(Z; z)-Solutions

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Institute of Computer Science
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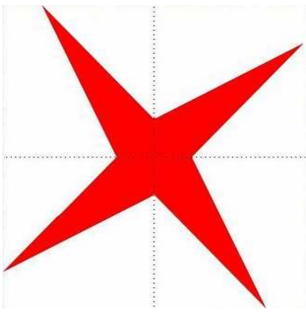
Abstract:

We give a simplified definition of Shary's AE-solutions which leads to a simple proof of their characterization.[1]



Keywords:
Interval linear equations, (Z, z)-solution, AE-solution, characterization, tolerance solution, control solution.

[1] Above: logo of interval computations and related areas (depiction of the solution set of the system $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$, $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$ (Barth and Nuding [1])).

1 Introduction

In 1995 Sergey P. Shary proposed a new unifying view of different concepts of solutions of interval linear equations by introducing quantifications over interval coefficients. His definition is reformulated here in order to make it, as well as the main characterization, as simple as possible. E is the matrix of all ones and e is the vector of all ones.

Definition. Let $|Z| = E \in \mathbb{R}^{m\times n}$ and $|z| = e \in \mathbb{R}^m$. A vector $x \in \mathbb{R}^n$ is said to be a (Z,z)-solution of a system $\boldsymbol{A}x = \boldsymbol{b}$ if for each $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$ with $Z_{ij} = -1$ and for each $b_i \in [\underline{b}_i, \overline{b}_i]$ with $z_i = -1$ there exist $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$ with $Z_{ij} = 1$ and there exist $b_i \in [\underline{b}_i, \overline{b}_i]$ with $z_i = 1$ such that $Ax = b$ holds[2].

2 Description

Despite the complexity of this definition, it turns out that description of (Z,z)-solutions becomes wonderfully simple as soon as the Hadamard product "$\circ$" is employed. The following theorem constitutes a generalization of the Oettli-Prager theorem as well as of several other previous results.

Theorem 1. (Shary-Lakeyev-Rohn) *A vector $x \in \mathbb{R}^n$ is a (Z,z)-solution of $\boldsymbol{A}x = \boldsymbol{b}$ if and only if it satisfies*

$$|A_c x - b_c| \le (Z \circ \Delta)|x| + z \circ \delta. \tag{2.1}$$

Proof. Given Z and z with $|Z| = E$ and $|z| = e$, first define interval matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and interval vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ by

$$\begin{aligned}
\boldsymbol{A}_1 &= \{\tfrac{1}{2}(E - Z)\circ A \mid A \in \boldsymbol{A}\} = [A'_c - \Delta', A'_c + \Delta'],\\
\boldsymbol{A}_2 &= \{\tfrac{1}{2}(E + Z)\circ A \mid A \in \boldsymbol{A}\} = [A''_c - \Delta'', A''_c + \Delta''],\\
\boldsymbol{b}_1 &= \{\tfrac{1}{2}(e - z)\circ b \mid b \in \boldsymbol{b}\} = [b'_c - \delta', b'_c + \delta'],\\
\boldsymbol{b}_2 &= \{\tfrac{1}{2}(e + z)\circ b \mid b \in \boldsymbol{b}\} = [b''_c - \delta'', b''_c + \delta''].
\end{aligned}$$

As we can see, $\boldsymbol{A}_1$ is obtained from $\boldsymbol{A}$ by zeroing ijth interval coefficients with $Z_{ij} = 1$, $\boldsymbol{A}_2$ by zeroing those with $Z_{ij} = -1$, and an analogue holds for $\boldsymbol{b}_1$, $\boldsymbol{b}_2$. Then x is a (Z,z)-solution if and only if for each $A_1 \in \boldsymbol{A}_1, b_1 \in \boldsymbol{b}_1$ the equation

$$(A_1 + A_2)x = b_1 + b_2,$$

i.e., the equation

$$A_1 x - b_1 = b_2 - A_2 x,$$

is satisfied for some $A_2 \in \boldsymbol{A}_2$, $b_2 \in \boldsymbol{b}_2$, which is equivalent to

$$\{A_1 x - b_1 \mid A_1 \in \boldsymbol{A}_1,\, b_1 \in \boldsymbol{b}_1\} \subseteq \{b_2 - A_2 x \mid A_2 \in \boldsymbol{A}_2,\, b_2 \in \boldsymbol{b}_2\}. \tag{2.2}$$

But according to Proposition 2.27 in [2],

$$\{A_1 x - b_1 \mid A_1 \in \boldsymbol{A}_1,\, b_1 \in \boldsymbol{b}_1\} = [A'_c x - \Delta'|x| - b'_c - \delta',\ A'_c x + \Delta'|x| - b'_c + \delta']$$

[2] Thus "-1" corresponds to "$\forall$" and "1" to "$\exists$". It could be argued that the reverse order would be more natural, but we would have to pay for it by introducing minus signs into the main formula (2.1).

and

$$\{b_2 - A_2x \mid b_2 \in \boldsymbol{b}_2, A_2 \in \boldsymbol{A}_2\} = [-A_c''x - \Delta''|x| + b_c'' - \delta'', \, -A_c''x + \Delta''|x| + b_c'' + \delta''],$$

hence the inclusion (2.2) is equivalent to

$$-(\Delta'' - \Delta')|x| - (\delta'' - \delta') \leq (A_c' + A_c'')x - (b_c' + b_c'') \leq (\Delta'' - \Delta')|x| + (\delta'' - \delta'),$$

which gives

$$|(A_c' + A_c'')x - (b_c' + b_c'')| \leq (\Delta'' - \Delta')|x| + (\delta'' - \delta'). \tag{2.3}$$

Now, taking into account that $A_c' + A_c'' = A_c$, $b_c' + b_c'' = b_c$, $\Delta'' - \Delta' = Z \circ \Delta$, and $\delta'' - \delta' = z \circ \delta$, we obtain (2.1). □

In this way, some previously defined types of solutions become special cases of (Z, z)-solutions, and their descriptions turn out to be special cases of Theorem 1. So we obtain

- weak solutions for $Z = E$, $z = e$ (Theorem 2.9 in [2]),
- strong solutions for $Z = -E$, $z = -e$ (Theorem 2.16 in [2]),
- tolerance solutions for $Z = -E$, $z = e$ (Theorem 2.28 in [2]),
- control solutions for $Z = E$, $z = -e$ (Theorem 2.29 in [2]).

This shows that Theorem 1 (though little known so far) indeed offers a unified view of different types of solutions of interval linear equations. It could also be easily reformulated for interval linear inequalities, but we refrain from it here.

3 History

S. P. Shary presented his idea of (Z, z)-solutions, which he then called "$\forall\exists$-solutions" (today "AE-solutions"), at a conference in Wuppertal in 1995 and published it in [5]. His formulation of Theorem 1 contained, however, interval arithmetic operations. A proof not using these operations and based purely on the Oettli-Prager theorem was given in this author's letter to S. P. Shary and A. V. Lakeyev [4]. The final step towards utmost simplicity by employing the Hadamard product was done by A. V. Lakeyev in [3]. Our definition using Z and z instead of subsets of indices, as well as the proof, are new.

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