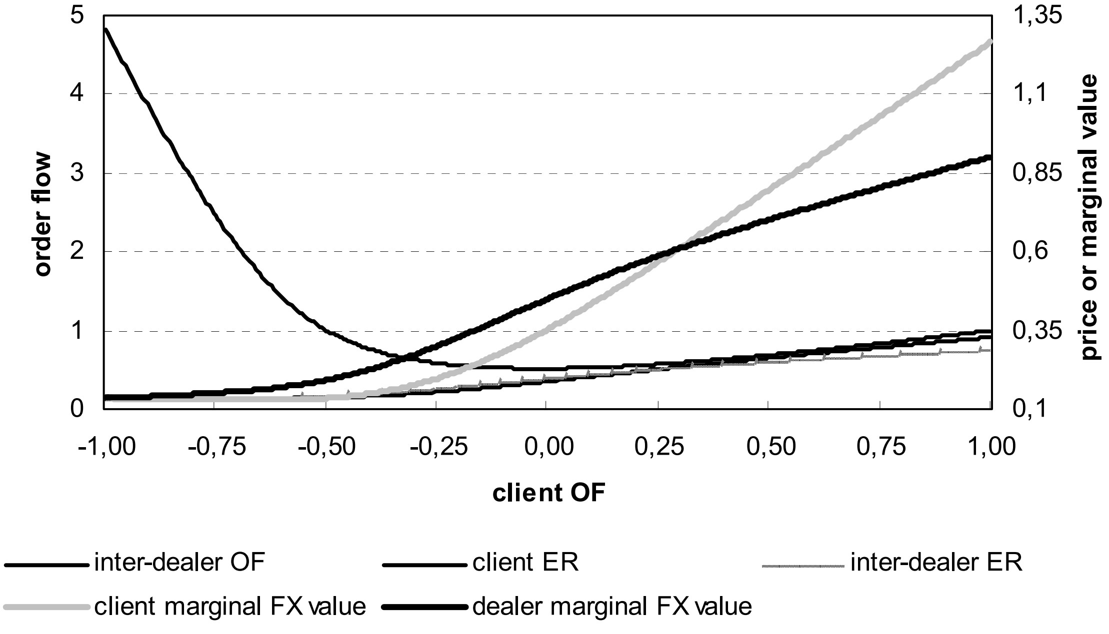
Note: The figure shows the trade pattern outcomes of the one-period quoting and trading game, corresponding to different values of the marginal foreign cash valuation of the representative non-dealer investor. The marginal foreign currency value is the endogenous indirect utility of the non-dealing investor obtained in the Nash equilibrium, and is in a one-to-one correspondence with this investor's per-dealer order flow, which is the negative of the exogenous investor foreign cash endowment.

Figure 3: Trade patterns as a function of the non-dealer investor order flow per dealer

(a) Direct inter-dealer market



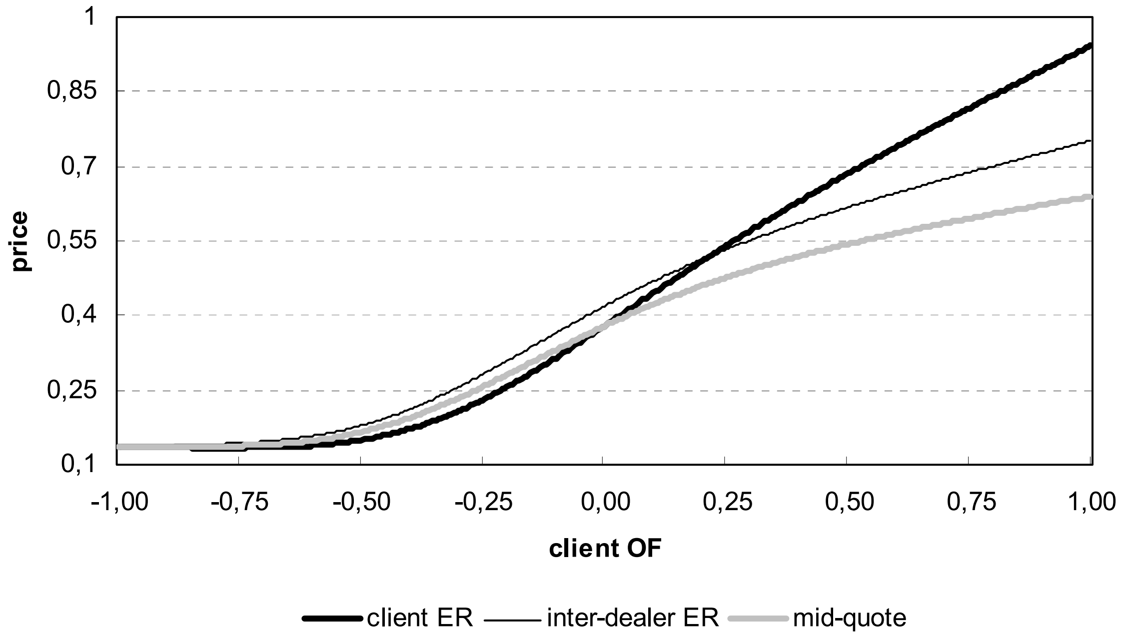
(b) Brokered inter-dealer market



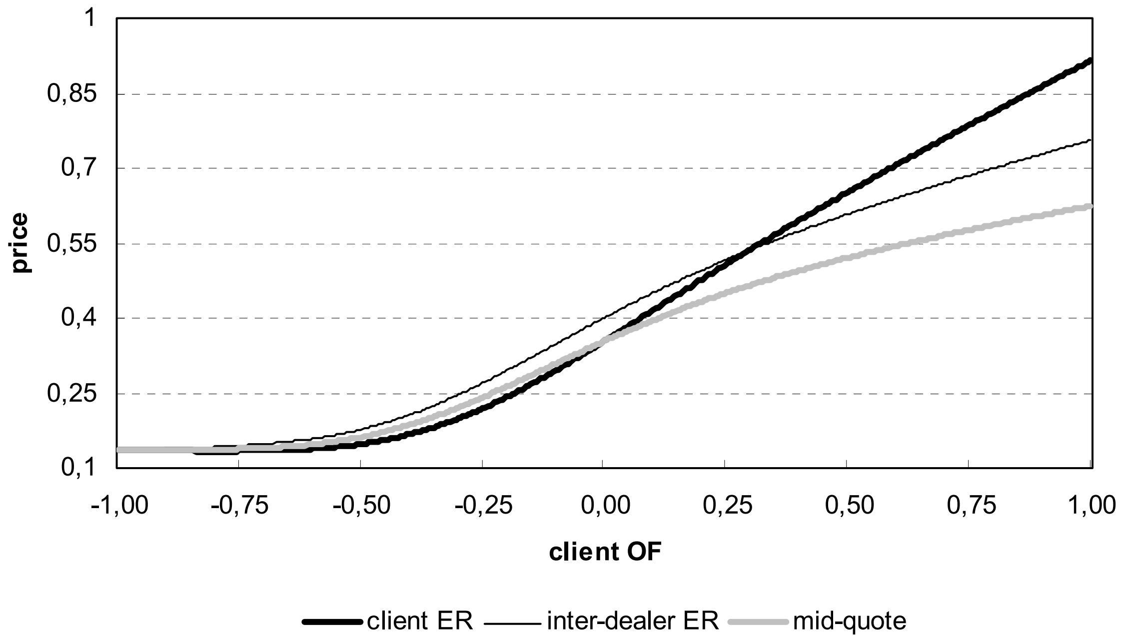
Note: The figure shows the trade pattern outcomes of the one-period quoting and trading game, corresponding to different values of the representative non-dealer investor per-dealer order flow, which is the negative of the exogenous investor foreign cash endowment.

