Theoretical Characterization of Enclosures

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Institute of Computer Science
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Dedicated to memory of Argentinean writer
Jorge Luis Borges (1899-1986)

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Technical report No. V-1158

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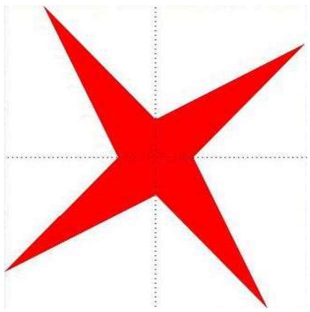
Abstract:

We give a theoretical characterization of enclosures of the solution set of interval linear equations formulated in terms of components of the solution set of the "dual" Oettli-Prager inequality.[1]



Keywords:
Interval linear equations, solution set, enclosure, component, characterization.

[1]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction and notation

Anyone interested in interval linear equations knows the inequality

$$|A_c x - b_c| \le \Delta|x| + \delta;$$

this is the Oettli-Prager inequality [2] describing the solution set of a system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$ with $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta] \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{b} = [b_c - \delta, b_c + \delta] \in \mathbb{IR}^n$. Very little, if anything at all, is known, however, of its "dual" inequality

$$|A_c x - b_c| \ge \Delta|x| + \delta.$$

In this report we show that these two inequalities are related in a peculiar way. If $\boldsymbol{A}$ is regular and $\delta > 0$, then the solution set of the first inequality is connected whereas that one of the second inequality consists of exactly 2^n components (nonempty connected subsets maximal with respect to inclusion), and an interval vector encloses the solution set of the first inequality if and only if it intersects all the 2^n components of the solution set of the second inequality. It is just this result that we call the "theoretical characterization of enclosures". The proof employs two nontrivial results from [3], [4], of which particularly the second one is little known.

Notation used: $Y = \{-1, 1\}^n$ is the set of all ±1-vectors in $\mathbb{R}^n$, and T_y denotes the diagonal matrix with diagonal vector y (used for $y \in Y$ only).

2 The result

Denote

$$\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) = \{\, x \mid |A_c x - b_c| \le \Delta|x| + \delta \,\} \tag{2.1}$$

and

$$\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b}) = \{\, x \mid |A_c x - b_c| \ge \Delta|x| + \delta \,\}. \tag{2.2}$$

Then we have the following main result.

Theorem 1. *Let $\boldsymbol{A}$ be regular and let $\delta > 0$. Then an interval vector $[\underline{x}, \overline{x}]$ is an enclosure of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ if and only if it intersects all the components of $\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$.*

Proof. The proof proceeds in three steps.

(a) For each $y \in Y$ define a set X_y by

$$X_y = \{\, x \mid T_y A_c x - \Delta t \ge T_y b_c + \delta,\ -t \le x \le t \text{ for some } t \,\}. \tag{2.3}$$

The set described by the right-hand side system of linear inequalities is a convex polyhedron, therefore X_y, as its projection onto the x-subspace, is again a convex polyhedron. Next we prove that $X_y \subseteq \boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$. Let $x \in X_y$, then it satisfies

$$T_y(A_c x - b_c) \ge \Delta t + \delta, \quad t \ge |x|, \tag{2.4}$$

hence

$$T_y(A_c x - b_c) \ge \Delta|x| + \delta \tag{2.5}$$

which in virtue of nonnegativity of the right-hand side implies that $T_y(A_cx - b_c) \geq 0$, thus $T_y(A_cx - b_c) = |A_cx - b_c|$, and (2.5) turns into

$$|A_cx - b_c| \geq \Delta|x| + \delta \tag{2.6}$$

which means that $x \in \boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$. Thus, $\bigcup_{y \in Y} X_y \subseteq \boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$. To prove the converse inclusion, take $x \in \boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$. Then it satisfies (2.6), thus also (2.5) for $y = \operatorname{sgn}(A_cx - b_c)$, and taking $t = |x|$, we see that it also satisfies (2.4) and (2.3), so that $x \in X_y$. In this way we have proved that

$$\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b}) = \bigcup_{y \in Y} X_y. \tag{2.7}$$

Finally we prove that all the X_y's are mutually disjoint. Suppose it is not so, so that $x \in X_y \cap X_{y'}$ for some $y \neq y'$, where $y_i = 1$ and $y'_i = -1$ for some i. Then from (2.5) we obtain both $(A_cx - b_c)_i \geq 0$ and $-(A_cx - b_c)_i \geq 0$, hence $(A_cx - b_c)_i = 0$ implying $(\Delta|x| + \delta)_i = 0$ which is a contradiction because $\delta > 0$ by assumption. Hence, (2.7) is a decomposition of $\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$ into a union of mutually disjoint convex (i.e., connected) polyhedra which, in turn, means that each X_y is a component of $\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$ (we shall see later that all the X_y's are nonempty, so that there are exactly 2^n of them).

