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ABSTRACT

This paper examines the parity conditions between assets denominated in different currencies, traded in a well-integrated segment of the international capital market, and derives the consequences for exchange rate expectations.

The main objective is to assess the uncovered asset return parity for the Czech koruna exchange rate. I argue that any reasonable, decision-theoretical foundation of the uncovered parity condition should be formulated in terms of secondary market yields on long-term instruments and not short-term money market rates. Specifically, this is how the international version of the Consumption-based Capital Asset Pricing Model (CCAPM), which is the universal (and, in fact, the single) message of any stochastic general equilibrium model of an open economy, should be interpreted. Accordingly, I replace the traditional uncovered interest rate parity (UIP) by the uncovered total return parity condition. The theoretical arguments in favor of the uncovered asset return parity are matched by a number of examples, of which the main group is formed by long-maturity government bonds. Data are examined for the Czech versus German 5-year government bonds, the CZK 10-year bonds of the European Investment Bank vs. German government 10-year bonds, as well as for the Austrian versus US-Treasury 10-year bonds. In both cases, parity seems to hold, although the time horizons and the measures of exchange rate movements for which it becomes visible are different.

The proposed micro foundation of the uncovered asset return parity uses a model of consumption and investment in an open economy under diffusion uncertainty with soft liquidity constraints. The model is solved by means of the stochastic maximum principle including the adjoint equations for the co-state variables. The general equilibrium of portfolio optimizing investors is used to derive a breakdown of the country risk premium present in the uncovered asset return parity relation between a selected pair of financial instruments. This allows one to analyze the prevailing beliefs about the long-term exchange rate path.

The basic consumption and investment model can be extended to the context of a productive firm issuing its own liabilities to cover liquidity needs. This extension leads to an analogous uncovered yield parity result. I show that the uncovered return parity from the producer perspective is the same as from the consumer and investor perspective. This opens a way for the exchange rate expectation-extraction by means of corporate bonds and other company-specific instruments. In addition, the disparity of returns in international equity markets possesses a certain explanatory power as regards expected competitiveness of domestic and foreign industries.

Keywords: Uncovered parity; Asset prices; Portfolio optimization; International CCAPM
JEL classification: E44, E52, G11, G12, C61

Contents

1 Exchange Rate, Uncovered Parity and Long Maturity Instruments

1.1 Introduction: the existence problem

The efforts by many economists to formulate the fundamental equilibrium for the exchange rate have been overturned by the reality of the foreign exchange (FOREX) market in many countries. The fundamental equilibrium in the FOREX market has somehow failed to show up yet. One should not be too surprised with this failure. After all, the equilibrium exchange rate is but an abstract academic concept, despite the attempts of a number of economic experts of "executive" category to treat it as a real world phenomenon.

The exchange rate equilibrium is, nevertheless, a useful didactic notion, operated with successfully since the emergence of international macroeconomics as an independent discipline. International finance finds much less use for the concept. Finally, those analysts and practitioners who feel no need for macroeconomics as such in their work (like both business school professors and their graduates), have a simple answer to the equilibrium exchange rate question ready at hand. Namely, they are content to state that the price of every currency in the FOREX market is the equilibrium price at any moment when it is being traded. The equilibrium is established *per definition* by the supply-demand equalization.

A way to appease the above statement with a broader economic understanding of the matter is to say that the equilibrium in the FOREX market is dynamic. Its trajectory does not have to converge to a single point, intuitively connected by most people to the (static) exchange rate equilibrium. As the first and easiest example, consider the equilibrium exchange rate that can be described by a log-normal diffusion process S giving the domestic price of a foreign currency unit, with the law of motion $dS=S(\mu dt+\sigma dZ)$ (Z is a Wiener process). In this case, the equilibrium cannot be connected to any particular *level* of the foreign currency price S. Instead, it is characterized by the value of its drift μ, which is the average appreciation or depreciation trend, and the volatility parameter σ. The famous models, such as the "honeymoon effect" one, for the exchange rate target zone by Krugman, 1991, as well as its collapse, by Krugman and Rotemberg, 1992, work with this specification of the freely floating exchange rate prevailing before the start and after the end of the target zone regime. During the time when the zone is effective, the exchange rate follows an even more involved law of motion (an Itô process obtained by a transformation of a diffusion process with reflecting barriers) which, again, has no preferred value to be pointed at as the equilibrium one.

A still more complicated equilibrium characterization would come about if, instead of a relatively simple log-normal diffusion we took a more general statistical model for the process S, like, for example, a continuous time analogue of GARCH (generalized auto-regressive conditional heteroskedasticity): $dS = Sbdt + \left(c_1 + c_2 S^2\right)^{\frac{1}{2}} dZ$. It is not difficult to construct a theory linking any of the three parameters in the above formula with the exchange rate equilibrium.

Whichever model will be eventually chosen, it is desirable to have in mind the whole path of the currency price when talking about equilibrium, while the currently observed value is just one point on that path. The exchange rate trajectory can, but does not have to, converge to a single point, which the "true believers" would like to associate with the fundamental exchange rate value. Just as imaginable are trajectories with a continuum of attractor values. The most important special case of such non-unique limit behavior is the "*multiple equilibria*" characterization of the market. The models dealing with multiple equilibria explain the observed irregularities in exchange rate behavior by sudden changes in the parameter value in the space of possible equilibria. Individual points in the latter space serve as focal points around which the FOREX market participants coordinate their behavior.

Any economist not satisfied with the preceding answers can try to establish the existence and the level of one of the following variables.

a) The exchange rate simultaneously guaranteeing the external and internal balance of the economy, other fundamentals unchanged
b) The exchange rate as the arbitrage-free price that clears a frictionless, efficient and complete FOREX market
c) The exchange rate that the general opinion in the market associates with the expected future equilibrium

Understanding of the equilibrium exchange rate, based on (a), corresponds to the condition of international macroeconomics in the early 1960s. At that time, the Neo-Keynesian model formulated by Mundell and Fleming, also known as the IS-LM-BP scheme, worked with a reduced form goods market equilibrium condition (the IS-curve) and the balance of payments condition (the BP-curve) under the assumption of their stability. Unfortunately, empirical evidence on the phenomenon of consumption smoothing, as well as other violations of the simultaneous income-consumption link, soon disclosed the faultiness of the IS-relation. At the same time, advances in international trade and expectation-formation theories undermined the confidence in the BP-relation. Accordingly, it was recognized that the explanatory power of reduced-form macro-econometrics of Fleming and Mundell has rather narrow limits.

Logically, any econometric model chosen to calculate the equilibrium exchange rate must be derived from some structural model. Economists often even invent underlying decision-theoretic foundations (*micro foundations*) additionally, long after the corresponding reduced form has gained recognition on its own. Good examples of this are the model of money demand under hyperinflation, Cagan, 1956, or the monetary model of exchange rates, Frenkel and Mussa, 1976 (see Obstfeld and Rogoff, 1983, for the micro foundation of the latter). Micro foundations become necessary in the cases when an intuitively appealing *ad hoc* model fails to verify empirically. Intuition can then be extended and fine-tuned to the point when the empirical evidence is no longer in disaccord with the theory. This paper provides an example of such a “micro-correction” of an intuitive no-arbitrage argument leading to the contradiction known as *Siegel’s paradox*.

At a first glance, arbitrage arguments and the rational expectation theory offer a way to derive the exchange rate value according to (b). However, the corresponding theories turn out to be too numerous, and too versatile. Besides, most of them have not attained the level of technical tractability to allow for an unambiguous numeric rule of the equilibrium exchange rate calculation. On the other hand, one of the best known analytically tractable examples in this category, the uncovered interest rate parity (UIP), illustrates the perils of intuitive no-arbitrage reasoning involving uncertainty and expectations, not supported by sufficient choice-theoretical foundations. As mentioned above, Siegel’s paradox (Siegel, 1972), discloses the inability of UIP to serve as an estimate for the future spot exchange rate.

This paradox has two sides, one empirical and the other theoretical. Empirically, evidence on the forward premia on practically all internationally traded currencies shows that the forward exchange rate is not an unbiased estimate of the future spot exchange rate. The UIP hypothesis was rejected in the studies covering the 1970s (Meese and Rogoff, 1983), the 1980s (Froot and Thaler, 1990), and the first half of the 1990s (Engel, 1996). Since, on the other hand, the covered interest rate parity (the equality of the international interest rate differentials and the corresponding forward exchange rates) is always satisfied for trivial deterministic arbitrage reasons, one arrives at the systematic failure of interest rate differentials to predict the spot rate correctly. In fact, as McCallum, 1994, points out, the sign of the coefficient in the regression of the *ex post* exchange rate change on the interest rate differential is *negative* more often than not![1] Thus, the textbook uncovered parity, a building block of almost every international macroeconomic model since Fleming and Mundell, lacks empirical grounds. To get around this problem, applied econometrics has to work with an additional variable representing the described disparity, generally called *forward premium*. Reported evidence on its statistical properties is highly controversial.

Theoretical explanations of forward premia are versatile. Nevertheless, an important contribution of Siegel's paradox is its demonstration of a flaw in the naive no-arbitrage theory that leads to the uncovered interest rate parity condition. By his observation, Siegel showed that the ability of the interest rate differential to predict the exchange rate was lacking sound theoretical grounding. Technical coverage of Siegel's paradox is given in Part 2 of the paper.

Thanks to the mentioned theoretical difficulties, one is no longer surprised to observe persistent non-zero interest rate differentials between currencies all over the world. The Czech koruna is no exception in this respect, as is clear from Fig. 1. Consequently, international equalization of interest rates in the UIP sense of elementary textbooks shall not be expected to hold. A deeper look into the problem suggests that uncovered parity is not a universal rule but rather, a feature of a particular pair of financial instruments and the market segment where they are being traded under no-arbitrage conditions. This specialization of notions is a necessary step towards theoretically and empirically reliable analysis.

The preceding discussion was meant to provide an intuitive argument for the choice of alternative (c) above. The following circumstances suggest that this alternative is a feasible one.

1. It is possible to find a micro-theoretically founded statement regarding the conditionally expected difference of *total returns* on any pair of comparable assets, paid out in two different currencies. This *generalized uncovered return parity* (GURP) condition is free of the flaw described by Siegel's paradox. This

[1] McCallum, 1994, provides a simple linear model involving monetary policy expectations to explain this outcome.

indirect verification allows one to hope that GURP will also fare better empirically than the textbook UIP.