Figure 4: Exchange rate setting and effective transaction prices as functions of the non-dealer investor order flow per dealer

(a) Direct inter-dealer market



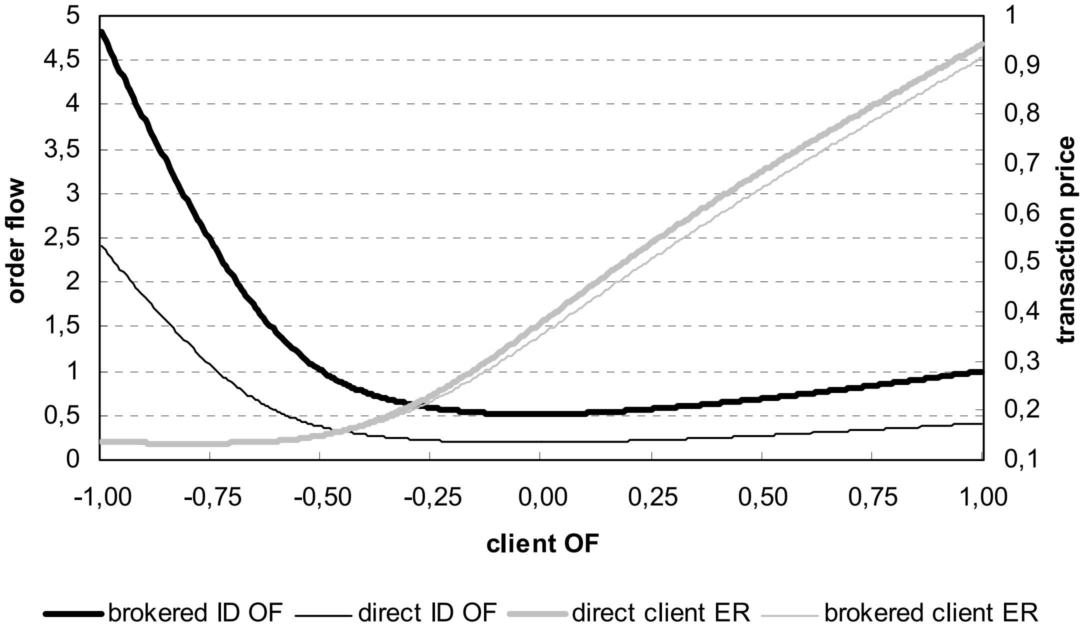
(b) Brokered inter-dealer market



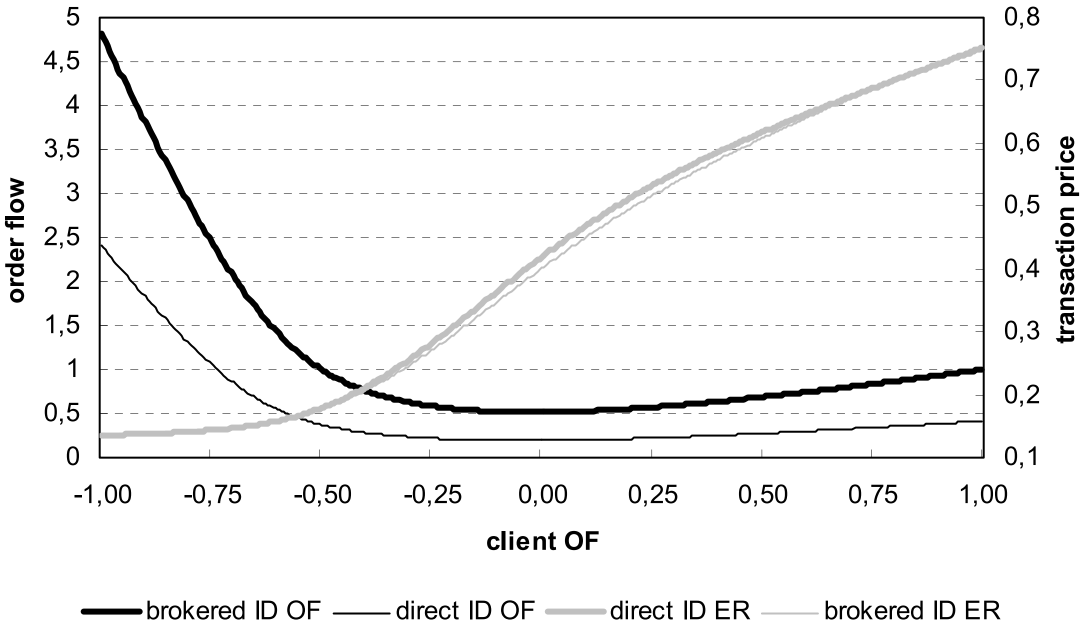
Note: The figure shows the effective customer–dealer and inter-dealer transaction prices and the exchange rate mid-quote as functions of the customer order flow.

Figure 5: Comparison of the direct and brokered trading mechanisms for a given level of the client's order flow per dealer

(a) Inter-dealer order flows and effective exchange rates paid by the client



(b) Inter-dealer order flows and effective exchange rates paid by the dealer



Note: The figure shows the dealer order flows and effective customer–dealer (first panel) and inter-dealer (second panel) transaction prices as functions of the customer order flow.