(b) Next we prove that if $[\underline{x}, \overline{x}]$ is an enclosure of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$, then it intersects all the components X_y, $y \in Y$. To see this, take an arbitrary $y \in Y$ and consider the absolute value equation

$$A_cx - T_y\Delta|x| = b_c + T_y\delta. \tag{2.8}$$

Since $\boldsymbol{A}$ is regular by assumption, by Theorem 2.2 in [3] the equation (2.8) has exactly one solution x_y which belongs to $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ and thus also to $[\underline{x}, \overline{x}]$. Rearranging the equation (2.8) to the form

$$T_y(A_cx - b_c) = \Delta|x| + \delta, \tag{2.9}$$

we can see that x_y satisfies (2.5), (2.4) and (2.3), hence $x_y \in X_y$. Thus $x_y \in [\underline{x}, \overline{x}] \cap X_y$ for each $y \in Y$, so that $[\underline{x}, \overline{x}]$ intersects all the components of $\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$.

(c) Finally we shall prove that if $[\underline{x}, \overline{x}] \cap X_y \neq \emptyset$ for each $y \in Y$, then $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) \subseteq [\underline{x}, \overline{x}]$. Take $x_y \in [\underline{x}, \overline{x}] \cap X_y$ for each $y \in Y$ and let $x \in \boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$. To prove that $x \in [\underline{x}, \overline{x}]$, we proceed as follows. Since $x \in \boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$, by definition of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ there exist $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ such that $Ax = b$. Now we have

$$|T_y(Ax_y - b) - T_y(A_cx_y - b_c)| = |(A - A_c)x_y + (b_c - b)| \leq \Delta|x_y| + \delta, \tag{2.10}$$

hence

$$T_y(Ax_y - b) \geq T_y(A_cx_y - b_c) - \Delta|x_y| - \delta \geq 0, \tag{2.11}$$

the nonnegativity being a consequence of (2.5) because $x_y \in X_y$. Thus we have proved that

$$T_y(Ax_y - b) \geq 0 \tag{2.12}$$

for each $y \in Y$. Now Theorem 2 in [4] tells us that this property implies existence of x^* such that $Ax^* = b$ and x^* belongs to the convex hull of the points x_y, $y \in Y$. Since each x_y, $y \in Y$, belongs to the convex set $[\underline{x}, \overline{x}]$, its convex hull is a part of $[\underline{x}, \overline{x}]$, hence $x^* \in [\underline{x}, \overline{x}]$. But since $Ax^* = b$ and $Ax = b$ and A is nonsingular, it must be $x^* = x$, hence $x \in [\underline{x}, \overline{x}]$. In this way we finally have that $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) \subseteq [\underline{x}, \overline{x}]$, which was to be proved. □

3 Conclusion

The result remains highly theoretical because in practice we will hardly ever be able to check that an interval vector intersects 2^n sets. But it is of certain interest because of its three features: first, that such a characterization exists at all; second, due to a special way in which inequalities $|A_c x - b_c| \leq \Delta|x| + \delta$ and $|A_c x - b_c| \geq \Delta|x| + \delta$ are related together; and third, due to the sole fact that the solution set of $|A_c x - b_c| \geq \Delta|x| + \delta$ has exactly 2^n components that are explicitly described by (2.3).

Bibliography

[1] W. Barth and E. Nuding, *Optimale Lösung von Intervallgleichungssystemen*, Computing, 12 (1974), pp. 117–125. 1

[2] W. Oettli and W. Prager, *Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides*, Numerische Mathematik, 6 (1964), pp. 405–409. 1

[3] J. Rohn, *Systems of linear interval equations*, Linear Algebra and Its Applications, 126 (1989), pp. 39–78. 1, 2

[4] J. Rohn, *An existence theorem for systems of linear equations*, Linear and Multilinear Algebra, 29 (1991), pp. 141–144. 1, 2