2. The chosen technique – optimal consumption and portfolio selection under uncertainty – is common for the majority of reduced form macromodels.
3. The existing variety of applied results found in state-of-the-art international macroeconomics can be reduced to just two fundamental theorems. One of them appears only in models with uncertainty and deals with the existence of self-fulfilling expectations and multiple equilibria. The other, present both in the stochastic and deterministic variants of the theory, is the parity of total returns on investment between all existing assets (in its abstract form, the said parity refers to marginal indirect utilities associated with the corresponding asset holdings). One of its best-known “incarnations” for closed economies is the Consumption-based Capital Asset Pricing Model (CCAPM) of Stephen Ross, 1976. Hence, it is logical to carry out an empirical verification of the possibly closest testable approximation of the theoretically relevant total return parity.
4. When the compared instruments in the generalized parity formula are denominated in different currencies, the expected rate of change in the exchange rate naturally enters the analysis. It is important to remember that the said log-exchange rate difference term in the parity formulae is *the only characterization of the equilibrium exchange rate following from the theory*. All other formulations appearing in international macroeconomic literature are but special cases or restatements of the above.
5. The stochastic general equilibrium models giving rise to GURP explain the price formation in all asset markets by the available information on the economy’s fundamentals, such as output, inflation, current account, etc. Due to this fact, it is no longer necessary for the policymaker to analyze the relationship of the exchange rate with fundamentals directly***. This analysis is being continuously performed by the market participants themselves***. Dynamic uncertainty resolution in the fundamentals is the source of dynamics of all asset prices, including the exchange rate.

The above arguments suggest that GURP will be a more efficient analytic tool for the exchange rate dynamic analysis and predictions than any macro-model based on fundamentals, particularly at times of big events in the FOREX market.

The paper is structured as follows. In Part 1, I proceed with the key statement of the generalized uncovered return parity. A number of simplifying conditions, allowing a check on data, are listed. Section 3 explains the choice of the market and financial instruments and data suitable for GURP-testing. Section 4 outlines the intuition behind the exchange rate predictions based on GURP and reports on the confrontation of GURP with Czech data on the exchange rate and asset returns, in relation to the currencies belonging to the European Monetary Union (EMU). Section 5 offers a comparison with the EMU-currencies in relation to the US dollar. This is followed by Part 2, a technical part dedicated to the stochastic portfolio optimization model underlying GURP. Section 2 briefly states the model, the individual optimization problem and the consequences of its solution for the equilibrium asset prices. Section 3 draws the main consequences of the model for the equilibrium exchange rate path, and establishes a relation to a number of traditional results of international finance, regarding asset pricing and exchange rate expectations. Section 4 presents a variation of the basic model solving the problem of a productive firm, and obtains a similar GURP statement in that case. Section 5 points at some possible extensions of the model. The conclusion of both parts of the paper sums up the results.

1.2 Testable implications of the asset return parity for the exchange rate

Researchers, who tried to overcome the missing support for uncovered interest rate parity, have observed that it is less markedly violated (even if not strictly satisfied in a statistical sense) in two groups of cases. The first refers to the economies with interventionist governments, both in the Third World and some European countries. There, rejection of UIP was less pronounced at times when the exchange rate was unsuccessfully protected from depreciation. Evidently, during such periods the currency speculators were being faced with much less uncertainty than usual. Therefore, the UIP came close to the generally valid covered parity. The second has to do with the time horizons for which UIP was tested.

Empirical investigations show that the worst results regarding UIP are generated by short money market rates. Well-known is the model by McCallum, 1994, which explains the “wrong” sign in the corresponding regression by the role of these rates in the monetary policy reaction function. In general, distinction between short-term and long-term maturities plays the key role in most models that were able to improve on the original UIP-failure. The longer the horizon of the instruments used in the tests, the closer the *ex post* exchange rate movement was to the one predicted by the interest differential. However, with long maturities, one is always faced with the non-trivial choice of the instrument and the rate to be used. In this respect, deposit rates perform worst. On the other hand, no alternative long-horizon instruments have zero-coupon property. One must replace the standard interest rate by the return rate of the individual instrument. Therefore, attempts are known to construct artificial measures of both return and time horizon in the UIP test, e.g. by using the duration measure in place of the maturity date (Alexius, 1998). Another possibility is to use the rates of return on a multi-annual horizon, implied by the synthesized yield curve (Meredith and Chinn, 1998). In all these cases, estimation results give the “right” signs and sometimes, do not reject the uncovered parity as such. Clearly, the said empirical transition from *interest rates* to the *rates of return* leads to the introduction of the same generalized uncovered return parity condition as was evoked on more theoretical grounds in the previous subsection. Full-fledged decision theoretic exposition of GURP is relegated to Part 2 of the paper.[2]

The basic equation of GURP assumes the existence of a representative optimizing agent – an international consumer and investor – who acts in continuous time under diffusion uncertainty. The corresponding formal model is defined in Derviz, 1997b. This model survives the first indirect test of consistency in that it does not suffer from the asymmetry described by Siegel’s paradox (see the proof in Part 2).

[2] The majority of traditional, open economy macroeconomic models, including the Mundell-Fleming one, tacitly leans upon the parity between the return differential and the exchange rate change, even if it is formally called interest rate parity. Indeed, as soon as one aggregates the domestic investment possibilities to one composite asset and the foreign – to another, the two parameters which carry the name of interest rate are, in fact, the total return rates on the resulting perpetuities. They shall not be, therefore, confused with the money market rates, whose highly specific role is usually played down in the textbook analysis.

The choice to work in continuous time offers a number of technical advantages over the models in discrete time (like the language of instantaneous conditional covariances and Itô's lemma). Qualitatively, however, the results are the same as in most stochastic open economy monetary models known from the literature. The differential equation system for the dynamic equilibrium in financial markets can be taken as a diffusion approximation of an analogous set of equations in discrete time.

Let *S* be the nominal exchange rate (price of one unit of foreign currency in domestic currency units). Consider one domestic asset X and another foreign asset X*. Their prices at time *t* are denoted by X_t and X^*_t, respectively. Also, let their dividend/coupon rates between times *t-1* and *t* be denoted by δ_t and δ^*_t. Volumes of new emissions of X and X* are denoted by ϕ_t and ϕ^*_t (if the emitters buy themselves up, then these rates are negative). Finally, z_t and z^*_t are the parameters measuring liquidity of the capital market segments where the corresponding assets are traded. The "hat" symbol for any variable *y* denotes its relative change rate: $\widehat{y}_t = \frac{y_t - y_{t-1}}{y_{t-1}}$.

Then, the model treated in Part 2 leads, after time parameter discretization and linearization, to the following discrete-time version of the generalized total return parity between X and X*:

$$\frac{\delta_t}{X_{t-1}} + \widehat{X}_t = \frac{\delta^*{}_t}{X^*{}_{t-1}} + \widehat{X}^*{}_t + \widehat{S}_t + a_0 + a_1\frac{\phi_t}{z_t} + a_2\frac{\phi^*{}_t}{z^*{}_t} + a_3\widehat{z}_t + a_4\widehat{z}^*{}_t + a_5\frac{z_t}{z^*{}_t} + \varepsilon_t\,. \tag{1}$$

Here, ε is a random error term.

If for no other reason, then at least thanks to its richer formal structure alone, the GURP expressed by equation (1) has a higher chance to fare well empirically than the standard UIP. Indeed, for each asset, it utilizes at least two parameters (the coupon rate and the secondary market price) instead of only one (the interest rate) in UIP. The coupon rate here is a formal analogue of the interest rate used in UIP. This parameter is always country-, market- and instrument-specific. It should be now clear that UIP is no more likely to hold in reality than a parity link between coupon or dividend rates in different countries. At this point, the applicability frontier of the textbook theory is attained. (The latter operates with the notion of capital mobility, requiring proportionality between the interest differential and the capital flow.) On the contrary, the second available parameter in (1), i.e. the price, serves to restore a dynamic equilibrium. A too high coupon rate in one country can be compensated by a higher secondary price, preventing a counter-intuitive difference in total returns. In addition, equation (1) allows for a difference in total returns for the reason of a non-zero country risk premium, and offers its breakdown. Therefore, an observed violation of the return parity does not mean inapplicability of the model as a whole, but simply calls for adjustment in the generality degree of the premium term in the equation.

In many cases, equation (1) can be simplified, provided one or several of the conditions named below are satisfied.

I. Markets for the involved securities clear at any given moment without the need of new emissions or withdrawals ($\phi_t = \phi^*_t = 0$ for all t).

II. The capital market segment where the domestic asset is traded is highly liquid on a permanent basis ($z_t = +\infty$ for all t).

III. The market segment where the foreign asset is traded is highly liquid on a permanent basis ($z^*_t = +\infty$ for all t).

IV. The total returns y and y^* of instruments X and X* are known (i.e. in the form of quoted yields to maturity). In that case, it is not necessary to account for capital gains and coupons/dividends separately.

In connection with condition IV, note the formal distinction between the current, or instantaneous yield used in (1), and the yield to maturity. This distinction vanishes completely in continuous time setup, to be used in Part 2. Accordingly, it is negligible in all examples with high frequency (i.e. daily) data, and remains small for weekly and even monthly time steps. Therefore, in the present paper I use the two yield notions interchangeably.

If conditions I-IV hold simultaneously, the GURP equation reduces to a simple and intuitively appealing form very closely resembling the textbook UIP:

$$y_t = a_0 + y^*_t + \widehat{S}_t + \varepsilon_t \,. \tag{2}$$

Equation (2), however, does not suffer from the asymmetry flaw described by Siegel's paradox. (See Subsection 2.3.2 of Part 2 for the proof.)

Equations (1) and (2) shall be understood in the following way. Fundamental information on the currency value is contained in the price/yield of the domestic asset. The same variable also reflects the current demand for domestic goods and other assets. The opportunity cost for the international investor, of investing in domestic products, is given by the foreign asset price/yield. Volatility of domestic fundamentals and their correlation with the relevant foreign fundamentals is contained in variable a_0, which is constant in the simplest cases.

The origins of (1) and (2) lie in the differential Euler equations of the underlying full model. For this reason, they do not rule out multiple equilibria in asset prices, including the exchange rate. Nevertheless, both these equations supply valuable information about the properties of the equilibrium exchange rate. First of all, if the yield differential is non-stationary, this signals an advent of the asset market turbulence. The reason may be either a reduction in liquidity, or an external intervention affecting the volumes of traded instruments, or a sharp change in the statistical properties of fundamental variables. In this respect, GURP provides important indirect evidence of the impending unrest. This evidence has an advantage over other known methods of expectation-extraction from financial instruments (Söderlind and Svensson, 1996, Campa and Chang, 1996) in that it does not require the use of "risk-neutral" probabilities. At times preceding pressures on the exchange rate correction, GURP is able to indicate its direction and extent.