7. Conclusion

We have investigated the question of the relationship between FX transaction prices and order flows by means of a static and dynamic model of the inter-dealer quoting and trading game. The pivotal feature of the model was the presence of a global market user who traded with two competing dealers by splitting her orders optimally between them. We have studied both the direct inter-dealer and the brokered inter-dealer FX market cases.

The main result of the analysis is the key role of a dealer's current marginal valuation of foreign cash (the ratio of the marginal indirect utilities of foreign and domestic cash holdings) for the impact that the incoming order flow has on his price-setting behavior. When dealers are symmetric, they still place orders with each other in equilibrium, generating what are known as "hot potato" trades. Asymmetry of marginal valuations, caused by either different endowments, information or customer base composition, is the reason why there exists inter-dealer trade other than hot potato transactions. The marginal currency value is also the variable through which new orders, coming from both other dealers and non-dealer customers, exercise an impact on the quoted prices, by channeling information about fundamental changes in currency supply and demand. Institutional differences between the direct and brokered market structures are reflected in the quantitative relationships between the marginal values and the patterns of trade, but do not affect the main qualitative link between the marginal values, prices and equilibrium customer order flow.

Specifically, we have shown that:

a) the price impact of the market user order flow is determined by the marginal indirect utility of foreign cash on the order placer and order recipient side;
b) hot potato inter-dealer trading is a part of the equilibrium trading pattern, i.e. it is present even when the dealers are completely identical. The reason is that the presence of at least one dealer makes it individually optimal for other dealers to unwind the FX position imbalances through trade with him, rather than try to coordinate a no-trade outcome (a phenomenon of the prisoner dilemma type). In the Nash equilibrium with hot potato trades, the latter forces dealers to reveal their marginal currency valuations to other dealers;
c) the institutional arrangement of the forex makes a quantitative difference: for a given level of customer indirect utility value of foreign cash, the brokered market features higher inter-dealer order flow and lower customer prices. The direct market exhibits a stronger price response to customer orders than the brokered market. On the other hand, the qualitative dependence of order flow and price (exchange rate) on the marginal indirect utility level distribution is similar;
d) the above feature contrasts with the "reduced form" view of the dependence between the customer order-flow, inter-dealer trades and transaction prices, established for the steady state Nash equilibrium (or the Nash equilibrium of the one-period game). We find that in the brokered market, the inter-dealer order flow is roughly twice as big as in the direct market for a given level of client demand or supply. With increasing client order flow, client effective prices surpass the inter-dealer effective prices earlier than in the brokered market. This comparative static assessment shows that the cost the investors have to bear for maintaining liquidity (and the price they have to pay for transparency) in the brokered

> market exists in the form of discouragement of big market orders. Moreover, by placing the same market order, the non-dealer investor in the brokered market achieves a lower adjustment of her marginal FX utility. Consequently, brokered markets limit the participants' ability to improve welfare by trade, compared to direct markets.

Under either market organization, the *observed time series of the exchange rate is not enough to construct a sufficient statistic* of the potential market imbalance. The reason is that the space of sufficient statistics has a higher dimension: it includes *the unobserved marginal currency values* of representative groups of forex participants. The observed time series from which these latent values can be extracted (e.g. by filtering techniques) can be either the *high frequency quote series* of individual representative market makers (in the direct inter-dealer market) or the *high frequency limit- and market-order series* of a representative broker (in the brokered market).

Unsolved Problems

Modeling direct and brokered FX market coexistence is technically and computationally difficult. The hypothesis we think is important to explore is that the co-existence of the two market structures is possible because the two mechanisms are utilized differently. The more routine, "technical" or arbitrage trading is probably more appropriately addressed at the brokered segment, whereas the "information-based", or simply large or "fundamentally" important, transactions are better served by the direct dealership segment.

Study of the models with full dealer rationality requires a degree of sophistication that is no longer realistic with regard to the actual information processing and formal analytical capacities of the market participants. It seems that models reverting to bounded rationality, e.g. adaptive and evolutionary learning, have a chance to produce more plausible results than the standard optimization paradigm of theoretical finance.

Looking at the model findings from the viewpoint of central bank interventions and communication with the forex, we think a relevant question is: will the central bank better pursue its goals by acting as a "local" player, with a fixed limited number of preferred counter-parties? Or should it act "globally", by reserving itself the option of addressing any market maker any time when it sees the right occasion to release its message by means of an (intervention) trade? Intuitively, it seems that the global role may sometimes be preferable, but a formal confirmation of this conjecture requires modeling heterogeneous investors on the customer side. This is a topic of future research.

Finally, since the model has disclosed the key role of the marginal utility (shadow) value of the currency in market participants' pricing and trade decisions, can the exchange rate policy hope to influence this value when it pursues a specific objective such as ERMII?

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Appendix

A1 Proof of Propositions 1 and 2

Direct Inter-Dealer Market

The problem of dealer 1 can be formulated as optimization of the objective function H^{q1} given by

$$H^{q1}(\rho,\sigma)=\frac{L^{s1}(\rho;\theta^1,\theta^2,\theta^U)}{\sigma}+f(\rho^{c1})h(q^{c1})-\theta^1 q^{c1}$$

(in the notation of Section 5), with respect to ρ, σ, with parameters θ defined in Subsection 4.1.1.

We shall make use of the auxiliary equalities $\dfrac{\partial\rho^{c1}}{\partial\rho^1}=\dfrac{1}{2}$, $\dfrac{\partial q^{c1}}{\partial\rho^1}=-\dfrac{1}{2\sigma^1}$, $\dfrac{\partial q^{c1}}{\partial\sigma^1}=\dfrac{\rho^1-\gamma^1}{2(\sigma^1)^2}=-\dfrac{q^{c1}}{\sigma^1}$.

Equations (4) for $\theta=\theta^2$ and $\theta=\theta^{\mathrm{U}}$, which yield each market user's optimal active trades, i.e., imply

$$\frac{\partial L^{s1}}{\partial\rho}=L^{s1}_{\rho}=L^{p1}.$$

Therefore,

$$\frac{\partial H^{q1}}{\partial\rho}=\frac{L^{p1}}{\sigma}-\frac{1}{2\sigma}\left[C^1(\rho,\sigma)+D^1(\rho,\sigma)\right],$$

and, for any fixed value of ρ, the above expression is equalized to zero for a single positive value of σ (this partial derivative is positive for σ below this critical value and negative for σ above it). This proves (10a).

Similarly, it is immediately checked that

$$\frac{\partial H^{q1}}{\partial\sigma}=\frac{1}{\sigma^2}\left[\frac{\rho-\gamma^1}{2}C^1(\rho,\sigma)-L^{s1}(\rho;\theta^1,\theta^2,\theta^U)\right],$$

proving (10b). This completes the proof of Proposition 1.