Application of GURP renders the value of an "equilibrium appreciation or depreciation trend" $\widehat{S}^e$, not the future exchange rate level *S* (recall the discussion in the introduction; in terms used there, $\widehat{S}^e$ corresponds to the drift parameter μ of a log-normal or another similar exchange rate diffusion process). Practical aspects of this equilibrium exchange rate evaluation are discussed next. For that purpose, one must choose the instruments and the capital market segment. I illustrate the choice for the case of the Czech koruna and an arbitrary EMU-currency.

1.3 Financial market data and the market views of the future exchange rate equilibrium

For a reliable estimate of the exchange rate trend in the GURP equation, it is desirable to use a pair of financial instruments with the longest possible maturity. The reason is that there is a distortion of the asset pricing formulae in models with a finite horizon. (The model in Part 2 of this paper deals with infinite horizon problems. Formal simplifications that allowed us to derive (1) or (2) also heavily rely on the absence of finite horizon effects.) In certain cases, the presence of a finite horizon, which divides the problem into the periods before and after a major event, is convenient. (See, e.g. Derviz, 1999, where such a time-axis split is used to obtain an American currency option pricing formula.) However, the shadow prices of assets in a finite T-horizon model may be decisively dependent on the expectations about the development at times after T. For these effects not to matter, T must be far enough from the present moment; then, the after T-effects are sufficiently discounted. This is another explanation of the poor predictive power of UIP based on short interest rates.

From the said point of view, the best instruments for the application of GURP would be perpetuities, i.e. common stocks of those Czech companies who also double float them internationally as GDR. Unfortunately, the stock market in the Czech case cannot offer either high liquidity or reliable price information to comply with the conditions under which GURP can be tested.

Another group of securities with sufficiently long maturities is formed by corporate bonds issued by the major Czech companies, provided they also issue bonds of the same maturity abroad (examples are ČEZ, KB, IPB, Škoda Plzeň). The empirical analysis of these instruments, however, is hampered by the presence of non-stationary individual risk premium of the issuer. Secondary market price distortions in these cases also originate from the complicated ownership in-breeds within industrial conglomerates (see Derviz, 1997c).

One more internationally integrated segment of the Czech capital market, namely, the market for koruna Eurobonds, would seem to provide a somewhat better chance of testable yield values. Underwritten by renowned international investment banks, these Eurobonds do not suffer from the individual risk premium distortion and allow an easy comparison with other similar bonds in other currencies. Unfortunately, the secondary market for koruna Eurobonds is rather shallow. Therefore, every new emission is able to shift an earlier attained equilibrium, so that condition I of the previous subsection is violated. In this sense, Eurobonds are not very suitable for applications of GURP to the exchange rate analysis. Nevertheless, these bonds remain an important object of future research regarding the views of foreign investors on the Czech currency.

We can conclude that, at the moment, the best candidate for the role of domestic asset X in the GURP equation is one of the long-horizon government bonds traded on international markets. These bonds comply with the requirement of small individual risk premium. Moreover, the liquidity of the corresponding market segment is highest among all instruments with similar maturity and coupon regime, bringing them close to the ideal state described by condition II.

There is more evidence – this time, historical evidence – supporting the suggested choice of X. The time series of yields on the Czech 5-year government bonds show the desirable dampened response on sharp short-term exchange rate movements. Specifically, the magnitude of the upward tilt that these yields experienced during the currency crisis of May 1997 was comparable with many other movements registered in later periods after the introduction of free floating of the Czech currency. In other words, right in the midst of the deepest turbulence in the FOREX market, the 5Y yield was sending a reassuring signal on the transient nature of this event, predicting the correct – and relatively modest – extent of the eventually realized depreciation. On the contrary, the yields on short-term instruments rocketed to prohibitive heights of several tens of percent during the same episode, thereby excluding any reasonable quantitative interpretation.

The choice of foreign instrument, X*, is pinned down by the currency which serves as the predominant “entry-exit” one for traders in Czech financial products, i.e. the German mark (the euro after 1 January 1999). We pick the German government 5Y bond as the asset most often traded by the relevant group of investors. The Czech and German 5Y government bond yields are featured in Fig. 2.

In recent months, an alternative pair of instruments with 10-year maturity has emerged which is a promising substitute for the discussed 5Y-pair. Now, on the "EU-side", asset X* is the index of 10Y government bonds of Germany and other Eurozone countries, i.e. the standard benchmark in the euro market of 10-year bonds. On the Czech side, the European Investment Bank has issued a 10-year bond in Czech korunas, for which price and yield quotes are available starting 26 February 1999. Although there is very little experience with this instrument at this time, and the degree of liquidity of the corresponding market segment is modest, the first confrontation of the 10Y yield differential implied by it, with the nominal CZK/EUR exchange rate, looks very promising (see Fig. 3a).[3]
A similar exercise can be made for 10Y US-government bond yields in the role of X*. The result (Fig. 3b) shows that the parity relationship of CZK with USD is much less direct than the one with the euro.

1.4 GURP application to the Czech koruna exchange rate

The first exercise assumes the time-step to be one week. In this way, one obtains a sufficiently long time series but, at the same time, reduces some of the extremely high volatility of the rate change $\widehat{S}$.

One observes immediately that the Czech yields are much more volatile than the German ones. Therefore, all visible changes in the yield differential can be attributed to the movements in the Czech government bond returns. By comparing them with the nominal CZK/DEM exchange rate (this will be done next), one easily recognizes co-movements in both variables. Even without formally calculating the correlation,

[3] There is one more category of Czech financial instruments with maturity of ten years and a secondary market which is more liquid than the EIB-bond segment, namely the interest rate swaps. Unfortunately, the information content of the quoted swap rates is different from what is needed for GURP testing. Standard theories claim that these rates are weighted sums of expected future values of short interest rates. Therefore, extraction of a yield measure from these quotes cannot go through directly without loss of information. That is why the present

one is able to register a rough correspondence in the *direction* of change of the yield differential and the exchange rate. This fact alone is a sizeable improvement compared to the short rate differentials, where no co-movements could be found (recall Section 2 and Fig. 1).

For the Czech currency, close correlation of *S* and the yield differential predicted by GURP holds even for daily data. The period from 1 July 1997 to 2 July 1999 is illustrated in Fig. 4. As this diagram shows, qualitative incidence of the short-term shifts in the Czech-German yield differential and the CZK/DEM nominal exchange rate is almost perfect. That is, ups and downs of the former are almost always accompanied by the tilts of the latter in the same direction, even if the magnitude of these movements can be different, so that the distance between the two lines varies

study would find it difficult to use swaps instead of bonds.

over time.[4] Especially significant are co-movements during the second half of 1997, i.e. the half-year immediately following the currency crisis. The Czech money market of that period was characterized by a long and difficult recovery of the short interest rates from the interventions and liquidity squeeze of May and June the same year. Parallel to this, yields of long maturity bonds already signaled the return to normality and even predicted the exchange rate movements correctly.

Going over from informal evaluation of the diagrams to quantitative estimates is less straightforward. As was mentioned before, the volatility of process $\widehat{S}$ is very high. Even for the time-step of one week, it oscillates between positive and negative values of 30–40% (annualized). Therefore, if one wants to work with the $\widehat{S}$-measure for the period of several months, it is not obvious which statistic to use.

The statistic chosen in this paper to deal with the CZK/DEM exchange rate is the following. The average appreciation/depreciation rate of the currency for a given period is identified with the average of relative change (i.e. $(S_t-S_{t-1})/S_{t-1}$ for every t) of nominal exchange rate value for that period. Fig. 5a and b show the comparison of this measure of $\widehat{S}$ with the yield differential, daily data. As one sees there, 1M changes in S bear a very weak relation to the yield differential. One may be tempted to say the same about 1Y changes (Fig. 5b) as well. However, it is important to notice the two sub-periods when the average annual movement in the exchange rate and the yield differential behaved roughly in line with one another. This happened, in particular, during the second half of 1997 (we have already observed this property in Fig. 4, where the co-movement is visible in even shorter time intervals), and, later, in the second quarter of 1998. Fig. 5b suggests that, during the first quarter 1998, a non-stationary process had been involved in a dynamic revision of the country risk premium, which eventually settled down on a lower level than before.
Intermediate time horizons for the averaged ER-change estimations show certain movements of the exchange rate autonomous from the yield differential and having

[4] Recall the very same effect observed on a shorter time series of 10Y bonds and shown in Fig. 3.

the form of smooth waves. The only trace of GURP is the ER-tendency to revert in the direction indicated by the yield values, in the mean. The time lag of this reversion is rather long (between 5 and 10 months).

Inside the period when GURP is described by the simplified equation (2) with a constant disparity, a simple procedure can be offered for a quick and rough ER-prediction: Let *YD* be the yield differential form equation (2), i.e. $YD=y\text{-}y^*$. One must choose a period when the country risk premium a_0 may be legitimately assumed constant. Equation (2) is then applied twice. Data from the first sub-period are used to obtain an estimate of a_0. Then, data from the second sub-period and the estimated a_0-value are used to assess the equilibrium expected depreciation or appreciation, $\widehat{S}^e$, for the remaining part of the sub-period with the assumed constant value of country risk. The latter is understood as the difference between the value implied by GURP and the actual value $\widehat{S}^1$ of $\widehat{S}$ (so far observed):

$\widehat{S}^e = YD^1 - YD^0 - \widehat{S}^1 - \widehat{S}^0$.

For example, average values for the second half of 1998 were: YD^1 =0.173, YD^0 =0.113 in the two sub-periods, $\widehat{S}^1$=-0.013, $\widehat{S}^0$=0.027. Accordingly, the expected weekly depreciation is 0.1%, i.e. 5.2% annual. Recall that the chosen notion of equilibrium works with the *slope of the ER-curve* as the parameter around which the market participants coordinate their expectations. In other words, the expectations are not formed on exchange rate *level*, which is a random variable with no privileged individual value.

Evidently, the above calculations do not claim much econometric sophistication. They are used barely as an example of a rule-of-thumb prediction in the absence of a better theory. The latter should be able to explain non-stationarity of the risk premia, so that it could deal with periods like the one of the Czech koruna in the first quarter of 1998. However, such a theory would require an even more radical departure from the standard stochastic GE-framework than the one undertaken in Part 2. One would need a model that would be able to capture the effects of future capital flows on the present asset demands. Its construction is a matter of current research.