Brokered Inter-Dealer Market

The problem of dealer 1 is equivalent to maximizing

$$H^1(x^m,x^i,q^1,\rho^1,\sigma^1,\xi_m,\xi_i)=v(x^m,x^i)+\xi_m\left\lfloor y^m+f(\rho^b)q^{b1}+f(\rho^{c1})h(q^{c1})\right\rfloor$$

$$+\xi_m\left[\frac{P(\rho^b,\theta^1)g(\rho^b,\theta^1)+P(\rho^b,\theta^2)g(\rho^b,\theta^2)+P(\rho^b,\theta^U)g(\rho^b,\theta^U)}{\sigma^1}-f(\rho^b+\sigma^b q^1)q^1\right],$$

$$+\xi_i\left[y^i-\frac{g(\rho^b,\theta^1)+g(\rho^b,\theta^2)+g(\rho^b,\theta^U)}{\sigma^1}+q^1-q^{b1}-q^{c1}\right]. \tag{A1}$$

w.r.t. ρ^1 and σ^1. Denote the right-hand side of (A1) by H(ρ^b,ρ^1,σ^1). The relevant partial derivatives of H^1 will be obtained with the help of the equality

$$\frac{\partial H^1}{\partial\rho^1}=\frac{\partial\rho^b}{\partial\rho^1}\frac{\partial\mathbf{H}}{\partial\rho^b}+\frac{\partial\mathbf{H}}{\partial\rho^1}. \tag{A2}$$

It can be easily checked that

$$\frac{\partial\rho^b}{\partial\rho^1}=\frac{\sigma^2}{\sigma^1+\sigma^2},\ \frac{\partial q^{b1}}{\partial\rho^1}=-\frac{1}{\sigma^1+\sigma^2},\ \frac{\partial\sigma^b}{\partial\sigma^1}=\left(\frac{\sigma^2}{\sigma^1+\sigma^2}\right)^2, \tag{A3}$$

$$\frac{\partial\rho^b}{\partial\sigma^1}=\frac{\sigma^2(\rho^2-\rho^1)}{(\sigma^1+\sigma^2)^2}=\frac{\sigma^2}{\sigma^1+\sigma^2}q^{b1},\ \frac{\partial q^{b1}}{\partial\sigma^1}=\frac{\rho^1-\rho^2}{(\sigma^1+\sigma^2)^2}=-\frac{q^{b1}}{\sigma^1+\sigma^2}. \tag{A4}$$

Next, invoking (8) for θ^1, θ^2, θ^U, we establish that

$$\begin{aligned}
\frac{\partial\mathbf{H}}{\partial\rho^b}&=\xi_m\left\{f'(\rho^b)q^{b1}+\frac{1}{\sigma^1}\left[\left(1+g_\rho(\rho^b,\theta^2)\right)\theta^2+\left(1+g_\rho(\rho^b,\theta^U)\right)\theta^U\right]\right\}\\
&-\xi_m\left\{\left(g_\rho(\rho^b,\theta^2)+g_\rho(\rho^b,\theta^U)\right)\theta^1+P(\rho^b,\theta^2)+P(\rho^b,\theta^U)\right\}+\xi_m\frac{P(\rho^b,\theta^1)-\theta^1}{\sigma^2}\\
&=\xi_m\left\{f'(\rho^b)q^{b1}+\frac{L^{p1}(\rho^b;\theta^1,\theta^2,\theta^U)}{\sigma^1}+\frac{P(\rho^b,\theta^1)-\theta^1}{\sigma^2}\right\},\\
\frac{\partial\mathbf{H}}{\partial\rho^1}&=\frac{\xi_m}{\sigma^1+\sigma^2}\left(\theta^1-P^b\right)-\frac{\xi_m}{2\sigma^1}\left[C^1(\rho^1,\sigma^1)+D^1(\rho^1,\sigma^1)\right].
\end{aligned} \tag{A5}$$

Using the first two equalities in (A3), we get the following:

$$\frac{\partial H^1}{\partial\rho^1}=\frac{\xi_m\sigma^2}{\sigma^1(\sigma^1+\sigma^2)}\left\{L^{p1}(\rho^b;\theta^1,\theta^2,\theta^U)-B^1(\rho^1,\rho^2,\sigma^1,\sigma^2)-\frac{\sigma^1+\sigma^2}{2\sigma^2}\left[C^1+D^1\right]\right\},$$

proving (11a). To prove (11b) observe that

$$\frac{\partial H^1}{\partial\sigma^1}=\frac{\partial\rho^b}{\partial\sigma^1}\frac{\partial\mathbf{H}}{\partial\rho^b}+\frac{\partial\mathbf{H}}{\partial\sigma^1}=\frac{\partial\rho^b}{\partial\sigma^1}\left(\frac{\partial\rho^b}{\partial\rho^1}\right)^{-1}\left[\frac{\partial H^1}{\partial\rho^1}-\frac{\partial\mathbf{H}}{\partial\rho^1}\right]+\frac{\partial\mathbf{H}}{\partial\sigma^1} \tag{A6}$$

(the second equality follows from (A2)). Observe also that $\frac{\partial \rho^b}{\partial \sigma^1}\left(\frac{\partial \rho^b}{\partial \rho^1}\right)^{-1} = \frac{\rho^2 - \rho^1}{\sigma^1 + \sigma^2} = q^{b1}$.

Next, we get by direct calculation that

$$\frac{\partial \mathbf{H}}{\partial \sigma^1} = \xi_m \left\{ f(\rho^b)\frac{\partial q^{b1}}{\partial \sigma^1} - \frac{g(\rho^b,\theta^1)P(\rho^b,\theta^1)}{\left(\sigma^1\right)^2} - f'(\rho^b + g(\rho^b,\theta^1))\left[g(\rho^b,\theta^1)\right]^2 \frac{1}{\left(\sigma^b\right)^2}\frac{\partial \sigma^b}{\partial \sigma^1} \right\}$$

$$- \xi_m \frac{L^{s1}(\rho^b;\theta^1,\theta^2,\theta^U)}{\left(\sigma^1\right)^2} + \xi_i \left\{ \frac{(\rho^1-\gamma^1)C^1(\rho^1,\sigma^1)}{2\left(\sigma^1\right)^2} - \frac{\partial q^{b1}}{\partial \sigma^1} \right\}.$$

Applying the last equality in (A3), the three equalities (A4) and the market order optimality condition $f'\left(\rho^b + g(\rho^b,\theta^1)\right)g(\rho^b,\theta^1) = \theta^1 - P(\rho^b,\theta^1)$, the last expression can be reduced to

$$\frac{\partial \mathbf{H}}{\partial \sigma^1} = \xi_m \left\{ \frac{(\rho^1-\rho^2)\left(P^b-\theta^1\right)}{\left(\sigma^1+\sigma^2\right)^2} - \frac{L^{s1}(\rho^b;\theta^1,\theta^2,\theta^U)}{\left(\sigma^1\right)^2} + \frac{(\rho^1-\gamma^1)C^1(\rho^1,\sigma^1)}{2\left(\sigma^1\right)^2} \right\}.$$

However, by (A5),

$$\frac{\partial \rho^b}{\partial \sigma^1}\left(\frac{\partial \rho^b}{\partial \rho^1}\right)^{-1}\frac{\partial \mathbf{H}}{\partial \rho^1} = \xi_m \frac{\rho^1-\rho^2}{\sigma^1+\sigma^2}\left\{ \frac{C^1+D^1}{2\sigma^1} + \frac{P^b-\theta^1}{\sigma^1+\sigma^2} \right\},$$

meaning that, by (A6),

$$\frac{\partial H^1}{\partial \sigma^1} = \frac{\xi_m}{\left(\sigma^1\right)^2}\left\{ \frac{\left(\sigma^1\right)^2(\rho^1-\rho^2)\left(P^b-\theta^1\right)}{\left(\sigma^1+\sigma^2\right)^2} - L^{s1} + \frac{(\rho^1-\gamma^1)C^1}{2} \right\}$$

$$+ \xi_m \frac{\rho^1-\rho^2}{\sigma^1+\sigma^2}\left[\frac{\theta^1-P^b}{\sigma^1+\sigma^2} - \frac{C^1+D^1}{2\sigma^1} \right] - \xi_m \frac{\sigma^2}{\sigma^1(\sigma^1+\sigma^2)}\left[L^{p1} - B^1 - \frac{(\sigma^1+\sigma^2)(C^1+D^1)}{2\sigma^2} \right]$$

$$= \frac{\xi_m}{\left(\sigma^1\right)^2}\left\{ \frac{(\rho^1-\gamma^1)C^1}{2} - L^{s1} - \frac{\sigma^1\sigma^2(\rho^1-\rho^2)}{\left(\sigma^1+\sigma^2\right)^2}\left(L^{p1}-B^1\right) \right\},$$

from which (11b) follows

A2 Trading Strategies and Equilibrium Outcomes in the Differential Inter-Dealer Game

We shall use the Hamiltonian characterization of the solution to the optimal control problem (13) of a given investor, with state-transition equation (14), (15), (16) or analogous. For example, the current value Hamiltonian for dealer 1 in the direct market is defined as

$$H^U(x^m,x^i,c,Q^1,Q^2,\xi_m,\xi_i)=u(c)+\beta v(x^m,x^i)$$

$$+\xi_m\left[y^m-f(\rho^1+\sigma^1Q^1)Q^1-f(\rho^2+\sigma^2Q^2)Q^2-c\right]+\xi_i\left[y^i+Q^1+Q^2\right], \tag{A7}$$

where ξ_m, ξ_i are adjoint variables of the problem. Their evolution is described by the Euler equations, to be featured shortly. We shall call these adjoint variables of the investor's optimization problem *shadow prices* of the domestic and foreign currency respectively. They constitute the currency valuation by means of the investor's indirect utility.

Under the made assumptions about the strict concavity of utility functions *u* and *v* and the growth properties of function *v* at infinity, the optimal policies of the global market user can be characterized by the first-order conditions following from the Maximum Principle (see e.g. Fleming and Rishel, 1975):

$$u'(c)=\xi_m,\ f'\left(\rho^j+\sigma^jQ^j\right)\sigma^jQ^j+f\left(\rho^j+\sigma^jQ^j\right)=\frac{\xi_i}{\xi_m},\ j=1,2. \tag{A8}$$

The shadow prices are characterized by the adjoint equations

$$\dot{\xi}_m=\beta\left(\xi_m-\frac{\partial v}{\partial x^m}\right),\ \dot{\xi}_i=\beta\left(\xi_i-\frac{\partial v}{\partial x^i}\right) \tag{A9}$$

for every market participant, established by direct inspection of the maximum principle. The initial conditions are implicitly pinned down by the initial conditions for the state variables x^m and x^i.

Analogously, in the brokered market, the current value Hamiltonian optimization by dealer 1 implies the first-order conditions of optimality

$$u'(c)=\xi_m,\ f'\left(\rho^b+\sigma^bq^1\right)\sigma^bq^1+f\left(\rho^b+\sigma^bq^1\right)=\theta^1$$

characterizing the dividend rate and own order size.

Propositions 1 and 2 of Section 3 are still valid, with the marginal currency values defined as

$$\theta^k=\frac{\xi_i^k}{\xi_m^k}=\frac{\partial V^k/\partial x^i}{\partial V^k/\partial x^m},\ k=1,2,U,$$

V^k being the value function of investor's problem (the result of maximization in (13)). These results are the two first-order conditions on dealer 1's quoting parameters ρ^1 and σ^1, and, by symmetry, two analogous conditions must be valid for dealer 2's quoting parameters ρ^2 and σ^2.

The quoting parameters themselves depend on current and future values of θ, i.e. to obtain the full characterization of equilibrium quotes, one needs to solve for the Nash equilibrium in the differential game. An embedded one-shot game at every time moment can be isolated if one fixes the marginal valuation vector $\theta = (\theta^1, \theta^2, \theta^U)$, since individual optimal orders and quoting parameters at each moment are well-defined functions of θ. We are mostly interested in the embedded game corresponding to the steady state.

The steady state conditions for the Euler equations (A9) imply

$$\frac{\partial V^U}{\partial x^m} = \xi_m = v_m(x^m, x^i)\,,\ \frac{\partial V^U}{\partial x^i} = \xi_i = v_i(x^m, x^i)\,,\ \theta^U = \frac{v_m(x^m, x^i)}{v_i(x^m, x^i)} \tag{A10}$$

(as usual, subscripts denote partial derivatives). If we denote by j the inverse function to the marginal utility u' of the investor's dividend rate, then optimization of the latter in the steady state implies $c=j(v_m(x^m,x^i))$. These facts will be used in the next two sections.