1.5 Comparison with the EMU currencies

It can be useful to compare the attained results on the GURP property of the CZK-DEM exchange rate with evidence on another pair of currencies for which long-term government bond yield data are available. For this purpose, I take the Euroland currencies and their relation to the US dollar. A number of European central banks had effectively pegged their currencies to the German mark for many years preceding the introduction of the euro (Austria, the Netherlands). Others did so in the final phase of their EMU-convergence effort in 1997/98 (e.g. France, Italy, or Ireland). The same convergence process has led to very close values of the benchmark (10Y) government bond yields in the EMU countries in recent years. As a result, it suffices to analyze GURP for one country and currency to be able to make conclusions about its validity in the whole Euroland, provided conditions I–IV of Subsection 2 are satisfied. In the present paper, I have used the data on the Austrian schilling (ATS) and German mark, compared with the 10Y government bond yields in Austria, Germany and the USA.

Fig. 6 shows the corresponding ten year yield differential and the *ex post* smoothed 3M change in the ATS/USD exchange rate, daily data between 1 Jan. 1994 and 30 Nov. 1998. Beside the average rate change statistic that was previously used for the CZK/DEM rate (here – Fig. 6a), I also experiment with an alternative one. In Fig. 6b, I compare daily values of the 10Y Austria-US yield differential with the slope of the log-exchange rate curve ATS/USD, for the three-month period starting on the current date. The immediate lesson to be learned from the two parts of the graph is that both exchange rate change measures give a qualitatively identical result, consisting in the rough compliance of the exchange rate with the GURP condition. However, the GURP property is manifested somewhat differently in the developed economies than in the transitional ones. Recall that the CZK exchange rate reacts on very short movements in the yield differential, while its 1–3 month smoothed changes hardly exhibit any reaction to it. At best, the smoothed ER-slope for the horizon of 4–6 months eventually reverts in the direction indicated by the differential. In contrast to this, daily ATS/USD movements are more or less unrelated to the moves in the benchmark yield differential (Fig. 6a). On the other hand, co-movements of the smoothed ATS exchange rate slope and the yield differential are visible in the 3–4 month horizons (Fig. 6b). A comprehensive theory explaining this phenomenon would exceed the scope of the present paper. The first intuitive hypothesis might be a difference in the typical holding times by international investors of government bonds of a developed country and a transitional country. Very short holding times for Czech securities may be the reason behind the elevated volatility of their secondary market yields.

Fig. 6 shows a period (from summer 1996 to summer 1997) when the smoothed ER-change curve significantly deviates from the yield differential curve. Although Austrian monetary policy has been for nearly twenty years closely tied to German monetary policy, Austrian and German benchmark bond yields are somewhat less synchronic than the external values of the two currencies. Therefore, one explanation of the increased ATS disparity is a structural change in the Austrian financial sector in the process of accommodating to the EU-requirements and preparations for Monetary Union. To find out whether this country-specific explanation is justified, one can take a look at the same time period, but another EMU-currency, namely, the German mark. The result is shown in Fig. 7. First of all, just like in the ATS case, it is impossible to trace any regularity in the absolute value of the nominal DEM/USD exchange rate and the 10Y yield difference between German and US government bonds (Fig. 7a). In Fig. 7b, we take the same log-exchange rate slope statistic as for the ATS/USD rate in Fig. 6b. Evidently, and contrary to the “Austria-specific” hypothesis expressed above, asynchronous movements of the German-US 10 yield difference and the 3M DEM/USD exchange rate slope are just as pronounced between July 1996 and July 1997 as in the case of the schilling. That is, one has good reason to attribute the temporary GURP-deviation of both the ATS and DEM value of the US-dollar to some EMS-wide factor.
With this comparison, I conclude the descriptive part of the paper and move on to the exposition of the model, proposed as an extended microfoundation of GURP.

2 The Shadow Asset Pricing Model and Generalized Total Return Parity

2.1 Motivation

This part of the paper introduces a dynamic stochastic model of investment, production and consumption in an open economy. I obtain equilibrium supplies and demands for goods and securities under the assumption of optimally behaving households, producers and foreign investors. The proposed model, among other things, covers the case of a transition economy with an ongoing privatization process, when new equity is continuously enlarging supply.

The traditional assumption of finance theory (Merton, 1971, Cox et al., 1985, and many others since then), positing zero net supply of bonds and a constant amount of equity, is inadequate in transition economies. Instead, the inflow of new securities must be regarded as one of the basic uncertainty factors of the economy, with consequences for the optimal portfolio choice, especially for the demand for money (see Derviz, 1997a, b, for details).

The model is solved by means of a stochastic maximum principle including the adjoint equations (Derviz, 1997b). The used continuous time stochastic calculus has an advantage over the discrete time approach, since the dynamics of asset prices can be calculated with the help of Itô's lemma. In particular, supply and demand of securities are derived in terms of their shadow prices, which are co-state processes of the optimization problems of the agents. The diffusion dynamics of the shadow price processes is derived in terms of utilities, production functions, asset returns and their growth/attrition rates. The obtained differential equations for the shadow prices lead to Itô equations for the equilibrium asset prices, including the exchange rate. The same shadow price method is able to clarify the impact of asset prices on consumption goods and export and import equilibrium prices – a link seldom made transparent in international trade models.

To model liquidity constraints, I work with domestic and foreign liquidity variables in the utility function. The derived equilibrium in the asset markets leads to testable relationships between the interest rates and the exchange rate. This seemingly involved method is very helpful under conditions of a transitional economy. That is, the Czech foreign liquidity preferences refute a number of popular macroeconomic stylized facts, often utilized in *ad hoc* empirical models. For example, there was a period – in mid-1995 – when the value of Czech imports exceeded foreign currency deposits of the private sector. In other words, a formal model with tight constraints would be unrealistic and should be replaced by another with soft constraints.

The proposed shadow price method has found a number of applications, e.g. pricing of American options on foreign currencies (Derviz, 1999), or the indirect measures of precautionary savings and other consumption function properties, exercised through the analysis of client interest rates (Derviz and Klacek, 1998). The present application is focused on the analysis of the expected future exchange rate path, based on comparison of financial instruments denominated in different currencies. Long-term exchange rate trends can be read off the yields of properly chosen long maturity instruments, one domestic and one foreign. These yields enter the generalized uncovered asset return parity (GURP) condition. The latter is closely related to the Consumption-based Capital Asset Pricing Model (CCAPM) with transaction costs. The technique used involves a variant of the CCAPM derived from a dynamic model of optimal consumption and investment in an open economy with diffusion uncertainty under soft liquidity constraints. If the studied market segment is perfectly internationally integrated, highly liquid and nearly frictionless, then GURP becomes very close to traditional uncovered parity statements of international macroeconomics.

A slight modification of the same model deals with optimizing by a productive firm. I derive the equilibrium prices for its domestically and internationally issued liabilities and analyze the parity of returns between them. Construction of empirical testing procedures in this part would be difficult for any transitional economy. Particularly, Czech firms have only paid out dividends exceptionally in the years following their privatization, and the perspective of future dividends is even more uncertain for most of them. Beside that, tax evasion behavior results in successful profit hiding, which often makes it impossible to derive correct earnings figures from officially available data. Asset markets, on the other hand, are pervaded by inside trading practices which, in turn, obscure price formation. There are many other distortions having to do with partial insulation and opaque ownership links between the financial and the real sectors. For this and other reasons of similar nature, the emerging Czech financial markets exhibit unstable individual risk premia. Therefore, one can only make qualitative preliminary estimates based on indirect price and profit measures.

In what follows, Section 2.2 defines the model, states the optimization problem of the representative agent, and outlines its solution and the resulting equilibrium dynamics of the exchange rate. Section 2.3 analyzes an internationally integrated financial market segment with negligible transaction costs and high liquidity, and explains why the suggested GURP condition is free of Siegel's paradox. Section 2.4 contains the optimizing producer and securities issuer model. Section 2.5 discusses the perspectives of the shadow price technique and the generalized uncovered asset return parity method for the analysis of other segments of the international capital market. I claim that the corresponding equity return disparity contains valuable information regarding the international competitiveness of certain domestic industries. By analyzing the consequences of exchange rate shocks for international asset valuation in the given industry, one would obtain an indirect measure of the production technology response to FOREX market movements.

2.2 The model and its solution

2.2.1 Agents, assets, markets and risk factors

The agents in the economy are households consuming one homogenous domestic consumption good and one homogenous foreign consumption good. I shall assume the existence of a representative household.

The economy offers four investment possibilities: cash and other domestic liquid assets M (real balances), international cash and liquid assets I, domestic financial securities K, numbered by $k=1, ..., K$, and foreign financial securities F, numbered by $f=1, ..., F$. I is understood as a synthetic international security aggregating those currencies which are held by the agents in the accounts at domestic banks. Therefore, when talking about the nominal exchange rate S, I have in mind the price in nominal domestic terms of this currency basket. Accordingly, $Q=S/P$ (the "real exchange rate") will be the price of this very basket in terms of *real domestic balances* M, P being the current domestic price level.

The agents trade in the markets for the four named assets as well as C^d – domestic consumption goods, and C^i – imported goods. M can be traded against C^d, I and K, while foreign goods and securities must be first exchanged for M before they can be transmitted to investment or consumption at home.

There is a stochastic instantaneous rate of return $d\pi^0$ on M (meaning the nominal rate of return on the sight deposit component of M less the inflation rate), and the M-dividend rates $d\Gamma^k$ ($k=1, ..., K$) on K. Foreign liquidity I grows at the rate $d\pi^i$. Domestic security $k \in$ K has the internal deterioration rate $d\pi^k$. (One can think of a random default rate, stock dilution, etc. Naturally, there are securities, such as government bonds, for which this rate is zero.) International asset $f \in$ F has the I-dividend $d\gamma^f$

and the internal deterioration rate $d\pi^f$. Cumulative growth and dividend variables π^0, π^i, Γ^k, γ^f, π^k and π^f are Itô processes.

The global risk factors represented by the Itô processes named above, have the diffusion parts spanned by a d-dimensional vector Z of standard mutually independent Brownian motions, generating the filtration F=(F$_t$)$_{t\in\mathbf{R}^+}$ satisfying "the usual conditions" (Elliott, 1982), whose tth element is the time-t publicly available information. All the processes appearing in the sequel are Markov diffusions with basis Z. For stochastic differentials of these Itô processes, I will use the following notations. For any Itô process x, define its drift by μ^x and its diffusion by σ^x. Also, for strictly positive processes x, put $\widehat{dx} = \frac{dx}{x}$ and let the drift and diffusion of $\widehat{dx}$ be denoted by δ^x and ρ^x, so that

$$dx = \mu^x dt + \sigma^x dZ = x\widehat{dx} = x\left(\delta^x dt + \rho^x dZ\right).$$

The M-prices in K-markets are denoted by P^k, the I-prices in F-markets, by P^{*f}. The representative household is assumed to be a price (Q, P^k and P^{*f}) and asset return (π^0, π^i, Γ^k, γ^f, π^k and π^f) taker.

Let us define the agent's state variables as follows: x^0 – number of held M-units, x^i – number of held I-units, x^k – number of held shares of security $k \in$ K, x^f – number of held shares of security $f \in$ F. Symbols x^K and x^F denote the vectors formed by components x^k, $k=1, ..., K$, and x^f, $f=1, ..., F$, respectively.

The agent's control variables are: c^d – domestic consumption good purchases per period (paid out of M-balances), c^i – imported consumption good per period (paid out of I-balances), φ^i – foreign currency purchase (sale if negative) rate, φ^k – number of purchased/sold shares of *k* per period, φ^f – number of purchased/sold shares of f per period.

Denote by x^l the total amount of domestic and foreign liquidity held by the agent: $x^l=x^0+Qx^i$. The transaction costs in the asset markets are defined as follows. For $\varphi^i > 0$, let $\varphi^0=j(x^l,Q\varphi^i)$ be the amount subtracted from the M-account in exchange for φ^i

purchased units of I ($\varphi^0 > Q\varphi^i$). For $\varphi^i < 0$, $-\varphi^0 = -j(x^l, Q\varphi^i) > 0$ is the increment in the M-account in exchange for $-\varphi^i$ sold units of I ($-\varphi^0 < -Q\varphi^i$). In both cases, the difference $j(x^l, Q\varphi^i) - Q\varphi^i$, a smooth increasing convex function of the absolute transaction volume $|\varphi^i|$, constitutes the transaction fee paid to an intermediary. This fee depends smoothly and negatively on the agent's solvency x^l: the wealthier the agent is, the easier it is for him to operate in the FOREX market. In aggregate (e.g. when the existence of a representative agent is assumed), one can also interpret x^l in the above formula as a measure of depth, or liquidity, of the corresponding financial market segment. Thus, the deeper the market, the lower the transaction costs. However, zero transaction costs are impossible unless there is no trade at all: $j(x, \psi) = 0$ if and only if $\psi = 0$.

The leading example of a transaction function *j* satisfying the above properties is the quadratic cost function similar to the physical capital installation cost often exploited by "Tobin's *q*" models in macroeconomics:

$$\varphi = j(x, \psi) = \psi - \frac{b}{2x}\psi^2, \ \psi = h(x, \varphi) = \left[\left(\frac{x}{b}\right)^2 + \frac{2x}{b}\varphi\right]^{\frac{1}{2}} - \frac{x}{b}. \qquad (1)$$

Here, b is a positive constant. The formula may be applied in the range of values of ψ such that $2b\varphi^0 / x^l > -1$. This means that the purchase rate of domestic currency shall not be too high compared to its current holdings.

For future calculations, it is necessary to know the *marginal transaction function* $v = 1/j_\psi = h_\varphi$ as a function of x and $\varphi = j(x, \psi)$ (subscripts denote partial derivatives). As follows from (1),

$$v(x^l, \varphi^0) = j_\psi(x^l, j(x^l, Q\varphi^i))^{-1} = \left(1 + \frac{bQ\varphi^i}{x^l}\right)^{-1} = \left(1 + \frac{2b\varphi^0}{x^l}\right)^{-\frac{1}{2}}. \qquad (2)$$

Transaction functions j^k and j^f in the K- and F-markets are defined similarly. For simplicity, I assume that the sole measure of liquidity is the same cash holding value $x^l = x^0 + Qx^i$, i.e. the richer the agent is, the easier it is for him to trade in all asset markets[5]. Marginal transaction processes v^k and v^f are defined analogously to (2), and, therefore, have the same functional form. If $P^k \phi^k = P^k j^k\left(x^l, \varphi^k\right)$, $P^{*f} \phi^f = P^{*f} j^f\left(x^l, \varphi^f\right)$ are liquidity expenditures on the corresponding assets, then $v^k = v^k(x^l, \varphi^k)$, $v^f = v^f(x^l, \varphi^f)$.

Under the above notations, the laws of motion of the components of the state vector $x=[x^0, x^l, x^K, x^F]^T$ look like

$$dx^0 = x^0 d\pi^0 + x^K \cdot d\Gamma - c^d dt - j\left(x^l, Q\varphi^i\right)dt - \sum_k P^k j^k\left(x^l, \varphi^k\right)dt\ , \tag{3a}$$

$$dx^i = x^i d\pi^i + x^F \cdot d\gamma - c^i dt + \varphi^i dt - \sum_f P^{*f} j^f\left(x^l, \varphi^f\right)dt\ , \tag{3b}$$

$$dx^k = -x^k d\pi^k + \varphi^k dt\ ,\ k=1,...,K, \tag{3c}$$

$$dx^f = -x^f d\pi^f + \varphi^f dt\ ,\ f=1,...,F. \tag{3d}$$

The limitations as regards debt in either domestic or foreign liquidity will be expressed in the “soft” form through the presence of variables x^0 and x^i in the period utility function. Partial derivatives of the utility with respect to x^0 and x^i will be decreasing, and too small cash holdings – penalized by the utility diminishing quickly

[5] There is strong evidence against the existence of the currency substitution effects in the Czech economy, which is our main example, see Derviz and Klacek, 1998. It is even less probable that such effects are relevant in the developed economies with which the Czech economy is compared. Therefore, it is not necessary to distinguish between domestic and foreign cash holdings as the measure of the agents’ ability to transact.

to zero (mimicking the cash in advance constraint). In this way, I prohibit negative positions in M and I, and ensure diminishing marginal utility effects of increased liquidity holdings. Specifically, I denote by x^* and c^* the M-values of foreign cash and foreign consumption: $x^*=Qx^i$, $c^*=Qc^i$. Let us define the CES liquidity index

$$A(x^0, x^*) = \left[\beta_x^{\frac{1}{\theta}} (x^0)^{\frac{\theta-1}{\theta}} + (1-\beta_x)^{\frac{1}{\theta}} (x^*)^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}}$$

and the CES consumption index

$$C(c^d, c^*) = \left[\beta_c^{\frac{1}{\eta}} (c^d)^{\frac{\eta-1}{\eta}} + (1-\beta_c)^{\frac{1}{\eta}} (c^*)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}},$$

where $\theta>0$ and $\eta>0$ are constant substitution elasticities. The period utility is posited to be of the form

$$u\left(x^0, x^i; c^d, c^i\right) = \frac{1}{1-\frac{1}{\rho}} \left[A(x^0, Qx^i)^{\alpha} C(c^d, Qc^i)^{1-\alpha} \right]^{1-\frac{1}{\rho}}$$

for some $\alpha \in (0,1)$, where $\rho>0$ is the intertemporal substitution elasticity.

As is clear from this definition, the liquidity index A and the consumption index C are imperfect complements in the Cobb-Douglas sense, while the domestic and foreign liquidity/consumption are imperfect substitutes in the CES sense.

2.2.2 The individual optimization problem

The representative agent maximizes

$$E\left[\int_0^{\infty} e^{-\vartheta t} u(x_t^0, x_t^i; c_t^d, c_t^i) dt \right], \tag{4}$$

with respect to the decision path $t \mapsto {}_t = \left[c_t^d, c_t^i, \varphi_t^i, \varphi^K{}_t, \varphi^F{}_t \right]^T$, subject to the state-transition equation (3), the initial asset values given.

The solution to a class of consumption/investment optimization problems containing (4) can be characterized in terms of the value function by dynamic programming methods (Fleming and Soner, 1993). Beside that, traditional consumption and portfolio optimization problems (Merton, 1991) can be usually analyzed by means of the “martingale” technique (Duffie, 1992). The problem discussed here contains the state process in the utility function and, therefore, cannot be treated with the latter method. Instead, I use the technique based on the adjoint Itô equation (an early example is in Bismut, 1976). The present version of the stochastic maximum principle has been developed in (Derviz, 1997). It synthesizes a number of earlier formulations found in stochastic control literature (Hausmann, 1981, Elliott, 1982, Peng, 1990, Cadenillas and Karatzas, 1995).

Allowing for variable time preference, I assume that for any $s>t$, the agents discount their time s-utilities at time *t* by means of the same locally riskless discount rate ϑ with $[\vartheta]_t^s = \int_t^s \vartheta_\tau d\tau$. Then the general form of any agent’s optimization problem can be symbolically written down as

$$\max_{\ell} E\left[\int_0^T e^{-[\vartheta]_0^t} U(X_t;\ell_t)dt + e^{-[\vartheta]_0^T} B(T, X_T)\right], \tag{5}$$

with respect to controls ℓ, subject to the state-transition equation

$$dX = \mu(X,\ )dt + \sigma(X,\ )dZ\,, \tag{6}$$

the value X_0 of the state process at time $t=0$ given. B is the so-called final bequest function. It must possess a limit if the time horizon of the optimization problem is infinite ($T=\infty$).

Problem (5), (6) can be solved by forming the current value Hamiltonian

$$\mathrm{H}(t, X,\ , \xi, \Xi) = U(X,\ ) + \xi\cdot\mu(X,\ ) - tr(\Xi\cdot\sigma(X,\ )),$$

to be maximized with respect to $_t$. Here, ξ and Ξ are the so-called *first-* and *second-order adjoint processes* (ξ is of the same dimension n as X and Ξ is an nxd-matrix), with $\Xi = \xi\cdot D_x\sigma$ along the optimal path. When state X stands for asset holdings, the adjoint process ξ can be called the *shadow price* vector of the corresponding group of assets.

Let $[f,g]$ define the predictable co-variation of diffusion processes f and g, and put $d[f,g]=\langle f,g\rangle dt$ (with the standard shorthand $\langle f\rangle$ for $\langle f,f\rangle$). Then the (first-order) adjoint process ξ satisfies the stochastic differential equation

$$d\xi=\xi\cdot\left(\vartheta\mathbf{1_n}\,dt-dA+\langle A\rangle dt\right)-D_X U dt\,, \tag{7}$$

with the nxn-matrix valued process A defined by

$dA=D_X\mu dt+D_X\sigma dZ.$

The final condition $\xi_T=D_XB(T,X_T)$, or an appropriate transversality condition if $T=\infty$, must be added to (7). The adjoint process ξ can be also described as the X-gradient of the value function of problem (5), (6), provided it exists.