A3 Proof of Proposition 3

Let $n=[\rho,\sigma]^{\mathrm{T}}$ be the 4-dimensional vector of the optimal quoting rules, considered a function of the 3-dimensional vector θ of the three marginal foreign currency valuations of the market participants: $n=N(\theta)$. Further, let $k(n)$ denote the terms in the cash position 6-equation system (2), (6) (in the direct market) or (7), (9) (in the brokered market), with the expressions (3) or (7) for the optimal active trades already substituted. That is, we write symbolically

$$\tilde{x}^m = \tilde{y}^m + k^m(n)\,, \tag{A11a}$$

$$\tilde{x}^i = \tilde{y}^i + k^i(n)\,, \tag{A11b}$$

where $\tilde{x}$ is the 6-dimensional vector of the cash positions of the two dealers and the non-dealer market user, whereas $\tilde{y}$ is the 6-dimensional vector of their start-of-the-period cash endowments. For a given $\tilde{y}$, we are looking for such a end-of-period cash holding vector $\tilde{x}$ that the Nash equilibrium defined at the end of Section 3 is attained. This means that the condition

$$\tilde{n} = N(\theta) = N \circ \vartheta(\tilde{x}) = M(\tilde{x})\,, \tag{A12}$$

must be satisfied for $\vartheta = \left(\vartheta^1, \vartheta^2, \vartheta^U\right)$, $\vartheta^l(x) = \dfrac{v_i(x)}{v_m(x)}$, l=1,2,U. Solution $\tilde{x}$ to the system (A11), (A12) is the Nash equilibrium of the one-period inter-dealer game. To prove Proposition 4, we shall find an endowment rate vector y in the dynamic game such that its steady state Nash equilibrium quoting rule $\bar{n} = N(\bar{\theta})$ is equal to $\tilde{n}$. This is equivalent to requiring that the steady state NE cash holding vector $\bar{x}$ satisfies the equality $M(\bar{x}) = M(\tilde{x})$, equivalent to $\vartheta(\bar{x}) = \vartheta(\tilde{x})$.

First observe that (A11) can be written as $k(\tilde{n}) = \tilde{x} - \tilde{y}$, whereas the steady state NE in the differential game (first-order conditions plus constancy of x and ξ) can be summarized as

$$k^m(\bar{n}) = j \circ v_m(\bar{x}) - y^m\,, \tag{A13a}$$

$$k^i(\bar{n}) = -y^i, \tag{A13b}$$

Note that (A13) is a non-trivial consequence of the maximum principle and the steady state conditions for variables x and ξ. It is at this point that the results of dynamic game theory help us to comply with the NE equilibrium conditions in the one-period game.

In order to obtain the one-period game NE from the steady state NE, we must make sure that the right-hand side of (A13) be equal to $k(\tilde{n}) = \tilde{x} - \tilde{y}$. This implies that the sought endowment rates y^m, y^i generate $\tilde{x}$ according to the rule

$$\tilde{x}^m = j \circ v_m(\bar{x}^m) + \tilde{y}^m - y^m,\ \tilde{x}^i = \tilde{y}^i - y^i, \tag{A14}$$

and, to prove our statement, we must find y that generate steady state NE cash holdings $\bar{x}$ that satisfy $\vartheta(\bar{x}) = \vartheta(\tilde{x})$. Recalling (1) and once again invoking (A13), we restate the latter condition in the form

$$\alpha_i k^i \circ N \circ \vartheta(\bar{x}) - \alpha_m k^m \circ N \circ \vartheta(\bar{x}) = \alpha_m \tilde{y}^m - \alpha_i \tilde{y}^i. \tag{A15}$$

Now, observe that one can generate any vector $\theta = \vartheta(\bar{x})$ with strictly positive components by varying $\bar{x}$. So, if we find $\theta = \vartheta(\bar{x})$ to satisfy (A5), endowment rates y can be reconstructed from $\bar{x}$ using (A13). But, by checking that the map $\theta \mapsto \alpha_i k^i \circ N(\theta) - \alpha_m k^m \circ N(\theta)$ has a full rank Jacobian at least on the open, everywhere dense subset of R^{3+}, we conclude that its range must be equal to the whole real line. Actually, there is a lot of freedom in the choice of $\bar{x}$, which means that one can generate different "convergence speeds" to the one-period game NE. This concludes the proof

A4 Proof of Proposition 4

We will proceed by deriving the steady state equilibrium equation system separately for the direct and brokered market mechanisms. Then we go over to the symmetric steady state case and prove the statement of Proposition 4 for the direct market. The proof for the brokered market is fully analogous.

Recall that the differential equations describing the optimal behavior of dealer 1, dealer 2 and the non-dealer market user are given by state-transition equations (14)–(16) (or their analogues) and (A9) (adjoint equations describing the evolution of the co-state variables for all three players).

The quoting rule f is exponential: $f(p) = e^{cp}$, $c>0$. Then the auxiliary function α introduced at the beginning of Section 3 reduces to

$$\alpha(\rho,\theta) = 1 + g_\rho(\rho,\theta) = \frac{f'(\rho + g(\rho,\theta))}{2f'(\rho + g(\rho,\theta)) + g(\rho,\theta) f''(\rho + g(\rho,\theta))} = \frac{1}{2 + cg(\rho,\theta)},$$

and the price function becomes

$$P(\rho,\theta) = f(\rho + g(\rho,\theta)) = e^{c(\rho + g(\rho,\theta))}.$$

We have assumed that the external matching happens according to the same rule with the same parameters for both dealers, so that the γ-parameter is common for them. Let us rewrite the other two auxiliary functions, C and D, in terms of variables q^c and σ instead of the original ρ and σ. They become

$$C^1(\rho,\sigma) = \left[1 - aq^{c1}\right] e^{c\frac{\rho+\gamma}{2}} - \theta^1,\ D^1(\rho,\sigma) = -c\sigma q^{c1}\left[1 - \frac{aq^{c1}}{2}\right] e^{c\frac{\rho+\gamma}{2}}.$$

Similar notations (index 2 replacing 1) will be used for dealer 2.

We shall now go over to choosing a more convenient set of variables for which the steady state will be characterized.

First, looking at the structure of the steady state conditions for our set of differential equations, we note that (4a), (15a), (16a) relate the steady state values of the domestic cash shadow price, ξ_m, with the remaining variables, whereas this variable does not appear alone in any other equation. Therefore, one can replace the pair (ξ_m, ξ_i) by the pair (ξ_m, θ) in further considerations, thus eliminating the necessity to involve the steady state versions of (14a)–(16a) in the calculations to follow.

Second, observe that for any agent, neither the right-hand side of the state-transition equation for x^i, i.e. (14b), (15b), or (16b), nor the first-order conditions of optimality, i.e. (3), (8), (10), or (11), depends on the state variables x. Therefore, these equations can be used to pin down the three marginal values θ and the two pairs of optimal quoting parameters, (ρ^k,σ^k), k=1,2. Subsequently, the steady state conditions for the adjoint equations (A9) can be used to determine the steady state values of the state variables x. Accordingly, one does not need to include (A9) in the intermediate calculations. One is left with variables q^{c1},σ^1, q^{c2},σ^2, θ^1, θ^2, θ^U and only deals with (14b)–(16b), (10) and (11).

Third, observe that, most of the time, the marginal currency values θ appear only as arguments in functions g. The only exceptions are functions L^p and L^s in (10), (11). In these expressions, it is more convenient to revert back from θ to terms containing g only, by using the left-hand side of the second equation in (A8). Instead of g, we shall use the trade volume variables q and Q directly, since it turns out to be simpler in terms of notation.

A4.1 Direct Inter-Dealer Market

Rewriting (A8), we see that $\theta^1 = e^{c\left(\rho^2+\sigma^2 q^{12}\right)}(1 + c\sigma^2 q^{12})$,

$$\theta^2 = e^{c(\rho^1+\sigma^1 q^{21})}(1 + c\sigma^1 q^{21}),\ \alpha(\rho^1,\theta^2)\theta^2 = (1 - \alpha(\rho^1,\theta^2))e^{c(\rho^1+\sigma^1 q^{21})},$$
$$\theta^U = e^{c(\rho^1+\sigma^1 Q^1)}(1 + c\sigma^1 Q^1),\ \alpha(\rho^1,\theta^U)\theta^U = (1 - \alpha(\rho^1,\theta^U))e^{c(\rho^1+\sigma^1 Q^1)},$$

and similar formulae can be written for dealer 2. This allows us to rewrite L^p and L^s as follows:

$$L^{p1} = \left(\frac{1 + c\sigma^1 q^{21}}{2 + c\sigma^1 q^{21}} + \frac{1 + c\sigma^1 Q^1}{2 + c\sigma^1 Q^1}\right) e^{c\left(\rho^2+\sigma^2 q^{12}\right)}\left(1 + c\sigma^2 q^{12}\right) - \frac{e^{c\left(\rho^1+\sigma^1 q^{21}\right)}}{2 + c\sigma^1 q^{21}} - \frac{e^{c\left(\rho^1+\sigma^1 Q^1\right)}}{2 + c\sigma^1 Q^1},$$

$$L^{s1}(\rho,\theta) = \sigma^1\left[e^{c\left(\rho^1+\sigma^1 q^{21}\right)} q^{21} + e^{c\left(\rho^1+\sigma^1 Q^1\right)} Q^1 - e^{c\left(\rho^2+\sigma^2 q^{12}\right)}\left(1 + c\sigma^2 q^{12}\right)\left(q^{21} + Q^1\right)\right]$$

(and similarly for dealer 2).

The next step is substitution of the above expressions into the first-order conditions (10), collecting terms that contain θ^1, on the left-hand side and dividing by $exp(c\rho^1)$. Use the fact that $\rho^k=\gamma\text{-}2\sigma^k q^{ck}$, k=1,2. The result is the following pair of optimality conditions:

$$\left(\frac{1}{2}+\frac{1+c\sigma^1 q^{21}}{2+c\sigma^1 q^{21}}+\frac{1+c\sigma^1 Q^1}{2+c\sigma^1 Q^1}\right)e^{2c\left(\sigma^1 q^{c1}-\sigma^2 q^{c2}\right)+c\sigma^2 q^{12}}\left(1+c\sigma^2 q^{12}\right)$$

$$=\frac{e^{c\sigma^1 q^{21}}}{2+c\sigma^1 q^{21}}+\frac{e^{c\sigma^1 Q^1}}{2+c\sigma^1 Q^1}+\frac{1-aq^{c1}-c\sigma^1\left(1-\frac{aq^{c1}}{2}\right)q^{c1}}{2}e^{c\sigma^1 q^{c1}}, \tag{A16}$$

$$e^{2c\left(\sigma^1 q^{c1}-\sigma^2 q^{c2}\right)+c\sigma^2 q^{12}}\left(1+c\sigma^2 q^{12}\right)\left(q^{21}+Q^1+q^{c1}\right)=e^{c\sigma^1 q^{21}}q^{21}+e^{c\sigma^1 Q^1}Q^1+e^{c\sigma^1 q^{c1}}(1-aq^{c1})q^{c1}. \tag{A17}$$

In addition, the optimal trade volumes of the non-dealer market user are linked, in accordance with (7), by the equality

$$e^{c(\gamma-2\sigma^1 q^{c1}+\sigma^1 Q^1)}(1+c\sigma^1 Q^1)=e^{c(\gamma-2\sigma^2 q^{c2}+\sigma^2 Q^2)}(1+c\sigma^2 Q^2). \tag{A18}$$

Note that (A16)–(A18) are valid outside the steady state as well; all we have done is change notations. To analyze the steady state, one must add to (A16), (A17) and the corresponding pair of first-order conditions for dealer 2 the three equations that state the constancy of the foreign cash holdings in the steady state. By denoting the foreign cash endowments of dealer 1, dealer 2 and the market user by y^{1i}, y^{2i} and Y^i for convenience, we write these conditions as

$$q^{21}+Q^1+q^{c1}=y^{1i}+q^{12}, \tag{A19}$$

$$q^{12}+Q^2+q^{c2}=y^{2i}+q^{21}, \tag{A20}$$

$$Q^1+Q^2=-Y^i. \tag{A21}$$

A4.2 Brokered Inter-Dealer Market

This time, (7) implies that

$$\theta^1=e^{c\left(\rho^b+\sigma^b q^1\right)}(1+c\sigma^b q^1),\ \theta^2=e^{c(\rho^b+\sigma^b q^2)}(1+c\sigma^b q^2),\ \theta^U=e^{c(\rho^b+\sigma^b Q)}(1+c\sigma^b Q),$$

and

$$L^{p1}=\left(\frac{1+c\sigma^b q^2}{2+c\sigma^b q^2}+\frac{1+c\sigma^b Q}{2+c\sigma^b Q}\right)\left(1+c\sigma^b q^1\right)e^{c\left(\rho^b+\sigma^b q^1\right)}-\frac{e^{c\left(\rho^b+\sigma^b q^2\right)}}{2+c\sigma^b q^2}-\frac{e^{c\left(\rho^b+\sigma^b Q\right)}}{2+c\sigma^b Q},$$

$$L^{s1}(\rho,\theta)=\sigma^b\left[e^{c\left(\rho^b+\sigma^b q^2\right)}q^2+e^{c\left(\rho^b+\sigma^b Q\right)}Q-e^{c\left(\rho^b+\sigma^b q^2\right)}\left(1+c\sigma^b q^1\right)\left(q^2+Q\right)\right].$$