For problem (3), (4), the adjoint process ξ is a vector of four shadow prices, ξ_0, ξ_i, ξ_K and ξ_F of asset categories M, I, K and F (the last two are vectors). The transversality condition in this case is $\lim\limits_{T\to\infty} e^{-\vartheta T}\left[\xi_0(T),\xi_i(T),\xi_K(T),\xi_F(T)\right]=\left[0,0,\mathbf{0}_K,\mathbf{0}_F\right]$.

Let $d\pi^0=\mu^0dt+\sigma^0dZ$, $d\pi^i=\mu^idt+\sigma^idZ$, $d\pi^k=\mu^kdt+\sigma^kdZ$, $d\pi^f=\mu^fdt+\sigma^fdZ$, $d\Gamma^k=\Delta^kdt+\varepsilon^kdZ$, $k=1, ..., K$, $d\gamma^f=\delta^fdt+\epsilon^fdZ$, $f=1, ..., F$. Interpreting (7) for problem (3), (4), one derives the following laws of motion of the shadow prices:

$$d\xi_0=\xi_0\left(\vartheta dt-d\pi^0+\left|\sigma^0\right|^2dt+j_x(x^l,Q\varphi^i)dt+\sum_k P^k j_x^k(x^l,\varphi^k)dt\right)+\xi_i\sum_f P^{*f}j_x^f dt-u_{x^0}dt\,, \tag{8a}$$

$$d\xi_i=\xi_0 Q\left(j_x+\sum_k P^k j_x^k\right)dt+\xi_i\left(\vartheta dt-d\pi^i+\left|\sigma^i\right|^2dt+Qj_x(x^l,Q\varphi^i)dt+Q\sum_f P^{*f}j_x^f dt\right)-u_{x^i}dt \tag{8b}$$

$$d\xi_k=\xi_0\left((\sigma^0-\sigma^k)\cdot(\varepsilon^k)^T dt-d\Gamma^k\right)+\xi_k\left(\vartheta dt+d\pi^k+\left|\sigma^k\right|^2dt\right),\ k=1,\ ...,\ K, \tag{8c}$$

$$d\xi_f = \xi_i\left((\sigma^i - \sigma^f)\cdot(\in^f)^T dt - d\gamma^f\right) + \xi_f\left(\vartheta dt + d\pi^f + \left|\sigma^f\right|^2 dt\right), f=1, ..., F. \tag{8d}$$

These Euler equations apply to both domestic and foreign agents and are symmetric with respect to the country of residence, except for the national liquidity preferences u_{x^0}, u_{x^i} that need an obvious formal correction. This symmetry property will be important in the context of Siegel's paradox, discussed in Subsection 2.3.2.

2.2.3 First order conditions and equilibrium asset prices

Let us take the FOREX, i.e. M/I-market, first. Optimal transactions here are described by the current value Hamiltonian optimization with respect to ϕ^j. The first order condition is $\xi_i - \xi_0 Q j_\psi\,(x^l, Q\phi^j) = 0$. This leads to the following expression for the real exchange rate Q as the optimal reservation price in the FOREX market:

$$Q = \frac{\xi_i}{\xi_0} j_\psi(x^l, Q\varphi^i) = \frac{\xi_i v(x^l, \varphi^0)}{\xi_0}. \tag{9}$$

Using the posited functional form (2) of the marginal transaction function v, let us rewrite this as

$$Q = \frac{\xi_i}{\xi_0}\left(1 + \frac{b\phi^i}{x^l / Q}\right)^{-1} = \frac{\xi_i}{\xi_0}\left(1 + \frac{2b\varphi^0}{x^l}\right)^{-\frac{1}{2}}. \tag{10}$$

Note that $x^d = x^l/Q$ is the liquidity measure for the international investor in M-instruments, because for this investor, I is the "domestic" currency. For domestic investors, (10) is the inverse I-demand function if $\phi^j > 0$, and the inverse I-supply function if $\phi^j < 0$. If written in the form

$$\frac{1}{Q} = \frac{\xi_0}{\xi_i}\left(1 + \frac{2b\varphi^0}{x^l}\right)^{\frac{1}{2}} = \frac{\xi_0}{\xi_i}\left(1 - \frac{2b(-\varphi^0)}{x^l}\right)^{\frac{1}{2}},$$

it can be regarded as the inverse M-supply for $\varphi^0 = j(x^l, \psi^j) > 0$ and the inverse M-demand in the opposite case.

Optimal transactions ϕ^k and ϕ^f in the asset markets are described by the first order conditions similar to (10):

$$P^k = \frac{\xi_k}{\xi_0} v^k(x^l, \varphi^k),\ k=1,\ ...,\ K;\ P^{*f} = \frac{\xi_f}{\xi_i} v^f(x^l, \varphi^f),\ f=1,\ ...,\ F\ . \qquad (11)$$

The Hamiltonian maximization with respect to c^d and c^i is characterized by the first order conditions

$$\xi_0 = u_{c^d}\ ,\ \xi_i = u_{c^i}\ . \qquad (12)$$

In equilibrium, one can impose the market clearing conditions $\phi^k=\varphi^k=0$, $\phi^f=\varphi^f=0$ for all k and f, as well as the FOREX market clearing: $\varphi^i=0$. According to our assumptions, this means $v \equiv v^k \equiv v^f \equiv 1$ for all k and f. The following characterization of the asset market equilibrium is obtained as a direct consequence of (11):

$$u_{c^d} = \xi_0 = \frac{\xi_k}{P^k},\ k=1,\ ...,\ K,\ \ u_{c^i} = \xi_i = \frac{\xi_f}{P^{*f}},\ f=1,\ ...,\ F. \qquad (13)$$

Equations (13) allow one to draw an analogy with continuous-time CCAPM (Duffie, 1992, Sec. 9), which must be corrected for the present model with non-trivial liquidity constraints.

As it turns out, the potential of (8), (10) and (13) is much stronger compared to Merton's and Ross' original results. Thanks to the explicit laws of motion of the shadow prices, one can decompose the risk premia existing in the relation between any pair of assets. The "parity violation" becomes a source of valuable information about the properties of the studied economy (e.g. the significance of the precautionary saving and currency substitution effects in the household behavior, as in Derviz and Klacek, 1998).

In the next subsection, equations (8) and (10) will be used to derive the equilibrium dynamic of the exchange rate in relation to the return rate differential on liquid assets at home and abroad.

2.3 Long horizon yield parity and the expected exchange rate

2.3.1 The generalized uncovered asset return parity statement

In this subsection, I am examining the mechanism by which the shadow asset prices reveal information about exchange rate expectations.

From now on, the real exchange rate Q is replaced by the nominal exchange rate, S. Understanding of all variables and equations shall be adjusted accordingly. This replacement can be justified by our focus on international investors who do not care about domestic inflation. From a purely technical point of view, when one goes over from real to nominal values of return rates and relative price changes, this leads only to changes in the diffusion parameters. The latter will not be analyzed explicitly.

Let us take an arbitrary domestic asset $k \in$ K and an arbitrary foreign asset $f \in$ F. Recall (13), which describes their equilibrium prices in terms of the shadow prices, and combine the two conditions of (13) with (9) as follows:

$$\xi_0 = \frac{\xi_k}{P^k} = \frac{\xi_f v}{SP^{*f}} . \tag{14}$$

Note that the FOREX market clearing without outside supply is not assumed here, i.e. I do not impose the restriction $v \equiv 1$.

To be able to differentiate (14), one needs the expressions for relative changes in the shadow prices ξ_k, ξ_f. As follows from the adjoint equations (8c), (8d) and the equilibrium conditions (9), (13),

$$\widehat{d}\xi_k = \vartheta dt + d\pi^k - \frac{d\Gamma^k}{P^k} + \left|\sigma^k\right|^2 dt + \frac{(\sigma^0 - \sigma^k)\cdot(\varepsilon^k)^T}{P^k} dt , \tag{15a}$$

$$\widehat{d}\xi_f = \vartheta dt + d\pi^f - \frac{d\gamma^f}{P^{*f}} + \left|\sigma^f\right|^2 dt + \frac{(\sigma^0 - \sigma^f)\cdot(\in^f)^T}{P^{*f}} dt . \tag{15b}$$

Rewriting (14) as $\xi_k S P^{*f} = \xi_f P^k v$ and applying Itô's lemma together with (15a), (15b), we obtain

$$\frac{d\Gamma^k}{P^k} + \widehat{d}P^k - d\pi^k = \frac{d\gamma^f}{P^{*f}} + \widehat{d}P^{*f} - d\pi^f + \widehat{d}S - \widehat{d}v + h\left(P^k, P^{*f}, \frac{\varphi^0}{x^l}\right)dt \,. \tag{16}$$

Here, h is shorthand for the term containing diffusion vector norms and scalar products that are not analyzed in detail.

The left hand side of this equation can be understood as the total return rate (instantaneous yield) dY^k on asset k: the dividend/coupon rate relative to the current price, plus the relative capital gain/loss, minus the internal value loss/attrition rate. In the same way, the first three terms on the right hand side form the total return rate dY^f on asset f. The last two terms in (16), which I shall denote $d\Lambda^{kf}$, originate in transaction costs and risk-adjustment. Collectively, these terms can be dubbed the *disparity rate* between assets k and f. This name points at the fact that in the absence of $d\Lambda^{kf}$, (16) would reduce to the standard *uncovered total return parity* condition on assets k and f.

2.3.2 Siegel's paradox

Siegel, 1972, observed that the uncovered interest rate parity (UIP), if understood quite generally (i.e. without a narrower qualification of instruments and rates of return on them) in a naive non-arbitrage way, cannot be valid simultaneously for international and domestic investors.

UIP is often associated with the statement that the forward rate is the conditional expected spot rate. Due to the universal validity of the *covered interest rate parity* (which is a no-arbitrage relation between three *certain* variables), this is equivalent to the claim that the interest rates at home, *i*, and abroad, *i**, between times *t-1* and *t* satisfy the relation

$$\frac{1+i_t}{1+i*_t}=E_{t-1}\left[\frac{S_t}{S_{t-1}}\right]=\frac{E_{t-1}[S_t]}{S_{t-1}}.$$

As usual, E_{t-1} means expectation conditioned on the generally available information on date *t*-1.

This is the domestic investor perspective. Had we applied the perspective of a foreign investor, the uncovered parity would look like

$$\frac{1+i*_t}{1+i_t}=E_{t-1}\left[\frac{S_{t-1}}{S_t}\right]=S_{t-1}E_{t-1}\left[\frac{1}{S_t}\right].$$

(It is assumed that the information structure of the internationally integrated FOREX market is the same for the residents of all countries.) Indeed, the roles of interest rates i and i^* are inverted for the foreign investor compared to the domestic one, while the exchange rate for the foreigner is $1/S$ instead of S.