In the same way as in A4.2, we will use q^{c1}, q^{c2} instead of ρ^1, ρ^2 as unknown variables. It can be easily checked that $\rho^b=\gamma-2\sigma^b(q^{c1}+q^{c2})$ and the auxiliary function B^1 appearing in the first-order conditions (11) can be expressed as

$$B^1 = e^{c\rho^b}\left\{\frac{\sigma^1}{\sigma^2}\left(1-e^{c\sigma^b q^1}\right)+\frac{2c\sigma^1\left(\sigma^2 q^{c2}-\sigma^1 q^{c1}\right)}{\sigma^1+\sigma^2}\right\}.$$

Therefore, after minor transformations, the first-order conditions (11) can be written, analogously with (A16), (A17), as

$$\left(\frac{\sigma^1+\sigma^2}{2\sigma^2}+\frac{1+c\sigma^b q^2}{2+c\sigma^b q^2}+\frac{1+c\sigma^b Q}{2+c\sigma^b Q}\right)e^{c\sigma^b q^1}\left(1+c\sigma^b q^1\right)+\frac{\sigma^1}{\sigma^2}\left(e^{c\sigma^b q^1}-1\right)$$

$$=\frac{e^{c\sigma^b q^2}}{2+c\sigma^b q^2}+\frac{e^{c\sigma^b Q}}{2+c\sigma^b Q}+\frac{2c\sigma^1\left(\sigma^2 q^{c2}-\sigma^1 q^{c1}\right)}{\sigma^1+\sigma^2}$$

$$+\frac{\sigma^1+\sigma^2}{2\sigma^2}\left[1-aq^{c1}-c\sigma^1\left(1-\frac{aq^{c1}}{2}\right)q^{c1}\right]e^{2c\sigma^b\left(q^{c1}+q^{c2}\right)-c\sigma^1 q^{c1}}, \tag{A22}$$

$$e^{c\sigma^b q^2}q^2+e^{c\sigma^b Q}Q+e^{2c\sigma^b\left(q^{c1}+q^{c2}\right)-c\sigma^2 q^1}(1-aq^{c1})\left(q^{c1}+q^{c2}\right)=e^{c\sigma^1 q^{c1}}\left(1+c\sigma^b q^1\right)\left(q^2+Q+q^{c1}+q^{c2}\right)$$

.

$$+e^{2c\sigma^b\left(q^{c1}+q^{c2}\right)-c\sigma^1 q^{c1}}c\sigma^1\frac{\sigma^2 q^{c2}-\sigma^1 q^{c1}}{\sigma^2}q^{c1}\left(1-\frac{aq^{c1}}{2}\right). \tag{A23}$$

Similar equations must hold for dealer 2.

These equations must be completed with the three steady state conditions following from (16b) and the (trivial) brokered-market analogue of (14b):

$$\frac{\sigma^1}{\sigma^1+\sigma^2}q^1+\frac{\sigma^2}{\sigma^1+\sigma^2}\left(q^2+Q\right)+\frac{(3\sigma^1+\sigma^2)q^{c1}-\sigma^2 q^{c2}}{\sigma^1+\sigma^2}=y^{1i}, \tag{A24}$$

$$\frac{\sigma^2}{\sigma^1+\sigma^2}q^2+\frac{\sigma^1}{\sigma^1+\sigma^2}\left(q^1+Q\right)+\frac{(3\sigma^2+\sigma^1)q^{c2}-\sigma^1 q^{c1}}{\sigma^1+\sigma^2}=y^{2i}, \tag{A25}$$

$$Q=-Y^i. \tag{A26}$$

In the direct market, (A16)–(A21) is a system of eight equations for eight unknown variables, namely q^{12}, q^{21}, Q^1, Q^2, q^{c1}, q^{c2}, σ^1, σ^2. In the brokered market, the system (A22)–(A25) has six equations for six unknowns: q^1, q^2, q^{c1}, q^{c2}, σ^1, σ^2. These systems must be solved to obtain the steady state Nash equilibria. In full generality this can only be done numerically.

A4.3 Dealer-Symmetric Steady State, Direct Market

The complete symmetry of both the parameters and behavior of dealer 1 and dealer 2 implies that the market user's trades with the two market makers are also identical. Accordingly, we have only four unknown variables instead of the original eight: q – the inter-dealer trade volume, Q – the customer trade volume, q^c – the volume of trade in external matching (looking for q^c is equivalent to looking for ρ, the dealer mid-quote), and σ – the slope of the dealer pricing schedule. Since (A18) is now vacuous and (A19), (A20) become one, we are also left with only four equations: (A16), (A17) (both simplified), (A19) and (A21). Denoting by y^i the now common per-period dealer endowment with foreign cash, we observe that Q and q^c are fully determined by the steady state conditions (A19), (A21): $q^c = y^i + \frac{Y^i}{2}$, $Q = -\frac{Y^i}{2}$. The corresponding simplified versions of (A16) and (A17) look like

$$\left\{\left(\frac{1}{2}+\frac{1+c\sigma q}{2+c\sigma q}+\frac{1+c\sigma Q}{2+c\sigma Q}\right)(1+c\sigma q)-\frac{1}{2+c\sigma q}\right\}e^{c\sigma q}=\frac{e^{c\sigma Q}}{2+c\sigma Q}+\frac{1-aq^c-c\sigma\left(1-\frac{aq^c}{2}\right)q^c}{2}e^{c\sigma q^c} \quad \text{(A27)}$$

$$\left[(1+c\sigma q)(q+y^i)-q\right]e^{c\sigma q}=e^{c\sigma Q}Q+e^{c\sigma q^c}(1-aq^c)q^c\,. \quad \text{(A28)}$$

This is a system of two equations for two unknowns, q and σ. Except for an exceptional combination of exogenous parameters, including the condition Y^i=0 that was excluded from consideration in Proposition 4, q=0 cannot be a part of the solution for the system (A27), (A28). That is, generically, “hot potato” trade cannot be avoided

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Czech National Bank
Economic Research Department
Na Příkopě 28, 115 03 Praha 1
Czech Republic
phone: +420 2 244 12 321
fax: +420 2 244 14 278
e-mail: eva.grenarova@cnb.cz