The two parity equalities above together imply the equality

$$E_{t-1}\left[\frac{1}{S_t}\right]=\frac{1}{E_{t-1}[S_t]}.$$

However, in any but a perfectly deterministic environment, the left-hand side of the latter must be strictly greater than the right-hand side, as follows from Jensen's inequality (function $x\mapsto\frac{1}{x}$ is strictly convex). Therefore, the discussed general form of the uncovered interest rate parity cannot hold simultaneously for investors living at home and abroad.[6]

Now, let us show that Siegel's paradox does not arise for any pair of assets $k\in K$ and $f\in F$ of the present model. Exactly speaking, if the *k-f* disparity Λ^{kf} and the *f-k* disparity Λ^{fk} are defined by (16), then they satisfy the necessary symmetry/consistency condition $d\Lambda^{fk}=-d\Lambda^{kf}-|\rho|^2\,dt$, so that the equalities

[6] The strict convexity of the function transforming the exchange rate of the domestic agents into that of their international counterparts, relates the analytical background of the paradox with that of Itô's lemma for the same function.

$$dY^k = dY^f + \widehat{d}S + d\Lambda^{kf}\ , \quad dY^f = dY^k + \widehat{d}\left(\frac{1}{S}\right) + d\Lambda^{fk}$$

hold simultaneously.

The trivial proof is a testimony of the power of the shadow asset price method. The relation between dY^k and dY^f was obtained from (14). The latter is the fundamental asset pricing formula of the model when "seen with the eyes" of domestic investors. The same equation can be restated in the form corresponding to the foreign perspective:

$$\xi_i = \frac{\xi_f}{P^{*f}} = \frac{\xi_k S}{P^k v}\ .$$

Now observe that not only S becomes $1/S$, but also v becomes $1/v$ if expressed from the point of view of international investors. Indeed, the marginal transaction process v has a symmetrical definition. It is the derivative of the other country cash with respect to the own country cash involved in the FOREX market transaction:

$v = d\psi / d\varphi$.

It follows that (16) must be just as symmetric and insensitive to the change of perspective from domestic to foreign as the equation from which it was derived, i.e. (14). This concludes the proof.

In connection with the disparity equation $d\Lambda^{fk} = -d\Lambda^{kf} - |\rho|^2 dt$ it is interesting to observe that the perceived forward exchange rate premium, differently from the theoretically derived GURP condition, does not have to be symmetric. That is, even if there is no forward premium for domestic investors buying foreign currency instruments, there is a non-zero premium for the foreign investors buying domestic currency instruments. This is true as long as the exchange rate has a non-zero volatility ρ.

2.4 The producer model and the international uncovered parity of returns

The final application of the shadow price technique deals with a productive firm characterized by four state variables. Let y^0 be the liquidity, i.e. number of cash (M) units on the current account. Further, y^a is the amount of the capital input (A); y^i – amount of the company's currently outstanding foreign liabilities, denoted J (typically, a GDR or Eurobonds) and y^e – amount of currently outstanding shares E issued by the company.

The amount h^c of output C is produced per period at a variable (labor) cost of $L(y^0,y^a;h^a,h^c)$, where h^a is the newly installed capital. The dependence on y^0 means that the firm needs free liquidity to run the production, with decreasing marginal benefits, i.e. L goes to ∞ when y^0 falls to zero, while the partial derivative L_{y^0} decreases to zero as y^0 increases. Dependence of L on y^a means that the production requires inputs of physical capital (the natural requirement here is a positive but decreasing partial derivative L_{y^a}). The positive dependence of production costs on h^a means a disruption of the production process caused by the new capital installation, analogous to Tobin's *q* models. The dependence on h^c is assumed to possess the usual properties, so that it generates a smooth increasing marginal cost function $h^c \quad L_c(y^0,y^a;h^a,h^c)$, giving rise to the traditional component of the C-market inverse supply function of the producer. There is another component of this supply function resulting from intertemporal optimality considerations, as will be shown below.

The labor costs will have one more, stochastic, component, $\zeta(h^c)dZ$, with ζ an increasing (in the vector norm) function of output h^c. Altogether, the variable production costs of the firm are $dw^c=Ldt+\zeta dZ$.

International liabilities J have the stochastic rate of interest (coupon, dividend) $d\gamma^j$. The amount $d\pi^j+h^i dt$ of new international debt is issued per period. Here, h^i is the decision variable and $d\pi^j$ is the new issue noise rate generated by the foreign issue managers, on which the company has no influence.

The dividends are paid in cash at rate h^d per share, so that, in total, the company pays $y^e h^d dt$ per period. The amount of issue of new equity per period is $y^e d\pi^e + h^e dt$, of which h^e is the decision variable and $d\pi^e$ is the new issue noise rate reflecting the technical conditions in the market beyond the company's control.

Let P^a be the market price of one unit of capital good A and P^e – of the newly issued equity share, both in M-terms. Also, define P^i as the I-price of a newly issued unit of international liabilities. Issues of both J and E give rise to issue costs $j^i(y^0/S,h^i)$, $j^e(y^0,h^e)$, with properties analogous to those of the transaction costs defined in Section 2.

Altogether, the vector of controls in the company's decision-making problem is $h=[h^a,h^i,h^c,h^d,h^e]^{\mathrm{T}}$.

From the given definitions follows the transition equation for the state process $y=[y^0,y^a,y^b,y^e]^{\mathrm{T}}$:

$$dy^0 = y^0 d\pi^0 - Sy^i d\gamma^i - y^e h^d dt + \left(ph^c - L(y^0, y^a; h^a, h^c)\right)dt - \varsigma\left(h^c\right)dZ$$

$$- P^a h^a dt + SP^i j^i\left(\frac{y^0}{S}, h^i\right)dt + P^e j^e(y^0, h^e)dt\,, \tag{18a}$$

$$dy^a = -y^a d\pi^a + h^a dt\,, \tag{18b}$$

$$dy^i = y^i d\pi^i + h^i dt\,, \tag{18c}$$

$$dy^e = y^e d\pi^e + h^e dt\,. \tag{18d}$$

Next, I define the aggregator function for instantaneous valuation of the firm's assets and operations in a given moment of time. It is assumed to have the form

$$F(y,h) = f(y^0, y^e; h^d) - \frac{\beta_c}{2}\left|\zeta(h^c)\right|^2. \tag{19}$$

Here, function f measures the firm's ability to pay dividends and finance operations out of current cash holdings y^0, relative to the ownership structure, i.e. number of outstanding shares y^e. The second term on the right hand side of (19) expresses the negative valuation of the production process risk given by ζ.

For simplicity, assume that the firm discounts the future by means of the same rate ϑ as the investors in the previous sections. Then the producer's optimization problem is

$$\max_h E\left[\int_0^T e^{-[\vartheta]_0^t} F(y_t, h_t)dt + e^{-[\vartheta]_0^T} B(T, y_T)\right], \tag{20}$$

subject to the state-transition equation

$$dy=\mu(y,h)dt+\sigma(y,h)dZ, \tag{21}$$

which is the symbolic abbreviation of (18), the holdings of assets y_0 at time $t=0$ given. The final bequest function B is equal to zero by definition if the time horizon of the optimization problem is infinite ($T=\infty$).

When $T<\infty$, there exists a natural final bequest function, which can be useful in many applications. Define by V_t the time t-value function of problem (20) with an infinite horizon:

$$V_t(y) = \max_h E\left[\int_t^\infty e^{-[\vartheta]_t^s} F(y_s, h_s)ds \middle| \mathrm{F}_t\right], \tag{22}$$

where $y_t=y$ is an F_t-measurable initial condition for the transition equation (21). Assume that this value function exists and depends smoothly on all variables. Then, if one defines $B(T,y_T)=V_T(y_T)$, the optimization problem can be split into two periods. Within the planning horizon $0\leq t<T$, there may exist an additional state variable y^*. Typically, y^*_t is the time t-volume of an additional security (e.g. a derivative) with maturity T. At $t=T$, the final cash flow generated by this security is included in $y^0{}_T$ in accordance with its definition. After that, the firm is faced with the optimization problem (21), (22) with the originally defined set of state variables.

The solution to (20), (21) can be obtained by forming the current value Hamiltonian

H$(t,y,h,\lambda,\Lambda)=F(y,h)+\lambda\bullet\mu(y,h)-tr(\Lambda\bullet\sigma\,(y,h))$,

where λ and Λ are the first- and second-order adjoint processes, λ is of the same dimension as y and Λ is a matrix, with $\Lambda=\lambda\bullet D_y\sigma^y$. The Hamiltonian must be maximized with respect to h_t. The co-state process λ, which appears in the above formulae, can be also described as the y-gradient of the value function V defined in (22).

As was discussed in Subsection 2.2, the (first-order) adjoint process λ satisfies the s.d.e.

$$d\lambda = \lambda \cdot \left(\vartheta 1\,\mathrm{dt} - \mathrm{d}A^{\mathrm{c}} + \left\langle A^{c}\right\rangle dt\right) - D_y F dt\ , \tag{23}$$

with the matrix-valued stochastic process A^c of the y-linearization of (21) defined by

$$dA^c = \begin{bmatrix} d\pi^0 - L_{y^0}dt + P^i j_y^i + P^e j_y^e dt & -L_{y^a}dt & -Sd\gamma^i & -h^d dt \\ 0 & -d\pi^a & 0 & 0 \\ 0 & 0 & d\pi^i & 0 \\ 0 & 0 & 0 & d\pi^e \end{bmatrix}.$$

Another consequence of the stochastic maximum principle is the Hamiltonian optimization result. Skipping the first order conditions, I write it out in terms of the relevant inverse supply and demand functions. The inverse supply of C is

$$p = L_c\left(y^0, y^a; h^a, h^c\right) + \left[\frac{\beta_c}{\lambda_0}\varsigma(h^c) - \sigma^0\right] \cdot \left[\varsigma'(h^c)\right]^T .$$

The inverse demand for A and the inverse supplies of J and E are given by

$$P^a = \frac{\lambda_a}{\lambda_0} + L_a \, ; \; SP^i = -\frac{\lambda_i}{\lambda_0}\frac{1}{j_h^i} \, ; \; P^e = -\frac{\lambda_e}{\lambda_0}\frac{1}{j_h^e} \tag{24}$$

(cf. (11) in Subsection 2.3).

Equation (24) allows one to derive the international parity of investment returns for the modeled company by repeating the procedure of Subsection 3.1. Indeed, (24) implies

$$\lambda_0 = -\frac{\lambda_e}{P^e j_h^e} = -\frac{\lambda_i}{SP^i j_h^i}.$$

Note that $R^e = P^e j^e{}_h(y^0, h^e)$ and $R^i = P^i j^i{}_h(y^0, Sh^i)$ are the *effective prices* cashed in by the issuing firm (i.e. they include a marginal issue discount factor j_h). By applying Itô's lemma and the adjoint equations for λ_e and λ_i, which are part of (23), we get (cf. (16) in Subsection 3.1)

$$\widehat{d}R^e + \frac{h^d dt}{R^e} + d\pi^e = \widehat{d}R^i + \frac{d\gamma^i}{R^i} + d\pi^i + \widehat{d}S + g_0 dt + g_1\left(\frac{h^e}{y^0}, \frac{Sh^i}{y^0}\right)dt + g_2 \widehat{d}y^0. \tag{25}$$

Denote by dY^e the left-hand side of this equation, i.e. the instantaneous total return on one E-share, by dY^i – the first three terms on the right-hand side (the instantaneous total return on one unit of J), and the last three terms (i.e. the disparity), by $dG(y^0, h^e, Sh^i)$. One can point at three cases when the uncovered disparity dG is close to its traditional constant (volatility dependent) component $g_0 dt$. This can happen if either of the below named conditions is satisfied.

- The transaction volumes h^e and h^i are close to zero.
- The company liquidity is very high compared to the traded volumes of E and J ($y^0 \rangle\rangle h^e, Sh^i$) as well as stable in time ($E_t\left[\widehat{d}y_t^0\right]/ dt \approx 0$).
- The foreign participants in the markets for E and J have significantly lower transaction costs than the transaction costs faced by the company.

In all named cases the prices P^e and P^i and dividend rates $h^d dt$, $d\gamma^i$ are related by a simple uncovered parity condition (cf. (27) below)

$$dY^e = dY^i + \widehat{dS} + g_0 dt. \qquad (26)$$

In such a market, the expected exchange rate movements $\widehat{dS}$ perfectly internationally equalize the expected returns on the company's productive assets. Interpreting this condition, one must bear in mind that the exchange rate is exogenous with respect to the prices of the company's liabilities. Therefore, one should speak of the forced asset price adjustment in accordance with the expected dividends and the exchange rate movements. Anyway, the theory predicts the instantaneous exchange rate-adjusted yield differential $dY^e - dY^i - \widehat{dS}$ to be a randomly disturbed constant. This is quite in line with the exchange rate "neutrality" predicted by neoclassical economics.

Naturally, if one looks at specific sectors and individual firms with "dually floated" stock or bonds, the above parity condition is often violated. For companies dependent on external foreign financing, the theory outlined above suggests that a higher-than-average or variable disparity may be caused by a liquidity squeeze. The latter complicates the financial situation of the company every time the exchange rate moves in the direction disadvantageous for its business. Thus, export-dependent sectors should experience a stock price reduction if the market expects an appreciation of the currency, while the stock prices in the sectors that use a lot of imported inputs, are hit by its depreciation. Both effects should be visible if the relevant segments of the stock market are sufficiently integrated with the outside world.

2.5 Extensions and final remarks

The price equations (13) make up two "modified CCAPM's", one for domestic securities and the other for international securities. They are tied together by (9), giving *an international generalized CCAPM*. In view of the disparities reflected in (16), our international CCAPM explains violation of the textbook uncovered parity. Very similar results based on martingale techniques can be found in Zapatero, 1995.

Equation (16) can be used to extract information about long-term exchange rate trends. To do this, it is best to find assets k and f such as were described by conditions I-IV of Section 1.2 (high liquidity of the secondary market for k and f, low transaction costs, etc.) As was explained there, GURP reduces to a simpler equation in such cases. Namely, the generalized parity condition (16) reduces to

$$dY^k = dY^f + \widehat{d}S + h^0 dt, \tag{27}$$

where h^0 is a term containing scalar products of diffusion vectors. The basic message of (27) is that the processes Y^k-Y^f and $s=logS$ are perfectly positively correlated. This is exactly what one finds in the Czech koruna vs. euro case (recall Fig. 3a) for very short time intervals (roughly, between several days and two weeks). The relationships of other examined currencies suggest that the exact understanding of the stochastic differential term ds can differ depending on the time interval and the noise filtering procedure proper for the particular pair of currencies. Recall the example of the DEM/USD exchange rate (Fig. 7), for which the short time step was evidently the wrong one, while smoothing over a three month period worked much better, as well as for the ATS/USD rate (Fig. 6).

Equation (27) has a natural, discrete time analogue

$$Y^k{}_t = Y^f{}_t + \widehat{S}_t + a_0 + \varepsilon_t \tag{28}$$

(a restatement of equation (2) of Part 1), which can be examined for various periods, time step lengths, etc.

Market equilibria in the present model were derived as price and trade volume supply/demand schedule functions of the shadow prices of the agents. Some of these schedules depend on the current value of the exchange rate. However, a much more surprising result is that they are *seemingly independent of the statistics of exchange rate movements*. Parameters of the exchange rate process happen to cancel out of the supply and demand schedules. They remain present in the dynamics of the shadow prices. As is well known, the latter may contain *sunspots*, i.e. nontrivial spurious values at infinity. A consequence is the possibility of multiple self-fulfilling equilibria in the asset markets, including the FOREX market. Particularly, multiple rational equilibrium paths are not excluded.

The main result of Sections 2.2 and 2.3 is the generalized uncovered total asset return parity condition, comparable to an international CCAPM with transaction costs and liquidity constraints. Every pair of assets that have to do with the economy's external sector has its own advantages and disadvantages as regards the analysis of exchange rate expectations. Highly liquid default-free long maturity bond markets are closest to the ideal textbook property of uncovered parity and can be used to extract these expectations most efficiently.

Looking at the data of numerical examples of Part 1, one always finds a non-zero yield differential. The positive sign of this differential seems to be logical, the country premium compensating for high variability and transaction expenditures. However, regarding the returns of Czech producers relative to the foreign ones, it is hard to justify a final country premium in most segments of the Czech equity market. Indeed, abstracting from occasional price changes, there is no way of expecting a yield on Czech equity to be sufficiently high to compensate for its high risk, given negligible dividends that have been a tradition since the beginning of the privatization process.[7] Equation (16) of Section 2.3 explains that there is no contradiction. Although, in general, Γ^k is lower than γ^f, the secondary market price level P^k lies even deeper below P^{*f}, so that the inequality $\frac{d\Gamma^k}{P^k} > \frac{d\gamma^f}{P^{*f}}$ is still satisfied, justifying foreign investment in k in the absence of foreign exchange turbulence.

In Section 4, an analogue (25) of the GURP condition (16) was derived in the optimizing model of a productive firm. It indicates the way to use equity returns of internationally traded domestic companies (e.g. in the GDR form) for estimating the competitiveness of the corresponding industry vis-a-vis its international rivals. The disparity shall be understood as a measure of over- or under-valuation of the

[7] The situation is slowly changing in recent months, but involves only a small number of most successful firms, all of them foreign-owned.

currency, seen in the context of competitiveness of the involved productive industry. A positive deviation of the disparity term in the yield differential from the average level is a sign of a competitively cheap domestic currency, expected to get more expensive in the long run, and vice versa. Note that by doing this analysis, one would generalize the well-known calculations of the expected increase/decrease of the real exchange rate, this time applied to a specific partial market. From another point of view, which takes the exchange rate to be exogenous for the market segment in question, the generalized parity provides a measure of expected growth of the industry represented by k, relative to comparable foreign industries.

The results of Section 2.3 show a way to replace the analysis of fundamentals with the *utilization of the financial market analysts' work* with these fundamentals. Although it seems to be a convenient shortcut, one cannot hope to make this the unique method of extracting exchange rate expectations. If everyone relied on the market beliefs as described in the paper, sunspots and unpredictable switches between equilibrium paths would occur even more frequently than at present. If the fundamentals are evaluated by means of asset prices alone, which are, in turn, formed on the basis of the analysts' view of fundamentals, the sunspots seem to be inevitable. A way out of this vicious circle would be to reserve the generalized uncovered asset yield parity as a supplementary technique. It must be accompanied by the study of international markets for goods and services. Then, although aggregate PPP is hardly to be found, accurate analysis of specific partial markets under the no-arbitrage condition can provide clarity where the financial market bubbles cause obscurity. Besides, it might be useful to look closer at *primary* intermediaries between the real economy and the financial markets, i.e. commercial banks.

3 Conclusion

Information about the opinions of international investors on the expected direction in the exchange rate move can be extracted from a properly quantified property of generalized uncovered return parity between two identical assets that pay returns one in domestic currency and the other in a foreign currency. This information includes, among other things, the expectations of those fundamental variables which the investors consider relevant for their domestic currency demand. The fundamental variable analysis exercises an influence on the exchange rate formation through the asset prices, and is materialized in the form of the secondary market yields. In sum, the equilibrium relationship between the expected exchange rate movement and the yield differential is manifested in the Generalized Uncovered Return Parity (GURP) formula, which finds both theoretical and empirical support and does not suffer from inconsistencies characteristic of the traditional uncovered interest rate parity.

In the exchange rate analysis based on GURP, it is desirable to use instruments with long maturity, traded in a liquid secondary market and equally available to residents and non-residents, with minimal transaction costs. Since individual (issuer-specific) risk factors are to be minimized, the best candidates are government bonds with the longest possible maturities. From this viewpoint, the most appropriate foreign instrument in the GURP analysis of the Czech koruna is a 5-year German government bond. Recently, 10 year CZK-denominated bonds were issued by the European Investment Bank. Since the rating of the latter exceeds that of the Czech government, these bonds are a possible temporary benchmark to be confronted with the 10-year benchmarks of Europe and North America, until a better candidate appears. It remains to hope that Czech government bonds with 10 year and longer maturities will one day occupy their legitimate place in the market, and their liquidity will exceed that of the EIB-bonds.

In principle, expected exchange rate movements can be estimated by means of other capital market segments as well. If data from several segments are available, they can be compared to improve the quality of results. In any case, it is essential to have at least one segment as a part of the domestic financial market, because the proximity of a price formation playground to the domestic monetary authority can speed up the registration of signals on possible corrections in the expected exchange rate. Therefore, financial liberalization is one of the means to ameliorate the exchange rate signals coming from the capital market. Conversely, an artificial slowdown of the financial account deregulation, particularly checks on the freedom of domestic residents to invest in foreign fixed income instruments, means chasing the external valuation of the domestic currency away from one’s own territory, therewith contributing to its possible distortion.

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